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CLOSED ULTRAFILTERS AND REALCOMPACTNESS

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We introduce some conditions which are closely related to closed ultrafilters and establish interconnections among these conditions and characterize realcompactness, almost realcompactness, c-realcompactness and weak-cb^*-ness $\cdots$.

We introduce some conditions which are closely related to closed ultrafilters and establish interconnections among these conditions and characterize realcompactness, alomost realcompactness, c-realcompactness and weak-cb^*-ness, $\cdots$.

Throughout this paper, by a space we mean a completely regular Hausdorff space and all functions are continuous and we assume familiarity with [3] whose notation and terminology will be used throughout. For a given space X, we denote by βX (or υX) the Stone-Čech compactification (or realcompactification) of X. In §1, we give definitions and preliminaries and introduce some conditions which are closely related to closed ultrafilters. In §2, we establish interconnections among conditions introduced in §1. In §3, we characterize realcompactness, almost realcompactness c-realcompactness and weak-cb^*-ness and give some examples in §4.

Notations and terminologies. N = the set of positive integers, nbd = neighborhood, ω = the first countable ordinal, Ω = the first uncountable ordinal, $C(X)$ = the ring of all continuous functions on X, $Z(f)$ = the zero set of $f \in C(X)$ where we assume $0 \leqq f \leqq 1$, $Z(X)$ = the set of all zero sets, $X^* = \beta X - X$. $\mathscr{F}$ ($\mathscr{U}$ or $\mathscr{R}$ resp.) = a free closed (open or regular closed resp.) ultrafilter. $\mathscr{F}^p(\mathscr{Z}^p)$ = a free closed (Z) ultrafilter converging to $p \in X^*$. $\mathfrak{F}$ = the set of all $\mathscr{F}$ (similarly define $\mathscr{U}^p$, $\mathfrak{U}$ and $\mathfrak{R}$ resp.), cl$\mathscr{U} = \{\mathrm{cl}\, U;\ U \in \mathscr{U}\}$ and $\{F_n\}_{\mathrm{cl}} \downarrow$ ($\{F_n\}_{\mathrm{cl}} \downarrow \varnothing$) = a decreasing sequence of closed sets (with the empty intersection). Similarly we define $\{R_n\}_{rc} \downarrow$ and $\{Z_n\}_{ze} \downarrow \cdots$ where "rc" and "ze" denote "R_n is a regular closed set" and "Z_n is a zero set" respectively.

1. Definitions and preliminaries.

A family $\mathscr{A}$ of subsets of X is said to be *stable* if for any $f \in C(X)$ there is $A \in \mathscr{A}$ such that $f \mid A$ is bounded. Mandelker ([10], Th. 5.1) has proved that X is realcompact iff any stable closed family $\mathscr{A}$ with the finite intersection property has non-empty intersection and Hardy and Woods ([4], Lemma 2.6) have obtained that $\mathscr{R}$ is stable iff there is $p \in \upsilon X - X$ and $\mathscr{R} \to p$. We say that $\mathscr{U}$ or $\mathscr{R}$ has *CIP* if $\cap\, \mathrm{cl}\, A_n \neq \varnothing$ for any

$A_n \in \mathscr{U}$ or $\mathscr{R}$ respectively.

1.1. (1) *$\mathscr{A} \in \mathfrak{F}$ (or $\mathfrak{U}$) is stable iff $\mathscr{A}$ converges to some point of $\upsilon X - X$.*

(2) *X is realcompact iff no $\mathscr{F}$ is stable.*

(3) *For $p \in \beta X - \upsilon X$, no prime closed (resp. open) filter $\mathscr{A}$ converging to p has CIP.*

Proof. (1) From the same method used in the proof of Lemma 2.6 in [4]. (2) From (1). (3) Let $p \in \beta X - \upsilon X$. Then there is $f \in C(\beta X)$ with $p \in Z(f) \subset \beta X - \upsilon X$. Since $\mathscr{A}$ is prime, $\mathscr{A} \ni F_n = \{x \in X;\ f(x) \leqq 1/n\}$(resp. $U_n = \{x \in X;\ f(x) < 1/n\}$) and hence $\mathscr{A}$ does not have *CIP*.

The following are well known (e.g., p. 649 line 4 in [4]).

1.2. (1) $\mathfrak{U} \ni \mathscr{U} \Rightarrow \mathscr{R} = \mathrm{cl}\,\mathscr{U} \in \mathfrak{R}$. *If $\mathscr{U}$ has CIP, so has $\mathscr{R}$.*

(2) $\mathfrak{R} \ni \mathscr{R} \Rightarrow \mathscr{U}(R) = \{U;\ \mathrm{int}\,R \subset U$ *for some $R \in \mathscr{R}$ and U is open*$\} \in \mathfrak{U}$. *If $\mathscr{R}$ has CIP, so has $\mathscr{U}(\mathscr{R})$.*

(3) $\mathfrak{U} \ni \mathscr{U} \Rightarrow \mathscr{U}(\mathrm{cl}\,\mathscr{U}) = \mathscr{U}$.

(4) $\mathfrak{R} \ni \mathscr{R} \Rightarrow \mathrm{cl}(\mathscr{U}(\mathscr{R})) = \mathscr{R}$.

1.3. We will divide X^*, especially $\upsilon X - X$, into three domains.

$\mathfrak{F}(0) = \{p \in X^*;$ any $\mathscr{F}^p$ has *CIP*$\}$.

$\mathfrak{F}(0, \triangle) = \{p \in X^*;$ there are $\mathscr{F}_1^p$ with *CIP* and $\mathscr{F}_2^p$ without *CIP*$\}$.

$\mathfrak{F}(\triangle) = \{p \in X^*;$ no $\mathscr{F}^p$ has *CIP*$\}$.

$\mathfrak{F}(\upsilon, \triangle) = \mathfrak{F}(\triangle) \cap (\upsilon X - X)$.

Similarly we define $\mathfrak{U}(0)$, $\mathfrak{U}(0, \triangle)$, $\mathfrak{U}(\triangle)$, $\mathfrak{U}(\upsilon, \triangle)$, $\mathfrak{R}(0)$, $\mathfrak{R}(0, \triangle)$, $\mathfrak{R}(\triangle)$ and $\mathfrak{R}(\upsilon, \triangle)$. From 1.1(3) and 1.2 we have $\mathfrak{U}(0) = \mathfrak{R}(0)$, $\mathfrak{U}(0, \triangle) = \mathfrak{R}(0, \triangle)$, $\mathfrak{U}(\triangle) = \mathfrak{R}(\triangle)$ and $\beta X - \upsilon X \subset \mathfrak{U}(\triangle)$.

1.4. As generalization of realcompactness, almost realcompactness [1], c-realcompactness [2] and a-realcompactness [2] are introduced. X is said to be *almost realcompact* if any open ultrafilter $\mathscr{U}$ with *CIP* is fixed, that is, $\cap\,\mathrm{cl}\,\mathscr{U} \neq \varnothing$[1]. X is *a-realcompact* (= *closed-complete*) if any closed ultrafilter $\mathscr{F}$ with *CIP* is fixed [2]. We say that X is *c-realcompact* if for each point $p \in \beta X - X$, there exists $\{R_n\}_{rc} \downarrow \varnothing$ with $p \in \cap\,\mathrm{cl}_{\beta X} R_n$ [5]. From 1.2 and the definition we have

(1) *X is almost realcompact iff $\mathfrak{U}(0) \cup \mathfrak{U}(0, \triangle) = \varnothing$.*

(2) *X is c-realcompact iff $\mathfrak{U}(0) = \varnothing$.*

(3) *X is a-realcompact iff $\mathfrak{F}(0) \cup \mathfrak{F}(0, \triangle) = \varnothing$.*

From these results and 1.2, it is natural to introduce the notion of wa-realcompactness, that is, we say that X is *wa-realcompact* if $\mathfrak{F}(0) = \emptyset$. Since $\mathfrak{F}(0, \triangle) = \emptyset$ for a normal space, a normal wa-realcompact space is a-realcompact. If X is either c-realcompact or a-realcompact, then X is wa-realcompact (cf. 2.1(1) below). The converse is not necessarily true (see 4.1 and 4.2 below). As another generalization of cb (weak-cb)-ness, we introduced the notion of cb^* (weak-cb^*) spaces [8]. X is said to be cb^* (*weak*-cb^*) if $\bigcap \mathrm{cl}_{\upsilon X} F_n = \emptyset (\bigcap \mathrm{cl}_{\upsilon X} R_n = \emptyset)$ for any $\{F_n\}_{\mathrm{cl}} \downarrow \emptyset (\{R_n\}_{rc} \downarrow \emptyset)$. In [8] we proved that X is cb^* iff any perfect map onto X is hyper-real. The following are easily seen by 1.1(3), 1.3 and the definitions.

(4) *X is weak-cb^* iff $\mathfrak{U}(\upsilon, \triangle) \cup \mathfrak{U}(0, \triangle) = \emptyset$.*

(5) *X is cb^* iff $\mathfrak{F}(\upsilon, \triangle) \cup \mathfrak{F}(0, \triangle) = \emptyset$.*

(6) *X is realcompact iff X is wa-realcompact and cb^*.*

(7) *X is countably compact iff $\mathfrak{F}(0, \triangle) \cup \mathfrak{F}(\triangle) = \emptyset$.*

(8) *X is pseudocompact iff $\mathfrak{U}(0, \triangle) \cup \mathfrak{U}(\triangle) = \emptyset$.*

DEFINITION 1.5. $\mathfrak{F}$ is said to be countably paracompact ($= cp$) (*weakly countably paracompact* ($= wcp$)) if for any $\{F_n \in \mathscr{F}\} \downarrow \emptyset$ there are $\{E_n \in F\} \downarrow \emptyset$ and $\{U_n\}_{\mathrm{open}} \downarrow$ such that $E_n \subset F_n$, $E_n \subset U_n (E_n \subset \mathrm{cl}\, U_n)$ and $\bigcap \mathrm{cl}\, U_n = \emptyset$. Obviously we have

(1) *If $\mathscr{F}$ is either unstable or has CIP, then it is cp.*

(2) *A cp $\mathscr{F}$ is wcp.*

DEFINITION 1.6. For $\mathscr{F}$ (or $\mathscr{U}$), we denote by $\mathfrak{U}(\mathscr{F})$ (or $\mathfrak{F}(\mathscr{U})$) the set of open (or closed) ultrafilters containing $\{U; U \supset F$ for some $F \in \mathscr{F}$ (or $\mathrm{cl}\, \mathscr{U}$)$\}$. In the sequel, "$\mathfrak{F}(\mathscr{U}) \to p$" means "any $\mathscr{F} \in \mathfrak{F}(\mathscr{U})$ converges to p" and similarly we use "$\mathfrak{F}(\mathscr{U})$ has CIP" and so on. Then we have

(1) *$\mathscr{F} \to p$ implies $\mathfrak{U}(\mathscr{F}) \to p$ and $\mathrm{cl}\, \mathfrak{U}(\mathscr{F}) \subset \mathscr{F}$. If $\mathscr{F}$ has CIP, so has $\mathfrak{U}(\mathscr{F})$.*

(2) *$\mathscr{U} \to p$ implies $\mathfrak{F}(\mathscr{U}) \to p$. If $\mathscr{U}$ does not have CIP, then any $\mathscr{F} \in \mathfrak{F}(\mathscr{U})$ is wcp but does not have CIP.*

(3) *For a given $\mathscr{U}$, $\mathscr{U} \in \mathfrak{U}(\mathscr{F})$ for any $\mathscr{F} \in \mathfrak{F}(\mathscr{U})$.*

Proof. (1) and (2) are obvious. (3) Let $\mathscr{F} \in \mathfrak{F}(\mathscr{U})$ and let W be an open set containing some $F \in \mathscr{F}$. If $W \notin \mathscr{U}$, then there is $U \in \mathscr{U}$ with $U \cap W = \emptyset$, so $\mathscr{F} \ni F \cap \mathrm{cl}\, U = \emptyset$, a contradiction. Thus $W \in \mathscr{U}$ and hence $\mathscr{U} \in \mathfrak{U}(\mathscr{F})$.

DEFINITION 1.7. We consider the following conditions.

(α)((β)) *For any $p \in X^*$, there is a wcp(cp)$\mathscr{F}^p$.*

($s\alpha$)(($s\beta$)) *For any $p \in X^*$, any $\mathscr{F}^p$ is wcp(cp).*

The following (1) and (2) follows from the definitions and the fact that if X is normal, then the closed ultrafilter converging to $p \in X^*$ is only one.

(1) $(s\beta) \Rightarrow (\beta) \Rightarrow (\alpha)$ *and* $(s\beta) \Rightarrow (s\alpha) \Rightarrow (\alpha)$.

(2) *If X is normal, then* $(\beta) \Rightarrow (s\beta)$ *and* $(\alpha) \Rightarrow (s\alpha)$.

From $\beta X - \upsilon X \subset \mathfrak{U}(\triangle)$, 1.1(3), 1.4(5), 1.6(1) and the result that X is countably paracompact iff for any $\{F_n\}_{\mathrm{cl}} \downarrow \varnothing$, there is $\{U_n;\, F_n \subset U_n\}_{\mathrm{open}} \downarrow \varnothing$ with $\cap \operatorname{cl} U_n = \varnothing$ [7], we have

(3) *If X is either countably paracompact or cb^* then X has* $(s\beta)$.

There is a normal space which has neither (α) nor (β) (see 4.1 below).

1.8. As one of the nice property of the zero sets, it is known that a Z-ultrafilter $\mathscr{Z}$ has *CIP* iff $\mathscr{Z}$ contains a prime Z-filter with *CIP*. Unfortunately this is not necessarily true for closed ultrafilters. N. Dykes [1] has, however, proved that if X is a cb-space, F has *CIP* iff $\mathscr{F}$ contains a prime filter with *CIP*. In the following we treat the related problem above. We consider the following conditions.

(*CPC*) *If $\mathscr{F}$ contains a prime closed filter with CIP, then $\mathscr{F}$ has CIP.*

(*OPC*) *If $\mathscr{A}$ is a prime open filter with CIP and* $\operatorname{cl} \mathscr{A} \subset \mathscr{F}$, *then $\mathscr{F}$ has CIP.*

(*OPO*) *If $\mathscr{U}$ contains a prime open filter with CIP, then $\mathscr{U}$ has CIP.*

(*WOPC*) *If $\mathscr{U}$ has CIP, then $\mathfrak{F}(\mathscr{U})$ has CIP.*

(*ZC*) *If $Z(X) \cap \mathscr{F} = \mathscr{Z}^p$ for $p \in \upsilon X - X$, then $\mathscr{F}$ has CIP* (cf. 2.2(3)).

Notice that (1) $\mathscr{Z}^p$ has *CIP* iff $p \in \upsilon X - X$ and (2) if $\mathscr{F}$ has *CIP*, then $\mathscr{F} \cap Z(X) = \mathscr{Z}^p$ for some $p \in \upsilon X - X$ by 2.2(3) below. From 1.1(3), 1.4(4.5) and the definition, we have the following implications:

$$\begin{array}{ccl} cb^* & \Longrightarrow & \{ZC, OPC, CPC\} \\ \Big\Downarrow & & OPC \Longrightarrow WOPC \\ & & \quad\Big\Downarrow \\ \text{weak-}cb^* & \Longrightarrow & OPO \quad (\text{cf. } 2.3(4))\,. \end{array}$$

There is a weak-cb normal space without ZC (see, 4.1 below).

2. Interconnections among conditions introduced in §1.

2.1. (1) $\mathfrak{F}(0) \subset \mathfrak{U}(0)$ *and* $\mathfrak{U}(\triangle) \subset \mathfrak{F}(\triangle)$.

(2) *Let $p \in X^*$. Then there is $\mathscr{U}^p$ without CIP iff there is wcp $\mathscr{F}^p$ without CIP. Equivalently let $p \in X^*$, then $p \in \mathfrak{U}(0)$ iff no $\mathscr{F}^p$ without CIP is wcp.*

(3) *$p \in \mathfrak{U}(0) \cap \mathfrak{F}(\triangle)$ iff no $\mathscr{F}^p$ has CIP and no $\mathscr{F}^p$ is wcp.*

(4) *$p \in \mathfrak{U}(0) \cap \mathfrak{F}(0, \triangle)$ iff there are $\mathscr{F}_1^p$ with CIP and $\mathscr{F}_2^p$ without CIP and no $\mathscr{F}^p$ without CIP is wcp.*

(5) *$p \in \mathfrak{F}(0, \triangle) \cap \mathfrak{U}(0, \triangle)$ iff there are $\mathscr{F}_1^p$ with CIP and a wcp $\mathscr{F}_2^p$ without CIP.*

Proof. (1) Let $p \in \mathfrak{F}(0)$ and $\mathscr{U} \to p$. By 1.6(2), $\mathfrak{F}(\mathscr{U}) \to p$, $\mathrm{cl}\, \mathscr{U} \subset \mathscr{F}$ for each $\mathscr{F} \in \mathfrak{F}(\mathscr{U})$ and $\mathfrak{F}(\mathscr{U})$ has *CIP*, and hence $\mathscr{U}$ has *CIP*, so $p \in \mathfrak{U}(0)$. Now suppose that $p \in \mathfrak{U}(\triangle)$ and $\mathscr{F} \to p$. Then $\mathfrak{U}(\mathscr{F})$ does not have *CIP*. By 1.6(1) $\mathscr{F}$ does not have *CIP*, so $p \in \mathfrak{F}(\triangle)$.

(2) $\Rightarrow$ From 1.6. (2) $\Leftarrow$. By the assumption, there is an open set $U_n (n \in N)$, $\mathrm{cl}\, U_n \in \mathscr{F}$ with $\bigcap \mathrm{cl}\, U_n = \emptyset$. Let $N(p)$ be the open nbd system of p in βX and let $\mathscr{U} \supset \{U_n \cap V;\, n \in N,\, V \in N(p)\}$. Obviously $\mathscr{U}$ converges to p but does not have *CIP*.

(3) $\Rightarrow$ Obvious. $\Leftarrow$. From (2).

(4) $\Rightarrow$ From $p \in \mathfrak{F}(0, \triangle)$ and (2). $\Rightarrow$. From (2) and the assumption.

(5) $\Rightarrow$ Since $p \in \mathfrak{F}(0, \triangle)$, there is $\mathscr{F}_1^p$ with *CIP*. By (2), $p \in \mathfrak{U}(0, \triangle)$ implies that there is a *wcp* $\mathscr{F}_2^p$ without *CIP*. $\Leftarrow$. By the assumption, we have $p \in \mathfrak{F}(0, \triangle)$ and $p \in \mathfrak{U}(0, \triangle)$ by (1) and (2).

LEMMA 2.2. (1) *If $\mathscr{U}^p$ has CIP, then $\mathscr{A} = \{F;\, F$ contains some $U \in \mathscr{U}$ and F is closed$\}$ is a prime closed filter with CIP and $\mathrm{cl}\, \mathscr{U}^p \subset \mathscr{F}$ whenever $\mathscr{A} \subset \mathscr{F}$.*

(2) *Let X is normal and let $p \in \upsilon X - X$. Then $\mathscr{A} = \{U;\, U$ contains some $Z \in \mathscr{Z}^p$ and U is open$\}$ is a prime open filter with CIP. If $\mathscr{U} \to p$, then $\mathscr{A} \subset \mathscr{U}$.*

(3) *If $\mathscr{F}$ has CIP, then $Z(X) \cap \mathscr{F} = \mathscr{Z}^p$ for some $p \in \upsilon X - X$.*

Proof. (1) It is sufficient to show that $\mathscr{A}$ is prime. Let E and F be closed, $E \cup F \in \mathscr{A}$ and $E \notin \mathscr{A}$. Then $E \cup F \supset \mathrm{int}(E \cup F) = A \in \mathscr{U}$ and $\mathrm{int}\, E \notin \mathscr{U}$. Since $\mathscr{U}$ is a ultrafilter and $\mathrm{int}\, E \cup (X - E)$ is dense in X, we have $X - E \in \mathscr{U}$, so $A \cap (X - E) \in \mathscr{U}$. Since $A \cap (X - E) \subset \mathrm{int}\, F$, we have $F \in \mathscr{A}$. The latter part is obvious.

(2) It suffices to show that $\mathscr{A}$ is prime. Suppose not; Let W and V be open and let $W \cup V \in \mathscr{A}$, $W \notin \mathscr{A}$ and $V \notin \mathscr{A}$, then there is $Z \in \mathscr{Z}^p$ such that $Z \subset W \cup V$. As $W \notin \mathscr{A}$ and $V \notin \mathscr{A}$, $Z - W \neq \emptyset$ and $Z - V \neq \emptyset$. Since X is normal, there are zero sets Z_1 and Z_2 such that $X - V \subset X - Z_1$, $Z - W \subset X - Z_2$ and $(X - Z_1) \cap (X - Z_2) = \emptyset$. Thus $Z_1 \subset V$, so as $V \notin \mathscr{A}$, $Z_1 \notin \mathscr{Z}^p$. Thus $Z_2 \in \mathscr{Z}^p$. Similarly, there are zero sets Z_3 and Z_4 such that $X - W \subset X - Z_3$, $Z - V \subset X - Z_4$ and $(X - Z_3) \cap (X - Z_4) = \emptyset$. As above, $Z_4 \in \mathscr{Z}^p$. Thus $Z \cap Z_2 \cap Z_4 \in \mathscr{Z}^p$. But $Z \cap Z_2 \cap Z_4 \subset W \cap V$, so W and $V \in \mathscr{A}$, which is a contradiction. Hence $\mathscr{A}$ is prime.

(3) In general, it is evident that $\mathscr{A} \cap Z(X)$ is a prime Z-filter for a prime closed filter $\mathscr{A}$. Suppose that there is $Z \in Z(X)$ which intersects each member of $Z(X) \cap \mathscr{F}$, but $Z \notin \mathscr{F}$. Let $Z = Z(f)$, $f \in C(X)$, $A_n = \{x; f(x) \geqq 1/n\}$ and $B_n = \{x: f(x) \leqq 1/n\}$. Since $A_n \cup B_n = X$, $\mathscr{F}$ contains A_n or B_n. If $B_n \in \mathscr{F}$ for infinitely many n, then $Z = \cap B_n \in \mathscr{F}$, a contradiction. Thus $A_n \in \mathscr{F}$ for some n, and hence $A_n \in \mathscr{F} \cap Z(X)$, a contradiction, so $\mathscr{F} \cap Z(X) = Z^p$ for some $p \in X^*$. By 1.1(3) we have $p \in \upsilon X - X$.

Notice that the assumption *CIP* in 2.2(3) is essential as is shown in 4.3 below.

2.3. (1) *X has (α) iff $\mathfrak{U}(0) \cap \mathfrak{F}(\triangle) = \emptyset$.*

(2) *X has $(s\alpha)$, then $\mathfrak{U}(0) = \mathfrak{F}(0)$.*

(3) *X has WOPC iff $\mathscr{F}$ does not have CIP, neither has $\mathfrak{U}(\mathscr{F})$.*

(4) *If X has CPC, then X has WOPC.*

(5) *If X has $(s\beta)$, then X has WOPC.*

(6) *If X has WOPC, then $\mathfrak{F}(0) = \mathfrak{U}(0)$, $\mathfrak{F}(0, \triangle) = \mathfrak{U}(0, \triangle)$ and $\mathscr{F}(\triangle) = \mathfrak{U}(\triangle)$.*

(7) *If X is normal, then X has WOPC iff $\mathfrak{F}(0) = \mathfrak{U}(0)$ and $\mathfrak{F}(\triangle) = \mathfrak{U}(\triangle)$.*

(8) *If X has ZC, then $\mathfrak{F}(\upsilon, \triangle) = \emptyset$ (and hence $\mathfrak{U}(\nu, \triangle) = \emptyset$).*

(9) *If $\mathscr{F} \cap Z(X)$ has CIP, so has $\mathscr{Z}^p$ (and hence $p \in \upsilon X - X$).*

Proof. (1) $\Rightarrow$ From 2.1(3). $\Leftarrow$. Since any $\mathscr{F}$ with *CIP* is *wcp*, we consider only a point $p \in \mathfrak{F}(\triangle)$. By the assumption, $p \notin \mathfrak{U}(0)$, so there is $\mathscr{U}^p$ without *CIP*. Thus there is a *wcp* $\mathscr{F}^p$ without *CIP* by 2.1(2).

(2) $(s\alpha)$ implies $\mathfrak{U}(0) \cap \mathfrak{F}(0, \triangle) = \emptyset$ by 2.1(4), so $\mathfrak{U}(0) = \mathfrak{F}(0)$ by (1) and 2.1(1).

(3) $\Rightarrow$ Suppose that $\mathscr{F}_0$ does not have *CIP* but some $\mathscr{U}_0 \in \mathfrak{U}(\mathscr{F}_0)$ has *CIP*. Since X has *WOPC*, $\mathfrak{F}(\mathscr{U}_0)$ has *CIP*. On the other hand, $\mathrm{cl}\,\mathscr{U}_0 \subset \mathscr{F}_0$, so $\mathscr{F}_0 \in \mathfrak{F}(\mathscr{U}_0)$ which shows that $\mathscr{F}_0$ has *CIP*, a contradiction. $\Leftarrow$. Suppose that $\mathscr{U}_0$ has *CIP* but some $\mathscr{F}_0 \in \mathfrak{F}(\mathscr{U}_0)$ does not have *CIP*. By 1.6(3), $\mathscr{U}_0 \in \mathfrak{U}(\mathscr{F}_0)$ and $\mathfrak{U}(\mathscr{F}_0)$ does not have *CIP*, a contradiction.

(4) Take $\mathscr{U}^p$ with *CIP* and let $\mathscr{F} \supset \mathrm{cl}\,\mathscr{U}^p$. Then $\mathscr{F}$ contains a prime closed filter $\mathscr{A}$ described in 2.2(1), and hence $\mathscr{F}$ has *CIP*. Thus $\mathfrak{F}(\mathscr{U}^p)$ has *CIP*.

(5) Take $\mathscr{F} = \mathscr{F}^p$ without *CIP* and let $\mathscr{U} \in \mathfrak{U}(\mathscr{F})$. Since $U \supset \{W; W$ is open and $W \supset F$ for some $F \in \mathscr{F})$ and $\mathscr{F}$ is *cp*, there is $\{W_n; W_n \in \mathscr{U}\}$ with $\cap\, \mathrm{cl}\, W_n = \emptyset$. Thus $\mathfrak{U}(\mathscr{F})$ does not have *CIP* and X has *WOPC* by (3).

(6) Let $p \in \mathfrak{U}(0, \triangle) \cup \mathfrak{U}(0)$ and take $\mathscr{U}^p$ with *CIP*. By the assumption, $\mathfrak{F}(\mathscr{U}^p)$ has *CIP*, so $p \notin \mathfrak{F}(\triangle)$ which shows $\mathfrak{F}(\triangle) = \mathfrak{U}(\triangle)$

by 2.1(1). Let $p \in \mathfrak{F}(0, \triangle) \cup \mathfrak{U}(0)$ and take $\mathscr{F}^p$ without CIP. Then $\mathfrak{U}(\mathscr{F}^p)$ does not have CIP by (3), a contradiction. This shows $\mathfrak{U}(0) = \mathfrak{F}(0)$.

(7) $\Rightarrow$ From (6) and $\mathfrak{F}(0, \triangle) = \varnothing$. $\Leftarrow$. Since X is normal, $\mathfrak{F}(0, \triangle) = \varnothing$. Take $\mathscr{F}^p$ without CIP. Then $p \in \mathfrak{F}(\triangle) = \mathfrak{U}(\triangle)$, and hence $\mathfrak{U}(\mathscr{F})$ does not have CIP, so X has $WOPC$ by (3).

(8) Let $p \in \mathfrak{F}(\upsilon, \triangle)$ and take $\mathscr{F} \supset \mathscr{Z}^p$. Since X has ZC, $\mathscr{F}$ has CIP, a contradiction.

(9) Suppose that $\mathscr{Z}^p$ does not have CIP. Then $p \in \beta X - \nu X$ and hence $\mathscr{Z}^p$ contains F_n described in 1.1(3). Since $F_n \in \mathscr{F}^p$, $\mathscr{F}^p \cap Z(X)$ does not have CIP, a contradiction.

By 1.4(1, 2) and 2.3(1, 6), we have the following implications.

$$\begin{array}{ccc} \text{almost realcompactness} & \Longrightarrow & WOPC \\ \Big\Downarrow & & \Big\Downarrow \\ c\text{-realcompactness} & \Longrightarrow & (\alpha)\ . \end{array}$$

3. Characterizations of spaces of by means closed ultrafilters.

THEOREM 3.1. (1) *X is weak-cb^* iff no $\mathscr{F}^p$ without CIP is wcp for $p \in X^*$.*

(2) *A normal space X is weak-cb^* iff X has OPO.*

(3) *X is cb^* iff X is weak-cb^* and has $WOPC$.*

(4) *The following are equivalent for a normal space X:* (i) *X is cb^** (ii) *X has OPC.* (iii) *Any stable $\mathscr{F}$ is cp.* (iv) *X has ZC.*

(5) *A pseudocompact space X is countably compact iff X has $WOPC$.*

Proof. (1) From 2.1(2).

(2) Since a weak-cb^* space has OPO by the diagram of 1.8, it suffices to show the converse. Let $p \in (\upsilon X - X) - \mathfrak{U}(0)$ and take any $\mathscr{U}^p$ without CIP. X being normal, there is a prime open filter $\mathscr{A}$ with CIP and $\mathscr{A} \subset \mathscr{U}^p$ by 2.2(2), so $\mathscr{U}^p$ has CIP by the assumption which is a contradiction.

(3) A cb^* space is weak-cb and has $WOPC$ by 1.8. Conversely suppose that X is weak-cb^* and has $WOPC$. X being weak-cb^*, $\upsilon X - X = \mathfrak{U}(0)$, so $\mathfrak{F}(0) = \mathfrak{U}(0)$ by 2.3(6), and hence X is cb^*.

(4) (i) $\Rightarrow$ (ii) From the diagram of 1.8.

(ii) $\Rightarrow$ (iii) Let $\mathscr{F}$ be stable. Then $\mathscr{F} \to p$ for some $p \in \upsilon X - X$ by 1.1(1). By the diagram of 1.8 and 2.3(7), we have $\mathfrak{F}(0) = \mathfrak{U}(0)$ and $\mathfrak{F}(\upsilon, \triangle) = \mathfrak{U}(\upsilon, \triangle)$. Since X is normal, $\mathscr{Z}^p \subset \mathscr{F}^p$, and hence $\mathscr{F}$ contains $\mathscr{A}$ described in 2.2(2), so $\mathscr{F}$ has CIP by OPC which shows that $\mathscr{F}$ is cp by 1.5(1).

(iii) $\Rightarrow$ (iv) Since X is normal, any $\mathscr{F}^p$ contains $\mathscr{Z}^p$. We suppose that there are $p \in \upsilon X - X$ and $\mathscr{F} = \mathscr{F}^p$ without *CIP*. Then there is $\{U_n\}_{\text{open}} \downarrow$ such that $F_n \subset U_n$ for some $F_n \in \mathscr{F}$ and $\cap \operatorname{cl} U_n = \varnothing$. X being normal, it is easily seen that there is $Z_n \in Z(X)$ with $F_n \subset Z_n \subset U_n$, so $\cap Z_n = \varnothing$, a contradiction.

(iv) $\Rightarrow$ (i) For $p \in \upsilon X - X$, $\mathscr{Z}^p$ has *CIP* and hence any $\mathscr{F}$ containing $\mathscr{Z}^p$ has *CIP* by *ZC*. Thus $\mathfrak{F}(\triangle) = \varnothing$. On the other hand, X being normal $\mathfrak{F}(0, \triangle) = \varnothing$, and hence $\upsilon X - X = \mathscr{F}(0)$, so X is cb^*.

(5) If X is countably compact, then X is cb^* [8], so X has *WOPC* by (3). Conversely, if X has *WOPC*, then X is cb^* by (3) because a pseudocompact space is weak-cb^*. Thus X is countably compact [8].

X is said to be *almost normal* ((υ)-*almost normal*) if a closed subset F disjoint from a regular closed subset E, then $\operatorname{cl}_{\beta X} F \cap \operatorname{cl}_{\beta X} E = \varnothing (\operatorname{cl}_{\upsilon X} F \cap \operatorname{cl}_{\upsilon X} E = \varnothing)$. It is obvious that X is almost normal ((υ)-almost normal) iff $\operatorname{cl} \mathscr{U}^p \subset \mathscr{F}^p$ for each $\mathscr{F}^p$ and each $\mathscr{U}^p$ for $p \in X^* (p \in \upsilon X - X)$.

3.2. *Let X be (υ)-almost normal. Then we have*

(1) $\mathfrak{F}(0, \triangle) \cap \mathfrak{U}(0, \triangle) = \varnothing$ (*equivalently*, $\mathfrak{F}(0, \triangle) \subset \mathfrak{U}(0)$).

(2) *If X is c-realcompact, then X is a-realcompact.*

Proof. (1) If there are $\mathscr{F}^p$ with *CIP* and $\mathscr{U}^p$ without *CIP* for $p \in \mathfrak{F}(0, \triangle) \cap \mathfrak{U}(0, \triangle)$, then $\operatorname{cl} \mathscr{U}^p \subset \mathscr{F}^p$ because X is (υ)-almost normal, a contradiction.

(2) Since X is c-realcompact, $\mathfrak{U}(0) = \varnothing$, so $\mathfrak{F}(0, \triangle) = \varnothing$ by (1). Thus $\mathfrak{F}(0) \cup \mathfrak{F}(0, \triangle) = \varnothing$ by 2.1(1) and hence X is a-realcompact.

THEOREM 3.3. (1) *X is realcompact iff X is (υ)-almost normal and there is a cp $\mathscr{F}^p$ without CIP for every $p \in X^*$.*

(1′) *An almost normal space X is realcompact iff there is a cp $\mathscr{F}^p$ without CIP for every $p \in X^*$.*

(2) *A countably paracompact space X is realcompact iff X is a-realcompact and (υ)-almost normal.*

(2′) *An almost normal and countably paracompact space X is realcompact iff X is a-realcompact.*

(3) *X is realcompact iff X is an a-realcompact space with ZC.*

(4) *X is c-realcompact iff there is a wcp $\mathscr{F}^p$ without CIP for every $p \in X^*$.*

(5) *An a-realcompact space X is c-realcompact iff X has (α).*

(6) *An a-realcompact space X is almost realcompact iff X has WOPC.*

(7) *If a wa-realcompact space X has* ($s\alpha$), *then X is c-realcompact.*

(8) *A normal c-realcompact space is realcompact iff X has OPO.*

Proof. (1) $\Rightarrow$ Evident. $\Leftarrow$. By the assumption, we have $X^* = \mathfrak{F}(\triangle) \cup \mathfrak{F}(0, \triangle)$. Let $p \in \upsilon X - X$ and take a *cp* $\mathscr{F}^p$ without *CIP*. Then there is $\{F_n \in \mathscr{F}^p\} \downarrow \varnothing$ and $\{U_n\}_{\text{open}} \downarrow$ such that $F_n \subset U_n$ and $\cap \operatorname{cl} U_n = \varnothing$. We may assume that $X - \operatorname{cl} U_n \neq \varnothing$, $n \in N$. Since X is (υ)-almost normal and $p \in \operatorname{cl}_{\beta X} F_n$, we have $p \notin \operatorname{cl}_{\beta X}(X - \operatorname{cl} U_n)$. Thus there is $f_n \in C(\beta X)$ such that $p \in Z(f_n) \subset \operatorname{cl}_{\beta X} U_n$. Then $f = \sum (1/2^n) f_n \in C(\beta X)$, $p \in Z(f)$ and $Z(f) \cap X = \varnothing$. This is a contradiction because $p \in \upsilon X - X$, so $\upsilon X = X$.

(1′) From (1).(2) From 1.7(3) and (1). (2′) From (2).(3) $\Rightarrow$ Obvious. $\Leftarrow$. From 1.4(3) and 2.3(8), (4) From 1.1(3), 1.5 and 2.1(2).

(5) If X has (α), then $\mathfrak{U}(0) \cap \mathfrak{F}(\triangle) = \varnothing$ by 2.3(1). Since X is a-realcompact, we have $\mathfrak{F}(0) \cup \mathfrak{F}(0, \triangle) = \varnothing$, and hence $\mathfrak{U}(0) = \varnothing$, so X is c-realcompact. The converse follows from the diagram of 2.3.

(6) $\Rightarrow$ From the diagram of 2.3. $\Leftarrow$. From 1.4(1), $\mathfrak{F}(0, \triangle) \cup \mathfrak{F}(0) = \varnothing$, 2.3(6) and *WOPC*.

(7) Since X has ($s\alpha$), $\mathfrak{F}(0) = \mathfrak{U}(0)$ by 2.3(2). On the other hand $\mathfrak{F}(0) = \varnothing$ by wa-realcompactness, so $\mathfrak{U}(0) = \varnothing$, and hence X is c-realcompact.

(8) $\Rightarrow$ Obvious. $\Leftarrow$. Since X is normal and has *OPO*, X is weak-cb^* by 3.1(2). Thus $\upsilon X - X = \mathfrak{U}(0)$. On the other hand, X being c-realcompact. We have $\mathfrak{U}(0) = \varnothing$ by 1.4(2), and hence X is realcompact.

3.4. *In the following, we have* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6).

(1) *X is realcompact.*

(2) *X has* ($s\beta$) *and no F has CIP.*

(3) *X is an almost realcompact space with* ($s\beta$).

(4) *X is almost realcompact.*

(5) *X has* ($s\alpha$) *and no $\mathscr{F}$ has CIP.*

(6) *X is an a-realcompact space with* ($s\alpha$) (*and hence X is also c-realcompact by* 3.3(5)).

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) are evident. (4) $\Rightarrow$ (5). From 1.4(1), 2.1(1) and $\upsilon X - X = \mathfrak{U}(0)$.

(2) $\Rightarrow$ (3) Suppose that $\mathscr{U} = \mathscr{U}^p$ has *CIP*, $p \in \upsilon X - X$. Let $\mathscr{F} \in \mathfrak{F}(\mathscr{U})$. By the assumption, $\mathscr{F}$ is *cp* and does not have *CIP*. There are $\{F_n \in \mathscr{F}\} \downarrow \varnothing$ and $\{U_n\}_{\text{open}} \downarrow$ with $F_n \subset U_n$ and $\cap \operatorname{cl} U_n = \varnothing$. This implies $U_n \in \mathscr{U}$ and $\mathscr{U}$ does not have *CIP*, a contradiction.

(6) $\Rightarrow$ (5) Take $\mathscr{F} = \mathscr{F}^p$, $p \in \upsilon X - X$. Since $\upsilon X - X = \mathfrak{F}(\upsilon, \triangle)$, $\mathscr{F}$ does not have *CIP*, so $\mathscr{F}$ is *wcp* by ($s\alpha$).

3.5. In [13], the following theorem was communicated to P. Simon by Z. Frolik:

THEOREM F. *If X is a normal a-realcompact space, then the realcompactness of X is equivalent to the following condition:*
(*SZC*) *If $\mathscr{F}$ does not have CIP, neither has $\mathscr{F} \cap Z(X)$.*

Since *SZC* is equivalent to the condition: if $\mathscr{F} \cap Z(X)$ has *CIP*, so has F. We have $SZC \Rightarrow ZC$, and hence the normality in Theorem F is superfluous by 2.3(9) and 3.3(3).

3.6. Simon proved the following [13].

THEOREM S. *There is a $\mathscr{Z}^p$ with CIP in X such that no $\mathscr{F} \supset \mathscr{Z}^p$ has CIP where X is the Dowker space in* 4.1 *belows.*

Relating this theorem, we consider the following conditions:
(a) *If $\mathscr{Z}^p$ has CIP, then no $\mathscr{F}^p$ has CIP.*
(b) *If $\mathscr{Z}^p$ has CIP, then there is $\mathscr{F}^p$ without CIP.*
Then Theorem S is a direct consequence of the fact that a-realcompactness is equivalent to (a). For since $\mathscr{Z}^p$ has *CIP* iff $p \in \upsilon X - X$, it is easy to see that (a) is equivalent to "no $\mathscr{F}^p$ has *CIP* for each $p \in \upsilon X - X$" equivalently to "$\upsilon X - X = \mathfrak{F}(\upsilon, \triangle)$, i.e., X is a-realcompact". Similarly we have that (b) is equivalent to the wa-realcompactness of X.

4. Examples.

Dowker space 4.1. Let X be the Dowker space, constructed by M. E. Rudin [12], which is normal but not countably paracompact. X is, moreover, weak-cb [6] and a-realcompact [13] but not c-realcompact [5]. Since a-realcompactness $\Leftrightarrow \upsilon X - X = \mathfrak{F}(\upsilon, \triangle)$, c-realcompactness $\Leftrightarrow \mathfrak{U}(0) = \varnothing$, normality $\Rightarrow \mathfrak{F}(0, \triangle) = \varnothing$ and weak-cb^*-ness $\Leftrightarrow \upsilon X - X = \mathfrak{U}(0)$, we have $\upsilon X - X = \mathfrak{F}(\upsilon, \triangle) = \mathfrak{U}(0)$. This shows that X is not cb^* and hence X has neither *WOPC* nor *ZC* by 2.3(7) and 3.1(4) respectively. It is obvious that X does not have (α) by 2.3(1).

4.2. Let X be the countably paracompact space, constructed by Mack and Johnson [9] (or see, [11]) is c-realcompact [14] and $\upsilon X = X \cup \{p\}$ but X is not weak-cb^* [8]. But X is neither almost realcompact [14] nor a-realcompact [5]. Thus $\upsilon X - X = \mathfrak{F}(0, \triangle) = \mathfrak{U}(0, \triangle) = \{p\}$.

Tychonoff Plank 4.3. Let $T = [0, \Omega] \times [0, \omega] - \{p\}$ where $p = (\Omega, \omega)$. T is pseudocompact but not countably compact and $\beta T = \upsilon T = [0, \Omega] \times [0, \omega]$. Since a pseudocompact space is weak-cb^*, $\mathfrak{U}(0) \neq \varnothing$. It is easy to see that $\mathfrak{U}(0) = \mathfrak{F}(0, \triangle) = \{p\}$ and T is not cb^* [8] and moreover, T has OPC but does not have CPC. Let $\mathscr{F}$ be a closed ultrafilter containing the right edge. Obviously $\mathscr{F}$ does not have CIP and $Z(X) \cap \mathscr{F} \neq \mathscr{X}^p$ which shows that X does not have ZC.

4.4. Let $X = [0, \Omega]^2 - \{p\}$ where $p = (\Omega, \Omega)$. Then $\upsilon X = [0, \Omega]^2$. It is easy to see that $\mathfrak{F}(0) = \mathfrak{U}(0) = \{p\}$.

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