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POINTWISE PERIODIC HOMEOMORPHISMS ON CHAINABLE CONTINUA

EDWIN DUDA

We show that if X is a chainable continuum with on small indecomposable subcontinua and which admits a monotone map ϕ onto the unit interval such that no point inverse has interior points, then each pointwise periodic mapping T on X is periodic and must have period 1 or 2.

0. **Introduction.** Beverly Brechner [2] has shown there exists a chainable continuum X and a periodic homeomorphism T of X of period 4. The only other periodic homeomorphisms on chainable continua known at that time were of period 1 and 2. Wayne Lewis [4] has recently shown that for every positive integer n there exists a nonhereditarily indecomposable chainable continuum X with a homeomorphism T of period n. He observes that X could be constructed so as to be a pseudo-arc and still have a homeomorphism T of period n. Michel Smith and Sam Young [5] have shown that if a chainable continuum admits a homeomorphism of period greater than 2, then the continuum must contain an indecomposable continum.

We show that if X is a chainable continuum with no small indecomposable continua and which admits a monotone map ϕ onto the unit interval such that no point inverse has interior points, then each pointwise periodic homeomorphism T must be periodic and of period 1 or 2.

1. **Notation and background.** In this note X will represent a metric continuum and mapping will mean a continuous function. A mapping T of X into itself is said to be pointwise periodic if for each $x \in X$ there exists an integer N_x such that $T^{N_x}(x) = x$, where T^{N_x} means the composition of T with itself N_x times. By a result of R. H. Bing, [1], a hereditarily unicoherent atriodic continuum X in which no indecomposable continuum has interior points admits a monotone mapping ϕ onto the unit interval $I = [0, 1]$ and furthermore no point inverse of ϕ has interior points relative to X. In case X is hereditarily decomposable he showed that X is chainable. J. B. Fugate, [3], strengthened this result by showing that if each indecomposable subcontinuum of an atriodic-hereditarily unicoherent continuum X is chainable, then X is chainable. In this note we wish to consider X to be a chainable continuum which has no indecomposable subcontinua with interior points and which has no inde-

composable continua of diameter less than a fixed positive number. By the above X admits a monotone mapping ϕ onto the unit interval I and no point inverse of ϕ has interior points. Furthermore E. S. Thomas, [6], has shown that X has the following property: If U and V are disjoint open sets in X, then there exists an $x \in U$ and $y \in V$ and a continuum K_{xy} irreducible from x to y such that the composant of x in K_{xy} is $K_{xy} - \{y\}$.

2. Preliminary results.

LEMMA 1. *A pointwise periodic map T on a continuum X is a homeomorphism.*

Proof. We show that T is 1–1. Suppose $T(x) = T(y)$ for $x, y \in X$. There exist integers N_x and N_y so that $T^{N_x}(x) = x$ and $T^{N_y}(y) = y$. Now $x = T^{N_x N_y}(x) = T^{N_x N_y - 1}(T(x)) = T^{N_x N_y - 1}(T(y)) = T^{N_y N_x}(y) = y$.

LEMMA 2. *If $A \subset X$ and $T(A) \subset A (T(A) \supset A)$, then $T(A) = A$, where T is pointwise periodic on X.*

Proof. Suppose $x \in A$ and $T(A) \subset A$. There is an integer N_x with $T^{N_x}(x) = x$ and since $A \supset T(A) \supset T^2(A) \supset \cdots \supset T^{N_x}(A) \supset \cdots$, $x \in T(A)$. The case $T(A) \supset A$ follows from Lemma 1 and the case just proved.

LEMMA 3. *Let X be a chainable continuum which admits a monotone map ϕ onto the unit interval I such that no point inverse contains interior points relative to X. If K is a continuum in X which meets $\phi^{-1}(t_1)$ and $\phi^{-1}(t_2)$, then $K \supset \phi^{-1}(t)$ for all t between t_1 and t_2. Furthermore X is irreducible from any point of $\phi^{-1}(0)$ to any point of $\phi^{-1}(1)$.*

Proof. Assume $t_1 < t < t_2$. Let $L = \phi^{-1}[0, t] \cap K$ and $M = \phi^{-1}[t, 1] \cap K$. By monotoneity of ϕ and hereditary unicoherence of X, L and M are continua. The continua L and M have a point in common in $\phi^{-1}(t)$. If $\phi^{-1}(t)$ is not contained in $L \cup M$, then L, M and $\phi^{-1}(t)$ determine a triod in X which is impossible. Suppose K is a continuum irreducible from $x \in \phi^{-1}(0)$ to $y \in \phi^{-1}(1)$. Then $K \supset \phi^{-1}(0, 1)$ by the first part of the lemma and $\phi^{-1}(0, 1)$ is dense in X since no point inverse has interior points. Thus, $X = \overline{\phi^{-1}(0, 1)} \subset K$ or $X = K$.

3. Main result.

THEOREM. *Let T be a pointwise periodic mapping on a chain-*

able continuum X which has no small indecomposable continua. If X admits a monotone map ϕ onto the unit interval such that no point inverse has interior points relative to X, then T is periodic.

Proof. Lemma 1 and Lemma 2 imply that for each $t \in I$, $T(\phi^{-1}(t))$ is contained in $\phi^{-1}(s)$ for some $s \in I$. The mapping T thus induces a pointwise periodic map T_1 on $[0, 1] = I$ defined by $T_1(t) = (\phi T \phi^{-1})(t)$. By known results T_1 is periodic and either $T_1 =$ identity or $T_1^2 =$ identity on I. We assume $T_1^2 =$ identity on I. In this case T^2 maps each $\phi^{-1}(t)$ into itself.

Let $x \in \phi^{-1}(0)$ and $y \in \phi^{-1}(t)$, $t \in (0, 1)$, and let U and V be disjoint open sets containing x and y respectively chosen so that $\phi(U)$ is entirely to the left of $\phi(V)$ and $1 \notin \phi(V)$. By a result of E. S. Thomas, [6], there exists an $x_1 \in U$ and a $y_1 \in V$ and a continuum $K_{x_1y_1}$ which is irreducible from x_1 to y_1 and the composant of x_1 in $K_{x_1y_1}$ is $K_{x_1y_1} - \{y_1\}$. Define a new continuum K_{xy_1} by $K_{xy_1} = \phi^{-1}[0, \phi(x_1)] \cup K_{x_1y_1}$. The continuum K_{xy_1} is irreducible from x to y_1 and the composant of x in K_{xy_1} is the complement of y_1 in K_{xy_1}. Let $K_{xy_1} - \{y_1\} = \bigcup_{i=1}^{\infty} K_i$, where each K_i is a continuum containing x and $K_i \subset K_{i+1}$ for all i. If $y_1 \in T^2(K_{xy_1})$, then $T^2(K_{xy_1}) \supset K_{xy_1}$ since $x, y_1 \in T^2(K_{xy_1})$ and K_{xy_1} is irreducible between x and y_1. By Lemma 2, $T^2(K_{xy_1}) = K_{xy_1}$. If $y_1 \notin T^2(K_{xy_1})$ we show this leads to a contradiction. We must have $T^2(K_i) \subset K_{xy_1}$ for all i, otherwise there exists an integer N such that $T^2(K_i) \not\subset K_{xy_1}$, and in this case $H = \overline{K_{xy_1} - \phi^{-1}\phi(y_1)}$, $K = T^2(K_N) \cap \phi^{-1}\phi(y_1)$, and $L = K_{xy_1} \cap \phi^{-1}\phi(y_1)$ determine a triod. The containing relation $T^2(\bigcup_{i=1}^{\infty} K_i) \subset K_{xy_1}$ implies $T^2(K_{xy_1}) \subset K_{xy_1}$ and again by Lemma 2, $T^2(K_{xy_1}) = K_{xy_1}$.

No $T^2(K_i)$ can contain y_1 otherwise there exists an integer N with $T^2(K_N)$ properly containing K_{xy_1} which implies a contradiction. Therefore $T^2(y_1) = y_1$ and since T^2 is the identity on a dense set it follows that $T^2 =$ identity on X.

The argument for $T_1 =$ identity on I implying $T =$ identity on X is similar.

COROLLARY. *If T is pointwise periodic and X is as in the theorem, then either $T^2 =$ identity on X or $T =$ identity on X and the fixed point set is a continuum.*

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UNIVERSITY OF MIAMI
CORAL GABLES, FL 33124