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AN ACTION OF THE AUTOMORPHISM GROUP OF A COMMUTATIVE RING ON ITS BRAUER GROUP

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An action of the automorphism group of a commutative ring on its Brauer group is given. The action is characterized cohomologically. Relations with the Teichmuller cocycle map and the Schur subgroup are pointed out.

In [13] G. J. Janusz gave an action of the automorphism group of a field K on its Brauer group $B(K)$. For number fields he characterized this action in terms of Hasse invariants and applied his results to the problem of the existence of an outer automorphism of the rational group algebra of a finite group.

Here we give an action of the automorphism group of a commutative ring R on its Brauer group $B(R)$ and describe the action cohomologically. Let A be an Azumaya R-algebra and let σ be an automorphism of R. Define a new R-algebra ${}_{\sigma}A$ by letting $A = {}_{\sigma}A$ as rings and with R-module action given by $r*a = \sigma^{-1}(r)a$ for $r \in R$, $x \in A$ where multiplication on the right is in A. Proposition 2 is the assertion that the correspondence $A \to {}_{\sigma}A$ induces an action of the group $\mathrm{Aut}(R)$ of automorphisms of R on $B(R)$.

Let L be a finite Galois field extension of K with finite Galois group G and let $\mathrm{Aut}(K: L)$ be the group of automorphisms of K which can be extended to L. In [11] S. Eilenberg and S. MacLane gave an action of $\mathrm{Aut}(K: L)$ on $H^n(G, L^*)$ for $n \geqq 0$. This action corresponds under the natural identification between $B(L/K)$ and $H^2(G, L^*)$ with Janusz's action on $B(L/K)$. For a commutative ring R, $B(R)$ is given as the torsion subgroup of $H^2_{et}(R, U)$ [12] and $H^2_{et}(R, U)$ is a limit of Amitsur cohomology groups [17]. For a faithfully flat commutative extension S of R we give an action of $\mathrm{Aut}(R: S)$ on $H^n(S/R, U)$ (Amitsur cohomology) and show this action commutes with the natural homomorphism given, for example, in [17] from $B(R)$ into $H^2_{et}(R, U)$. We study the problem of extending an automorphism from R to an R-algebra A and its relation to normal algebras and the Teichmuller cocycle map. We show that if K is a field of characteristic $= 0$ then $\mathrm{Aut}(K)$ must always leave the Schur subgroup of $B(K)$ invariant, and we calculate some examples. Throughout all unexplained terminology and notation will be as in [14]. I would like to thank G. J. Janusz, D. Saltman, and D. Zelinsky for helpful remarks.

1. Let R denote a commutative ring, $\sigma \in \mathrm{Aut}(R)$, and let M be

an R-module. Form the new R-module ${}_\sigma M$ which is equal to M as an abelian group and with R-action given by $r*m = \sigma^{-1}(r)m$ for all $r \in R$, $m \in M$ where multiplication on the right-hand side is in M. This action is well known, see for example [9]. If A is an R-algebra then ${}_\sigma A$ is the R-algebra equal to A as a ring and ${}_\sigma A$ as an R-module.

LEMMA 1. *Let A and B be R-algebras and M, N be R-modules. Let $\sigma \in \mathrm{Aut}\,(R)$, then*

(a) ${}_\sigma(A \otimes B) \cong {}_\sigma A \otimes {}_\sigma B$.

(b) ${}_\sigma\mathrm{Hom}_R\,(M, N) \cong \mathrm{Hom}_R\,({}_\sigma M, {}_\sigma N)$.

(c) *M is an R-progenerator if and only if ${}_\sigma M$ is an R progenerator.*

(d) *A is a separable R-algebra if and only if ${}_\sigma A$ is a separable R-algebra.*

(e) *If σ induces an automorphism $\bar{\sigma}$ of $R.1$ then $A \cong {}_\sigma A$ as R-algebras if and only if $\bar{\sigma}$ extends to a ring automorphism of A.*

Proof. The isomorphisms in (a) and (b) are the identity map, (c) is well known (p. 115 of [9]).

For (d) the separability of A over R is equivalent to the existence of an idempotent $e \in A \otimes A$, $e = \sum_{i=1}^n a_i \otimes b_i$, such that $\sum_{i=1}^n a_ib_i = 1$ and $(1 \otimes x - x \otimes 1)e = 0$ for all $x \in A$. Let $e_\sigma = \sum_{i=1}^n a_i \otimes b_i$ in ${}_\sigma A \otimes {}_\sigma A^0$, then we still have $\sum_{i=1}^n a_ib_i = 1$. Let $\phi\colon {}_\sigma A \otimes {}_\sigma A^0 \to A \otimes A^0$ by $\phi(x \otimes y) = x \otimes y$. Since $A \otimes A^0 = {}_\sigma(A \otimes A_0)$ as rings it follows from (a) that ϕ is a ring isomorphism. For any $x \in {}_\sigma A$ we have

$$(x \otimes 1)e = \sum_{i=1}^n xa_i \otimes b_i \xrightarrow{\phi} \sum_{i=1}^n xa_i \otimes b_i = \sum_{i=1}^n a_i \otimes b_ix \xrightarrow{\phi^{-1}} \sum_{i=1}^n a_i \otimes b_ix$$
$$= (1 \otimes x)e \,.$$

This proves (d).

For (e) assume $A \cong {}_\sigma A$ and let $f\colon {}_\sigma A \to A$ be the given R-algebra isomorphism. Define τ by $\tau(a) = f(a)$ for all $a \in A (A = {}_\sigma A$ as rings). Then $\tau \in \mathrm{Aut}\,(A)$ and for any $r \in R$, $\tau(r\cdot 1) = f(r\cdot 1) = f(\sigma(r)^*1) = \sigma(r)\cdot f(1) = \bar{\sigma}(r1)$. Conversely, assume there is an element $\tau \in \mathrm{Aut}\,(A)$ extending $\bar{\sigma}$. Define $f\colon {}_\sigma A \to A$ by $f(a) = \tau(a)$ for all $a \in A$. For any $r \in R$ and $a \in {}_\sigma A$ we have $f(r*a) = f(\sigma^{-1}(r)\cdot a) = \tau(\sigma^{-1}(r)\cdot a) = \tau\bar{\sigma}^{-1}(r\cdot 1)\cdot\tau(a) = r\cdot\tau(a) = r\cdot f(a)$. It follows that f is an R-isomorphism.

We have let $\mathrm{Aut}\,(R\colon A)$ be the group of all automorphisms of R which can be extended to A, it follows from (e) that $\mathrm{Aut}\,(R\colon A) = \{\sigma \in \mathrm{Aut}\,(R) \mid A \cong {}_\sigma A$ as R-algebras$\}$. If A is a faithful R-algebra and $\mathscr{G}$ is the group of all automorphisms of A sending R into itself it also follows from (e) that $\mathrm{Aut}_R(A)$ is a normal subgroup of $\mathscr{G}$ with factor group $\mathrm{Aut}(R{:}A)$.

If A is an Azumaya (=central separable) R-algebra let $|A|$ denote the class of A in $B(R)$.

PROPOSITION 2. *Let R be a commutative ring, $\sigma \in \mathrm{Aut}(R)$, and A an Azumaya R-algebra. The correspondence $A \to {}_\sigma A$ induces an action $\sigma\cdot|A| = |{}_\sigma A|$ of* Aut (R) *as a group of Automorphism of $B(R)$.*

Proof. Since $\mathrm{Center}(A) = \mathrm{Center}({}_\sigma A)$ it follows from Lemma 1(d) that ${}_\sigma A$ is an Azumaya R-algebra if A is an Azumaya R-algebra. If P is an R-progenerator then ${}_\sigma P$ is an R-progenerator by Lemma 1(c). From Lemma 1(a) and (b) we have ${}_\sigma(A \otimes \mathrm{Hom}_R(P, P)) \cong {}_\sigma A \otimes \mathrm{Hom}_R({}_\sigma P, {}_\sigma P)$ so the given action on $B(R)$ is well defined. By Lemma 1(a) the action of σ on $B(R)$ is a homomorphism of $B(R)$. If $\tau \in \mathrm{Aut}(R)$ then ${}_{\sigma\tau}A \cong {}_\sigma({}_\tau A)$ as R-algebras by ϕ where $\phi(a) = a$. Thus the inverse of σ is σ^{-1} and σ acts as an automorphism of $B(R)$. Also, $\sigma(\tau\cdot|A|) = \sigma\tau\cdot|A|$. It follows that $\mathrm{Aut}(R)$ acts as a group of automorphisms of $B(R)$.

Let S, T be commutative, faithfully flat R-algebras, let $\sigma \in \mathrm{Aut}(R)$ and let $f\colon S \to T$ be an extension of σ. We show f induces an isomorphism from $H^n(S/R, F)$ to $H^n(T/R, F)$ (Amitsur cohomology) when F is the units functor denoted U or the functor Pic which associates to a commutative ring its group of isomorphism classes of invertible projective modules. Then we show that if g is another extension of σ from S to T then g induces the same isomorphism on cohomology groups f does. Notation is as on pg. 119 of [14]. Let $f^n\colon S^n \to S^n$ by $f^n(s_1 \otimes \cdots \otimes s_n) = f(s_1) \otimes \cdots \otimes f(s_n)$ for all $s_i \in S$. Since $f|_R \in \mathrm{Aut}(R)$ it follows that f^n is a well-defined ring isomorphism which extends σ. The restriction of f^n to $U(S^n)$ is a group isomorphism from $U(S^n)$ to $U(T^n)$ which we also denote f^n. Let $u \in U(S^n)$ and write $u = \sum_{i=1}^m s_{i,1} \otimes \cdots \otimes s_{i,n}$ with $s_{ij} \in S$. Then

$$\begin{aligned}
\Delta_{n-1}(f^n(u)) &= \Delta_{n-1}\Bigl(\sum_{i=1}^m f(s_{i,1}) \otimes \cdots \otimes f(s_{i,n})\Bigr)\\
&= \sum_{i=1}^m \sum_{j=1}^{n+1} (-1)^{j-1}\varepsilon_j(f(s_{i,1}) \otimes f(s_{i,n}))\\
&= f^{n+1}\Bigl(\sum_{j=1}^{n+1}\sum_{i=1}^m (-1)^{j-1}\varepsilon_j(s_{i,1} \otimes \cdots \otimes s_{i,n})\Bigr)\\
&= f^{n+1}\Delta_{n-1}(u)\ .
\end{aligned}$$

It follows that f induces an isomorphism from $H^n(S/R, U)$ to $H^n(T/R, U)$ by $v \to f^{n+1}(v)$ for any n-cocycle v. We denote the induced isomorphism by f^n_*. If $\tau \in \mathrm{Aut}(R)$ and g is an extension of τ from T to another commutative faithfully flat R-algebra W then $g^n_* f^n_* = (gf)^n_*$ since this is the case on the cocycle level. Thus if g is another extension of σ to an isomorphism from S to T then $f = g(g^{-1}f)$ and $g^{-1}f \in \mathrm{Aut}_R(S)$. It is a result of Amitsur (see, for example, 3.2 of [2]) that $(g^{-1}f)^n_*$ is the identity on $H^n(S/R, U)$ so $f^n_* = g^n_*$. Thus σ induces an isomorphism from $H^n(S/R, U)$ to $H^n(T/R, U)$. In particular, it follows that we have defined an action of $\mathrm{Aut}(R: S)$ on $H^n(S/R, U)$ for all $n \geqq 0$.

Let S be an étale R-algebra and let $\sigma \in \mathrm{Aut}(R)$. Then ${}_\sigma S$ is étale and $f: S \to {}_\sigma S$ by $f(s) = s$ for all $s \in S$ is an extension of σ. Thus σ sends elements of $H^n(S/R, U)$ to elements of $H^n({}_\sigma S/R, U)$. From what we have seen it is routine to check that $\mathrm{Aut}(R)$ acts on $\varinjlim H^n(S/R, U)$ where the limit is taken over all étale extensions S of R. Thus $\mathrm{Aut}(R)$ acts on $H^n_{et}(R, U)$.

We proceed as above with the functor Pic. Keeping the previous notation of this section the isomorphism $f: S \to T$ which extends σ induces an isomorphism from $\mathrm{Pic}(S)$ to $\mathrm{Pic}(T)$ by $f(|E|) = |T \otimes_S E|$ for each class $|E|$ in $\mathrm{Pic}(S)$. Now let $|E| \in \mathrm{Pic}(S^n)$, then

$$\begin{aligned}
\Delta_{n-1}(f^n(|E|)) &= \Delta_{n-1}(|T^n \otimes_{S^n} E|) \\
&= \prod_{i=1}^{n+1} |\mathrm{Pic}(\varepsilon_i)(T^n \otimes_{S^n} E)|^{(-1)^{i-1}} \\
&= \prod_{i=1}^{n+1} |T_i^{n+1} \otimes_{T^n} (T^n \otimes_{S^n} E)|^{(-1)^{i-1}} \\
&= \prod_{i=1}^{n+1} |T_i^{n+1} \otimes_{S^n} E|^{(-1)^{i-1}} \\
&= \prod_{i=1}^{n+1} |T^{n+1} \otimes_{S^{n+1}} S_i^{n+1} \otimes_{S^n} E|^{(-1)^{i-1}} \\
&= \prod_{i=1}^{n+1} f^{n+1} |\mathrm{Pic}(\varepsilon_i)(E)|^{(-1)^{i-1}} \\
&= f^{n+1}\Delta_n(|E|) .
\end{aligned}$$

In this calculation S_i^{n+1} is an S^n-algebra via ε_i and T_i^{n+1} is a T^n-algebra by ε_i. Also T^{n+1} is an S^{n+1}-algebra by f^{n+1}. It now follows exactly as in case "U" above that σ induces an isomorphism from $H^n(S/R, \mathrm{Pic})$ to $H^n(T/R, \mathrm{Pic})$ and that $\mathrm{Aut}(R: S)$ acts on $H^n(S/R, \mathrm{Pic})$. Moreover, $\mathrm{Aut}(R)$ acts on $\varinjlim H^n(S/R, \mathrm{Pic})$ where the limit is taken over all étale extensions S of R so $\mathrm{Aut}(R)$ acts on $H^n_{et}(R, \mathrm{Pic})$.

THEOREM 3. *The natural monomorphism* (*given for example in* [17]) *from* $B(R)$ *to* $H^2_{et}(R, U)$ *commutes with the action of* $\mathrm{Aut}(R)$.

Proof. First, following [17] pg. 153, we describe λ. Let A be an Azumaya R-algebra, and let S be an étale R-algebra such that $|A| \in B(S/R)$. Then $A \otimes S \cong \mathrm{Hom}_S(P, P)$ for some S-progenerator P. Define $\phi\colon \mathrm{Hom}_{S\otimes S}(S \otimes P) \to \mathrm{Hom}_{S\otimes S}(P \otimes S)$ by the commutative diagram

$$\begin{tikzcd} S \otimes A \otimes S \arrow[r, "1\otimes\tau"] \arrow[d, "\gamma\otimes 1"'] & \mathrm{Hom}_{S\otimes S}(S \otimes P) \arrow[d, "\phi"] \\ A \otimes S \otimes S \arrow[r, "\tau\otimes 1"'] & \mathrm{Hom}_{S\otimes S}(P \otimes S) \end{tikzcd} \tag{4}$$

where τ is the given isomorphism from $A \otimes S$ to $\mathrm{Hom}_S(P, P)$ and γ is the switch map. From the Morita theorem (Proposition 3.3, pg. 19 of [9]) ϕ is induced by an $S \otimes S$-isomorphism $S \otimes P \to (P \otimes S) \otimes_{S\otimes S} I$ where $|I| \in \mathrm{Pic}(S \otimes S)$. By Proposition 13.13 of [15] we can, by extending S, assume $I \cong S \otimes S$ so ϕ is induced by an isomorphism $\rho\colon S \otimes P \to P \otimes S$. The isomorphism ρ induces three $S \otimes S \otimes S$-isomorphisms: $\rho_1\colon S \otimes S \otimes P \to S \otimes P \otimes S$, $\rho_2\colon S \otimes S \otimes P \to P \otimes S \otimes S$, and $\rho_3\colon S \otimes P \otimes S \to P \otimes S \otimes S$ where $\rho_2 = \rho_3\rho_1$. Thus $\rho_2\rho_3\rho_1$ is multiplication by a unit $u(\tau, \rho)$ in $S \otimes S \otimes S$. The correspondence $A \to u(\tau, \rho)$ induces the monomorphism λ from $B(R)$ into $H^2_{et}(R, U)$. Given S as above and $\sigma \in \mathrm{Aut}(R)$ we have an action of σ from $H^2(S/R, U)$ to $H^2({}_\sigma S/R, U)$ and it suffices to compare the image of $|{}_\sigma A|$ in $H^2({}_\sigma S/R, U)$ with $f^3(u(\tau, \rho))$. Now ${}_\sigma A \otimes {}_\sigma S \cong {}_\sigma(A \otimes S) \overset{\tau}{\cong} {}_\sigma\mathrm{Hom}_S(P, P) \cong \mathrm{Hom}_{{}_\sigma S}({}_\sigma P, {}_\sigma P)$ and the composition of these isomorphisms is τ. The diagram corresponding to (4) for ${}_\sigma A$ is

$$\begin{tikzcd} {}_\sigma S \otimes {}_\sigma A \otimes {}_\sigma S \arrow[r, "1\otimes\tau"] \arrow[d, "\gamma\otimes 1"'] & \mathrm{Hom}_{{}_\sigma S\otimes{}_\sigma S}({}_\sigma S \otimes {}_\sigma P) \arrow[d, "\phi"] \\ {}_\sigma A \otimes {}_\sigma S \otimes {}_\sigma S \arrow[r, "\tau\otimes 1"] & \mathrm{Hom}_{{}_\sigma S\otimes{}_\sigma S}({}_\sigma P \otimes {}_\sigma S)\,. \end{tikzcd} \tag{5}$$

One can check that the $S \otimes S$ isomorphism $\rho\colon S \otimes P \to P \otimes S$ which gave rise to ϕ in (4) also is an ${}_\sigma S \otimes {}_\sigma S$ isomorphism from ${}_\sigma S \otimes {}_\sigma P$ to ${}_\sigma P \otimes {}_\sigma S$ which gives rise to ϕ in (5) (same ϕ!). Therefore, $\rho_2^{-1}\rho_3\rho_1$ is multiplication by the unit $f^3(u(\tau, \rho))$ in ${}_\sigma S \otimes {}_\sigma S \otimes {}_\sigma S$ on ${}_\sigma S \otimes {}_\sigma S \otimes {}_\sigma P$. That is, for any $x \in {}_\sigma S \otimes {}_\sigma S \otimes {}_\sigma P$, $\rho_2^{-1}\rho_3\rho_1(x) = u(\tau, \rho)x = f^3(u(\tau, \rho))\cdot x$. Therefore, ${}_\sigma A$ corresponds to $f^3(u(\tau, \rho))$ in ${}_\sigma S \otimes {}_\sigma S \otimes {}_\sigma S$ which proves the theorem.

Let S be a commutative finitely generated projective R-algebra. There is an exact sequence of Amitsur cohomology due to Chase and Rosenberg [2] which is

$$\begin{aligned} 0 \longrightarrow H^1(S/R, U) &\xrightarrow{\alpha} \mathrm{Pic}(R) \xrightarrow{\beta} H^0(S/R, \mathrm{Pic}) \xrightarrow{d} H^2(S/R, U) \\ &\xrightarrow{\gamma} (S/R) \xrightarrow{\gamma} H^1(S/R, \mathrm{Pic}) \xrightarrow{\rho} H^3(S/R, U)\,. \end{aligned} \tag{6}$$

The maps in this sequence have been explicitly given in [16].

THEOREM 7. *The homomorphisms in the exact sequence* (6) *of Chase and Rosenberg commute with the action of* Aut $(R\colon S)$.

Proof. Let $\sigma \in \operatorname{Aut}(R)$, let S and T be commutative finitely generated projective R-algebras and let f be an extension of σ from S to T. Then we obtain the diagram

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & H^1(S/R, U) \arrow[r, "\alpha"] \arrow[d, "f^1_*"] & \operatorname{Pic}(R) \arrow[r, "\beta"] \arrow[d, "\operatorname{Pic}(\sigma)"] & H^0(S/R, \operatorname{Pic}) \arrow[r, "d"] \arrow[d, "f^0_*"] & H^2(S/R, U) \arrow[r, "\gamma"] \arrow[d, "f^2_*"] & B(S/R) \arrow[r, "\tau"] \arrow[d, "B(\sigma)"] & H^1(S/R, \operatorname{Pic}) \arrow[r, "\rho"] \arrow[d, "f^1_*"] & H^3(S/R, U) \arrow[d, "f^3_*"] \\ 0 \arrow[r] & H^1(T/R, U) \arrow[r, "\alpha"] & \operatorname{Pic}(R) \arrow[r, "\beta"] & H^0(T/R, \operatorname{Pic}) \arrow[r, "d"] & H^2(T/R, U) \arrow[r, "\gamma"] & (B(T/R)) \arrow[r, "\tau"] & H^1(T/R, \operatorname{Pic}) \arrow[r, "\rho"] & H^3(T/R, U)\,. \end{tikzcd} \tag{8}$$

To prove the theorem it is sufficient that this diagram commutes. The method of proof is to write down each of the maps explicitly following [16] and to check the commutativity of the diagram on the cocycle level. This is a routine calculation. We carry out here only the proof of $B(\sigma)\gamma = \gamma f^2_*$. This will show the natural homomorphism $\gamma \cdot H^2(S/R, U) \to B(S/R)$ commutes with the action of Aut $(R\colon S)$.

Let $u \in U(S \otimes S)$ be a 1-cocycle and let $E = \{x \in S \mid 1 \otimes x = x \otimes 1 \cdot u\}$. Then E represents an element in $\operatorname{Pic}(S/R)$ and α is induced by the correspondence $u \to E$. Now $f^1_*(|u|) = |f^2(u)|$ in $H^1(T/R, U)$. Let $F = \{x \in T \mid 1 \otimes x = x \otimes 1 \cdot f^2(u)\}$. Define $\psi\colon {}_\sigma E \to F$ by $\psi(x) = f(x)$ (note ψ is well defined since ${}_\sigma E = E$ as abelian groups) for $r \in R$ and $x \in {}_\sigma E$; $\psi(r * x) = \psi(\sigma^{-1}(r)x) = f(\sigma^{-1}(r)x) = rf(x) = r\psi(x)$. Thus $\alpha f^1_*(|u|) = |{}_\sigma E| = \operatorname{Pic}(\sigma)\alpha(|u|)$.

Now $f^3(u) = \sum_i f(a_i) \otimes f(b_i) \otimes f(c_i) \in U(T^3)$. Let $Q = T \otimes T$ be the projective T-module obtained by letting T act on the first factor. Define $T \otimes T$ isomorphism $h\colon T \otimes Q \to Q \otimes T$ by $h(x \otimes y \otimes z) = \sum_i f(a_i)x \otimes f(c_i)z \otimes f(b_i)y$ for all x, y, z in T. The isomorphism h induces an isomorphism $\phi(f^3(u))\colon \operatorname{Hom}_{T\otimes T}(T \otimes Q, T \otimes Q) \to \operatorname{Hom}_{T\otimes T}(T \otimes Q, T \otimes Q)$ by $\phi(f^3(u))(\rho) = h\rho h^{-1}$ for all $\rho \in \operatorname{Hom}_{T\otimes T}(T \otimes Q, T \otimes Q)$. The Azumaya algebra $A(f^3(u)) = \{w \in \operatorname{Hom}_T(Q, Q) \mid \phi(f^3(u))\varepsilon_1(W) = \varepsilon_2(w)\}$ is a representative of the class $\gamma f^2_*(|u|)$ in $B(S/R)$. The algebra ${}_\sigma A(u)$ represents the class $B(\sigma)\gamma(|u|)$ so it suffices to show $A(f^3(u)) \cong {}_\sigma A$ as R-algebras. Now f^3 induces an isomorphism from $\operatorname{Hom}_{S\otimes S}(S \otimes P, S \otimes P)$ to $\operatorname{Hom}_{T\otimes T}(T \otimes Q, T \otimes Q)$ by $\rho \to f^3\rho(f^3)^{-1}$ for all $\rho \in \operatorname{Hom}_{S\otimes S}(S \otimes P, S \otimes P)$, and f^2 induces an isomorphism ψ from $\operatorname{Hom}_S(P, P) \to \operatorname{Hom}_T(Q, Q)$ by $\psi(w) = f^2 w (f^2)^{-1}$ for all $w \in \operatorname{Hom}_S(P, P)$. We show ψ is a ring isomorphism from $A(u)$ to $A(f^3(u))$. Let $w \in A(u)$. We need to check that $\psi(w) \in A(f^3(u))$. But

$$
\begin{aligned}
\phi(f^3(u))\varepsilon_1(\psi(w)) &= \phi(f^3(u))\varepsilon_1(f^2w(f^2)^{-1}) \\
&= \phi(f^3(u))f^3\varepsilon_1(w)(f^3)^{-1} \\
&= h(f^3\varepsilon_1(w)(f^3)^{-1})h^{-1} \\
&= \tau f^3(u)f^3\varepsilon_1(w)(f^3)^{-1}f^3(u)^{-1}\tau \\
&\quad \text{(where } \tau \text{ is twist map which exchanges} \\
&\quad \text{the last two factors)} \\
&= \tau f^3 u\varepsilon_1(w)u^{-1}(f^3)^{-1}\tau \\
&= f^3\tau u\varepsilon_1(w)u^{-1}\tau(f^3)^{-1} \\
&= f^3 g\varepsilon_1(w)g^{-1}(f^3)^{-1} \\
&= f^3\phi(u)\varepsilon_1(w)(f^3)^{-1} \\
&= f^3\varepsilon_2(w)(f^3)^{-1} \\
&= \varepsilon_2(f^2(w)(f^2)^{-1}) \\
&= \varepsilon_2(\psi(w)) .
\end{aligned}
$$

For any $r \in R$ and $w \in {}_\sigma A(u)$ we have $\psi(r*w) = f^2\sigma^{-1}(r)w(f^2)^{-1} = rf^2w(f^2)^{-1} = r\psi(w)$ so ψ is an R-isomorphism from ${}_\sigma A(u)$ to $A(f^3(u))$.

COROLLARY 9. *If S is a commutative finitely generated projective R-algebra and* $\mathrm{Pic}(S) = \mathrm{Pic}(S \otimes S) = 0$ *then γ is a natural* $\mathrm{Aut}(R\colon S)$ *isomorphism from $H^2(S/R, U)$ to $B(S/R)$.*

COROLLARY 10. *Let S be a Galois extension of R, and assume S has no idempotents other than 0 and 1. Let G be the Galois group of S over R. Then the homomorphisms in the seven-term sequence of Galois cohomology*

$$
\begin{aligned}
1 \longrightarrow H^1(G, U(S)) &\longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(S)^G \longrightarrow H^2(G, U(S)) \\
&\longrightarrow B(S/R) \longrightarrow H^1(G, \mathrm{Pic}(S)) \longrightarrow H^3(G, U(S))
\end{aligned}
$$

commute with the action of $\mathrm{Aut}(R\colon S)$. *In particular, if* $\mathrm{Pic}(S) = 0$ *then $H^2(G, U(S))$ is isomorphic to $B(S/R)$ by an isomorphism which commutes with the action of* $\mathrm{Aut}(R\colon S)$.

Proof. If S is a Galois extension of R with group G and S has no idempotents other than 0 and 1 then $\mathrm{Aut}_R(S) = G$. Thus the group $\mathscr{G}$ of automorphism of S leaving R setwise fixed acts on G by conjugation. If $\sigma \in \mathscr{G}$ and $\tau \in G$ we let $\tau^\sigma = \sigma^{-1}\tau\sigma$. Assume M is a $\mathscr{G}$-module. Let $f \in Z^n(G, M)$ be a Galois n-cocycle and let $\sigma f \in Z^n(G, M)$ be defined by $\sigma f(\tau_1, \cdots, \tau_n) = \sigma[f(\tau_1^\sigma, \cdots, \tau_n^\sigma)]$ for $\tau_i \in G$ and $n \geqq 1$ and let $\sigma \cdot m = \sigma(m)$ for $m \in Z^0(G, M)$. Then a direct calculation gives $\partial_n \sigma \cdot f = \sigma \cdot \partial_n f$ for all $n \geqq 0$. Thus $\mathscr{G}$ acts on $H^n(G, M)$ with the action induced by the correspondence $f \to \sigma \cdot f$.

By a tedious calculation with cocycles one can check that if $\sigma \in G$ then f and σf represent the same class in $H^n(G, M)$. It follows that $\mathrm{Aut}(R: S)$ acts on $H^n(G, U(S))$ and $H^n(G, \mathrm{Pic}(S))$ for all $n \geqq 0$. This action is given in [11] where these assertions are verified when S is a field. Proofs are the same in the general context.

Next we check following notation in [15], p. 120–125, that the actions we have defined on $H^n(S/R, F)$ and $H^n(G, F)$ correspond to one another. The isomorphisms $\phi_n: S^{n+1} \to K^n(G, S)$ given by $\phi_n(s \otimes \cdots \otimes s_n)(\tau_k, \cdots, \tau_n) = s_1\tau_1(s_2)\tau_1\tau_2(s_3)\cdots\tau_1\cdots\tau_n(s_{n+1})$ for $s_i \in S$ and $\tau_i \in G$ induce a homomorphism $F(\phi_n): F(S^{n+1}) \to F(K^n(G, S)) = K^n(G, F(S))$ which induces a morphism of complexes for any additive functor F. This morphism of complexes induces homomorphisms $\psi_n: H^n(S/R, F) \to H^n(G, F(S))$. Let $\sigma \in G$, then

$$\begin{aligned}\sigma\phi_n(s_1 \otimes \cdots \otimes s_n)(\tau_1, \cdots, \tau_n) &= \sigma[s_1\tau_1^\sigma(s_2)\cdots\tau_1^\sigma\cdots\tau_n^\sigma(s_{n+1})] \\ &= \sigma(s_1)\tau_1\sigma(s_2)\cdots\tau_1\cdots\tau_n\sigma(s_{n+1}) \\ &= \phi_n\sigma(s_1 \otimes \cdots \otimes s_{n+1}) .\end{aligned}$$

Thus if F is the units functor U then the induced homomorphism $\psi_n: H^n(S/R, U) \to H^n(G, U(S))$ is an $\mathrm{Aut}(R: S)$ homomorphism. In a similar way the induced homomorphism $\psi_n: H^n(S/R, \mathrm{Pic}) \to H^n(G, \mathrm{Pic}(S))$ can be checked to be $\mathrm{Aut}(R: S)$ homomorphisms since $\mathrm{Pic}(\phi_n)$: $\mathrm{Pic}(S^{n+1}) \to \mathrm{Pic}(K^n(G, S))$ is an $\mathrm{Aut}(R: S)$ isomorphism.

Now the result is a consequence of Theorem 7 and Corollary 5.5 of [1] which connects the exact sequence of Amitsur cohomology with the exact sequence of Galois cohomology.

We saw before that $\mathrm{Aut}(R)$ acts on $H^n_{et}(R, U)$. We now expose the corresponding result in Galois cohomology. Let R be a commutative ring with no idempotents other than 0 and 1 and let Ω be the separable closure of R (see p. 99 of [9]). For any $\sigma \in \mathrm{Aut}(R)$ it is easy to check that ${}_\sigma\Omega$ is another separable closure of R. By uniqueness of the separable closure we know $\Omega \cong {}_\sigma\Omega$ as R-algebras. By Lemma 1e, it follows that $\mathrm{Aut}(R) = \mathrm{Aut}(R: \Omega)$. Let S be a Galois extension of R in Ω, then ${}_\sigma S$ is a Galois extension of R in Ω and if G is the Galois group of S then G is also the Galois group of ${}_\sigma S$. Let f be an extension of σ from S to ${}_\sigma S$ and let g be an n-cocycle in $Z^n(G, U(S))$. There is a corresponding n-cocycle h in $Z^n(G, U({}_\sigma S))$ by $h(\tau_1, \cdots, \tau_n) = f(g(f^{-1}\tau_1 f, \cdots, f^{-1}\tau_n f))$. As with Amitsur cohomology σ induces an isomorphism from $H^n(G, U(S))$ to $H^n(G, U({}_\sigma S))$. This action is compatible with the maps in the direct limit system $\varinjlim H^n(G, U(S))$ where the limit is taken over all Galois extensions S of R in Ω. Thus $\mathrm{Aut}(R)$ acts on $H^n_{\mathrm{Gal}}(R, U)$, and this action commutes with the natural homomorphism from $H^2_{\mathrm{Gal}}(R, U)$ to $B(R)$.

Let $\mathscr{G}$ be the subgroup of $\mathrm{Aut}(A)$ which leaves $R\cdot 1$ setwise invariant. If Center $A = R\cdot 1$ then $\mathscr{G} = \mathrm{Aut}(A)$. If A is a faithful R-algebra and τ is the restriction map from $\mathscr{G}$ to $\mathrm{Aut}(R)$ then from Lemma 1e the sequence

$$(11) \qquad 1 \longrightarrow \mathrm{Aut}_R(A) \xrightarrow{i} \mathscr{G} \xrightarrow{r} \mathrm{Aut}(R\colon A) \longrightarrow 1$$

is exact.

Now the splitting of the exact sequence (11) over the finite subgroups $H \subset \mathrm{Aut}(R\colon A)$ for an Azumaya R-algebra is considered. The given sequence splits over H in case there is a monomorphism $s\colon H \rightarrow \mathrm{Aut}(A)$ such that $rs = 1_H$.

THEOREM 12. *Let A be an Azumaya R-algebra, let H be a finite subgroup of $\mathrm{Aut}(R\colon A)$, and let $K = \{r \in R \mid \tau(r)$ for all $\tau \in H\}$. Assume R is a Galois extension of K with group H, then the following are equivalent.*

(1) *The sequence* (11) *splits over H.*

(2) *There is an Azumaya K-algebra B such that $A \cong R \otimes_K B$ as R-algebras.*

(3) *A is a normal algebra and the class of A in $B(R)$ is in the kernel of the Teichmuller cocycle map* [3].

Proof. If $A \cong R \otimes_K B$ then each element $\sigma \in H$ induces the K-automorphism $\sigma \otimes 1$ of A and the splitting map is $s(\sigma) = \sigma \otimes 1$ so $2 \rightarrow 1$. Next assume the existence of a splitting map s. Then there is a group of automorphisms H' of A whose restriction to R is H. In this case all the hypotheses of Lemma 2 of [6] are satisfied. Let $B = \{a \in A \mid \sigma(a) = a$ for all $\sigma \in H'\}$. In the proof of Lemma 2 of [6] it is shown that B is an Azumaya K-algebra and $R \otimes_K B \cong A$ by $r \otimes b \rightarrow rb$ for all $r \in R$, $b \in B$ so $1 \rightarrow 2$. An Azumaya R-algebra A is called normal if H is a subgroup of $\mathrm{Aut}(R\colon A)$. By Corollary 5.2 of [3], the class in $B(R)$ represented by a normal algebra A has a trivial image under the Teichmuller cocycle map if and only if there is an Azumaya K-algebra B such that $A \cong R \otimes_K B$ as R-algebras. Thus $2 \leftrightarrow 3$.

We saw in Lemma 1 that if $\sigma \in \mathrm{Aut}(R)$ then σ extends to an automorphism of the Azumaya algebra A if and only if $A \cong {}_\sigma A$ as R-algebras. We have let $\mathrm{Aut}(R\colon A) = \{\sigma \in \mathrm{Aut}(R) \mid A \cong {}_\sigma A\}$. When does $\mathrm{Aut}(R\colon A) = \{\sigma \in \mathrm{Aut}(R) \mid |A| = |{}_\sigma A|\}$? We first have the following positive result.

PROPOSITION 13. *Let R denote a local ring or the ring of polynomials in one variable over a perfect field. For any Azumaya*

R-algebra A and any $\sigma \in \mathrm{Aut}(R)$ we have $A \cong {}_\sigma A$ as R-algebras if and only if the class of A and ${}_\sigma A$ are the same in $B(R)$.

Proof. If $A \cong {}_\sigma A$ as R-algebras then $|A| = |{}_\sigma A|$. Conversely, Corollary 1 of [7] implies $A \cong M_n(D)$ where D is the unique Azumaya R-algebra with no idempotents other than 0 and 1 in the same class as A in $B(R)$. Employing Lemma 1 we have the chain of isomorphisms, ${}_\sigma A \cong {}_\sigma(D \otimes \mathrm{Hom}_R(R^n, R^n)) \cong {}_\sigma D \otimes \mathrm{Hom}_R({}_\sigma R^n, {}_\sigma R^n) \cong M_n({}_\sigma D)$. If $|A| = |{}_\sigma A|$ then by the uniqueness of D we have $D \cong {}_\sigma D$ so $A \cong {}_\sigma A$.

Now we outline an example which shows the previous proposition fails even when R is a Dedekind domain. Let R be a Dedekind domain and let I be a fractional R-ideal. Let $M = R \oplus I$ and $A = \mathrm{Hom}_R(M, M)$. Then A is an Azumaya R-algebra. Let $\sigma \in \mathrm{Aut}(R)$, then ${}_\sigma A \cong \mathrm{Hom}_R({}_\sigma M, {}_\sigma M)$ by Lemma 1. By Moritia's theory (#7, pg. 37 of [9]) we have $A \cong {}_\sigma A$ if and only if ${}_\sigma M \cong J \otimes M$ for some fractional R-ideal J.

But ${}_\sigma M \cong R \oplus {}_\sigma I$ and $J \otimes M \cong R \oplus IJ^2$, so $M \cong {}_\sigma M$ if and only if there is a fractional R-ideal J so that ${}_\sigma I = IJ^2$. Now A is always equivalent to ${}_\sigma A$ in $B(R)$ and so to give a counter example to the conclusion of Proposition 13 if R is a Dedekind domain it suffices to give a Dedekind domain R such that $\mathrm{Pic}(R)$ is the group of order 4 and exponent 2 and so that $\mathrm{Aut}\, R$ acts nontrivially on $\mathrm{Pic}(R)$. Such an example is easy to construct using results of L. Claborn [5]. It follows that under the action of $\mathrm{Aut}(R)$ on $B(R)$ we have $B(R)^G$ is guaranteed to consist of those classes in $B(R)$ represented by normal algebras only when the hypotheses of Proposition 13 are satisfied.

PROPOSITION 14. *Let K be a field of characteristic $= 0$. Then the Schur subgroup is an* $\mathrm{Aut}(K)$ *invariant subgroup of $B(K)$.*

Proof. Let K be a field of characteristic $= 0$. A cyclotomic algebra is a crossed product over K of the form $\Delta(K\sqrt[n]{1}, G, f)$ where f is a 2-cocycle on G with its values in the cyclic group generated by $\sqrt[n]{1}$. The Schur subgroup of $B(K)$ consists of those classes in $B(K)$ represented by a cyclotomic algebra [19]. If $\sigma \in \mathrm{Aut}(K)$ then $\sigma \in \mathrm{Aut}(K\colon K\sqrt[n]{1})$ and ${}_\sigma\Delta(K\sqrt[n]{1}, G, f) = \Delta(K\sqrt[n]{1}, G, \sigma\cdot f)$. But $\sigma\cdot f(\tau, \rho) = \sigma[f(\tau^\sigma, \rho^\sigma)]$ so the values of $\sigma\cdot f$ are in the nth roots of unity and $\Delta(K\sqrt[n]{1}, G, \sigma\cdot f)$ is another cyclotomic algebra. Thus $\mathrm{Aut}(R)$ leaves the Schur subgroup of $B(K)$ invariant.

Finally, let $R = \boldsymbol{R}[x, y]/(xy - 1)$ where $\boldsymbol{R}$ denotes the real numbers. Then $B(R) = Z_2 \oplus Z_2$ [10]. Let $S = \boldsymbol{C} \otimes_{\boldsymbol{R}} R$, then S is a

Galois extension of R with Galois group G of order 2 with generator of G denoted τ. Let $\sigma \in \mathrm{Aut}(R)$ be given by $\sigma(x) = -x$ and $\sigma(y) = -y$. Let $\Delta(S: G: f)$ be the crossed product with $f(\tau, \tau) = x$. Then $\Delta(S: G: f)$ represents an element in $B(R)$ of order 2 and ${}_\sigma\Delta(S: G: f) = \Delta(S: G: \sigma \cdot f)$ where $\sigma \cdot f(\tau, \tau) = \sigma(x) = -x$. Thus ${}_\sigma\Delta(S: G: f)$ represents an element in $B(R)$ inequivalent to $\Delta(S: G: f)$. The crossed product $\Delta(S: G: g)$ with $g(\tau, \tau) = -1$ represents a nontrivial element in $B(R)$ such that ${}_\sigma\Delta(S: G: g) \cong \Delta(S: G: g)$ for all $\sigma \in B(R)$ so the image of $\mathrm{Aut}(R)$ in $\mathrm{Aut}(B(R))$ has order 2. Not every automorphism of $B(R)$ can be represented by an automorphism of R.

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