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UNIVERSAL CONNECTIONS: THE LOCAL PROBLEM

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It is well known that the canonical connection on the Stiefel bundle over the Grassman manifold is universal in a certain range of dimensions. We give some local results specifying necessary and sufficient conditions for the connection to be universal for particular dimensions.

1. Introduction. In this paper we consider connections on a principal G-bundle over a manifold M of dimension m, where G is $O(n)$, $U(n)$, or $\mathrm{Sp}(n)$. The universal examples are the bundles

$$O(N)/O(N-n) \longrightarrow O(N)/O(n) \times O(N-n)$$
$$U(N)/U(N-n) \longrightarrow U(N)/U(n) \times U(N-n)$$
$$\mathrm{Sp}(N)/\mathrm{Sp}(N-n) \longrightarrow \mathrm{Sp}(N)/\mathrm{Sp}(n) \times \mathrm{Sp}(N-n)$$

with their canonical connections. (See § 2 for the definitions.) These are the Stiefel and Grassman manifolds for $\boldsymbol{R}$, $\boldsymbol{C}$, and $\boldsymbol{H}$. It is a theorem of Narasimhan and Ramanan that the canonical connection is universal in the sense that any connection on a principal G-bundle over M^m is induced by a map into the appropriate Grassman manifold N is sufficiently large. According to [2], [4], and § 8, it suffices to take

$$N \geqq 2n(m+1)(2mn^2+1) \qquad \text{or} \qquad \frac{1}{2}[(n+m)^2 + 7(n+m) + 10]$$

for $O(n)$,

$$N \geqq n(m+1)(2mn^2+1)$$

for $U(n)$, and

$$N \geqq n(m+1)(4mn^2+2mn+1)$$

for $\mathrm{Sp}(n)$.

These inequalities are not sharp. The situation is analogous to the problem of finding isometric imbeddings of Riemannian manifold into Euclidean space. In that case, global results are available, but they are not sharp with respect to dimension. The only sharp results date back to L. Schläfli [3] who found the least dimensional Euclidean space for the local isometric imbeddability of a real-analytic Riemannian manifold. This was made more rigorous by M. Janet, C. Burstin, and E. Cartan. (See [5] for an account of this theorem.)

The principal result of this paper is to show how E. Cartan's theory of differential systems can be used to get local existence theorems for connection preserving maps into the appropriate Grassmanian. These results are sharp with respect to dimension.

I would like to thank my advisor, I. M. Singer, for suggesting this problem to me. I am also grateful to S. S. Chern for explaining differential systems. Finally, I am indebted to Uncle Ludwig for inspiration.

2. **Definition of the universal connection.** Let F be $\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$. The Stiefel manifold S is the set of N-by-N matrices P over F such that $P^*P = 1$. For $P \in S$, let

$$W_p = \operatorname{im} PP^*$$

so PP^* is the projection from F^N onto W_p. The map $P \mapsto W_p$ gives a fibration of S over the Grassman manifold

$$\{W \subseteq F^N\colon W \text{ is a subspace of dimension } n\} .$$

It is a principal bundle with the obvious action of $U(n, F)$ (i.e., $O(n)$, $U(n)$, or $\mathrm{Sp}(n)$). It has a canonical connection defined as follows: the horizontal space at P is $\hom_F(W_p, W_p^\perp)$ where T_pS is identified with the subspace of N-by-N matrices A such that $A + A^* = 0$.

The connection one-form is given by

$$\psi(A) = P^*AP\,, \quad A \in T_pS\,.$$

The curvature is

$$\Phi(A, B) = -P^*[A, B]P + [P^*AP, P^*BP]\,, \quad A, B \in T_pS\,.$$

Let o be the point

$$\begin{bmatrix} 1 & & & \\ & 1 & & \bigcirc \\ & & \ddots & \\ \bigcirc & & & 1 \end{bmatrix}$$

in S. The horizontal vectors at o can be indentified with $(N-n)$-tuples of vectors in F^n as follows: $(a_1, \cdots, a_{N-n}) \in F^n \times \cdots \times F^n$ is identified with

$$\begin{bmatrix} 0 & a_1 \cdots a_{N-n} \\ -a_1^* & \\ \vdots & \bigcirc \\ -a_{N-n}^* & \end{bmatrix}.$$

The curvature form is

$$\Phi((a_1, \cdots, a_{N-n}), (b_1, \cdots, b_{N-n})) = a_1 \wedge b_1 + \cdots + a_{N-n} \wedge b_{N-n}$$

where

$$a \wedge b = ab^* - b^*a \ , \quad a, b \in F^n$$

so $a \wedge b$ is an n-by-n matrix.

3. Cartan's method. Suppose we have a real-analytic connection on a principal $U(n, F)$-bundle over $\boldsymbol{R}^m$. With respect to a moving frame, the connection is given by a skew-hermitian matrix $\omega = (\omega_{ij})$ of one-forms on $\boldsymbol{R}^m$. The curvature is

$$\Omega = (\Omega_{ij}) = d\omega + \omega \wedge \omega \ .$$

We want to find a map

$$f: \boldsymbol{R}^m \longrightarrow U(N, F)/U(N-n, F)$$

such that in a neighbor hood of $o \in \boldsymbol{R}^m$,

$$\omega = f^*\psi \ .$$

The graph of f will be a submanifold of $\boldsymbol{R}^m \times U(N, F)/U(N-n, F)$ containing $(0, o)$, having dimension m, and transverse to $U(N, F)/U(N-n, F)$. The one-forms

$$\pi_1^*\omega_{ij} - \pi_2^*\omega_{ij} \ , \quad 1 \leqq i, j \leqq n$$

vanish on this submanifold, where π_1 and π_2 are the obvious projections. Furthermore, our given connection may be obtained from the universal connection if and only if we can find a submanifold with these properties.

Let $\mathscr{I}$ be the ideal of differential forms generated by

$$\pi_1^*\omega_{ij} - \pi_2^*\psi_{ij} \ , \quad 1 \leqq i, j \leqq n$$

and

$$\pi_1^*\Omega_{ij} - \pi_2^*\Phi_{ij} \ , \quad 1 \leqq i, j \leqq n \ .$$

LEMMA 3.1. (a) *$\mathscr{I}$ is closed under the exterior derivative.*

(b) *If the forms $\pi_1^*\omega_{ij} - \pi_2^*\psi_{ij}$ vanish on a submanifold, then all of $\mathscr{I}$ does.*

Proof. (a) We have, by definition,

$$\Omega_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}$$
$$\Phi_{ij} = d\psi_{ij} + \sum_k \psi_{ik} \wedge \psi_{kj} \ ,$$

so

$$\begin{aligned} d(\pi_1^*\omega_{ij} - \pi_2^*\psi_{ij}) &= \pi_1^* d\omega_{ij} - \pi_2^* d\psi_{ij} \\ &= \pi_1^*(\Omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}) - \pi_2^*(\Phi_{ij} - \sum_k \psi_{ik} \wedge \psi_{kj}) \\ &= \pi_1^*\Omega_{ij} - \pi_2^*\Phi_{ij} - \sum_k \pi_1^*\omega_{ik} \wedge (\pi_1^*\omega_{kj} - \pi_2^*\psi_{kj}) \\ &\quad - \sum_k (\pi_1^*\omega_{ik} - \pi_2^*\psi_{ik}) \wedge \pi_2^*\psi_{kj} \,. \end{aligned}$$

Also, the Bianchi identities are

$$\begin{aligned} d\Omega_{ij} + \sum_k \omega_{ik} \wedge \Omega_{kj} - \Omega_{ik} \wedge \omega_{kj} &= 0 \\ d\Phi_{ij} + \sum_k \psi_{ik} \wedge \Phi_{kj} - \Phi_{ik} \wedge \psi_{kj} &= 0 \,, \end{aligned}$$

so

$$\begin{aligned} &d(\pi_1^*\Omega_{ij} - \pi_2^*\Phi_{ij}) \\ &\quad = \pi_1^* d\Omega_{ij} - \pi_2^* d\Phi_{ij} \\ &\quad = \pi_1^* \sum_k \Omega_{ik} \wedge \omega_{kj} - \omega_{ik} \wedge \Omega_{kj} \\ &\qquad - \pi_2^* \sum_k \Phi_{ik} \wedge \psi_{kj} - \psi_{ik} \wedge \Phi_{kj} \\ &\quad = \sum_k (\pi_1^*\Omega_{ik} - \pi_2^*\Phi_{ik}) \wedge \pi_1^*\omega_{kj} + \sum_k \pi_2^*\Phi_{ik} \wedge (\pi_1^*\omega_{kj} - \pi_2^*\psi_{kj}) \\ &\qquad - \sum_k \pi_1^*\omega_{ik} \wedge (\pi_1^*\Omega_{kj} - \pi_2^*\Phi_{kj}) - \sum_k (\pi_1^*\omega_{ik} - \pi_2^*\psi_{ik}) \wedge \pi_2^*\Phi_{kj} \,. \end{aligned}$$

It follows easily that $d\mathscr{I} \subset \mathscr{I}$.

(b) Let f be the inclusion map of a submanifold. If

$$f^*(\pi_1^*\omega_{ij} - \pi_2^*\psi_{ij}) = 0 \,, \quad 1 \leqq i, j \leqq n \,,$$

then

$$\begin{aligned} f^*(\pi_1^*\Omega_{ij} - \pi_2^*\Omega_{ij}) &= f^* d(\pi_1^*\omega_{ij} - \pi_2^*\psi_{ij}) \\ &= d[f^*(\pi_1^*\omega_{ij} - \pi_2^*\psi_{ij})] \\ &= 0 \,. \end{aligned}$$ □

Thus $\mathscr{I}$ is a differential system and our problem is equivalent to finding an integral submanifold for $\mathscr{I}$. The Cartan-Kähler theorem gives a sufficient condition for such a submanifold to exist (see [1] or [5] for modern treatment). For the rest of this paper we assume that all connections are real-analytic.

Let $e_1, \cdots, e_m$ be the standard basis for $\boldsymbol{R}^m$. It is not hard to show that for N sufficiently large, every

$$A \in \Lambda^2(\boldsymbol{R}^m) \otimes u(n, F)$$

is of the form

$$A = \frac{1}{2} \sum_{i,j=1}^{m} \sum_{k=1}^{N-n} (e_i \wedge e_j) \otimes (a_i^k \wedge a_j^k) \tag{3.2}$$

for some $a_i^k \in F^n$. Let N_{mn} be the least such integer N.

We now discuss the problem of finding the least N such that every connection on a $U(n, F)$-bundle over $\boldsymbol{R}^m$ is obtained from a map

$$f: \boldsymbol{R}^m \longrightarrow U(N, F)/U(N-n, F) .$$

A necessary condition is that $N \geqq N_{mn}$ because if A is the value of $f^*\Phi$ at some point, then A satisfies equation (3.2) where $(a_i^1, \cdots, a_i^{N-n})$ corresponds to the horizontal part of $f_*(e_i)$. Conversely, at least in the generic case, the Cartan-Kähler theorem may be used to show that we may actually take $N = N_{mn}$. In the following four sections, this procedure is worked out for specific examples.

4. $O(n)$-Bundles over $\boldsymbol{R}^2$.

THEOREM 4.1. *Every $O(n)$-bundle over $\boldsymbol{R}^2$ is locally obtained from a map*

$$f: \boldsymbol{R}^2 \longrightarrow O(N)/O(N-n)$$

if and only if $N \geqq (3n-1)/2$.

Proof. Suppose n is $2k$ or $2k+1$, and $N = n + k$ so $N \geqq (3n-1)/2$. We are going to apply the Cartan-Kähler theorem, so the reader is referred to [1] or [5] for the relevant definitions.

Each point is a regular integral element, as its polar space always has dimension two.

Choose $a_1, \cdots, a_k, b_1, \cdots, b_k$ in $\boldsymbol{R}^n$ such that

$$\Omega(e_1, e_2) = a_1 \wedge b_1 + \cdots + a_k \wedge b_k .$$

That we can do this follows from the normal form theorem for skew-symmetric bilinear forms. Furthermore, we can arrange $a_1, \cdots, a_k$ to be linearly independent.

Let $Y_1, Y_2 \in T_0 O(n+k)/O(k)$ be the vectors with vertical parts $\omega(e_1), \omega(e_2)$ (resp.) and horizontal parts $(a_1, \cdots, a_k), (b_1, \cdots, b_k)$ (resp.). The polar space for (e_1, Y_1) is

$$\varepsilon(e_1, Y_1) = \{(X, Y): \omega(X) = \psi(Y), \Omega(e_1, X) = \Phi(Y_1, Y)\} .$$

It is not hard to see that this has constant dimension $k + 2 + k(k-1)/2$, so (e_1, Y_1) is a regular integral element.

Thus we can now apply the Cartan-Kähler theorem to get an integral submanifold tangent to (e_1, Y_1) and (e_2, Y_2). The existence of f follows. □

5. $SU(2)$ and $U(2)$ bundles over $\boldsymbol{R}$.

LEMMA 5.1. *Suppose $M \in u(2)$ has eigenvalues $\lambda_1 i$, $\lambda_2 i$ with* $\lambda_1 \leqq$

λ_2. *Then there exist* $a, b \in \boldsymbol{C}^2$ *such that* $a \neq 0$ *and*

$$M = a \wedge b \tag{5.2}$$

if and only if $\lambda_1 \leqq 0 \leqq \lambda_2$.

Proof. Suppose that (5.2) holds for some $a, b \in \boldsymbol{C}^2$. We can assume that $|a| = 1$. Choose $u \in U(2)$ such that

$$ua = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Define $\alpha, \beta, \gamma, \delta \in \boldsymbol{R}$ by

$$\begin{pmatrix} \alpha + i\beta \\ \gamma + i\delta \end{pmatrix} = ub .$$

Then

$$\begin{aligned} uMu^* &= u(a \wedge b)u^* \\ &= ua \wedge ub \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \alpha + i\beta \\ \delta + i\delta \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \alpha + i\beta \\ i\delta \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\alpha - i\beta \\ \alpha - i\beta & -2i\delta \end{pmatrix} \end{aligned}$$

has eigenvalues $\lambda_1 i$ and $\lambda_2 i$, so

$$\begin{aligned} \lambda_1 &= -\delta - \sqrt{\alpha^2 + \beta^2 + \delta^2} \\ \lambda_2 &= -\delta + \sqrt{\alpha^2 + \beta^2 + \delta^2} . \end{aligned} \tag{5.3}$$

It follows that $\lambda_1 \leqq 0 \leqq \lambda_2$.

Convesely, if $\lambda_1 \leqq 0 \leqq \lambda_2$ then we can find $\alpha, \beta, \delta \in \boldsymbol{R}$ such that (5.3) holds. Therefore

$$M = u\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \alpha + i\beta \\ i\beta \end{pmatrix}\right]u^*$$

for some $u \in U(2)$, and hence

$$M = u\begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge u\begin{pmatrix} \alpha + i\beta \\ i\delta \end{pmatrix} . \qquad \square$$

Now suppose we have a connection ω on a $U(2)$-bundle over $\boldsymbol{R}^2$, with curvature Ω. Let $\lambda_1 i, \lambda_2 i$ $(\lambda_1 \leqq \lambda_2)$ be the eigenvalues of $\Omega(e_1, e_2)$.

THEOREM 5.4. *On a neighborhood V of $0 \in \boldsymbol{R}^2$ the connection is obtained from a map*

$$f: V \longrightarrow U(4)/U(2) \times U(2) .$$

In order for there to exist a neighborhood V of $o \in \boldsymbol{R}^2$ such that the given connection is obtained from a map

$$g: V \longrightarrow U(3)/U(2) \times U(1)$$

it is necessary and sufficient that there exist a neighborhood of $o \in \boldsymbol{R}^2$ such that $\lambda_1 \leqq 0 \leqq \lambda_2$.

Proof. Suppose

$$u^*\Omega(e_1, e_2)u = \begin{pmatrix} \lambda_1 i & 0 \\ 0 & \lambda_2 i \end{pmatrix} .$$

If we let

$$a = u\begin{pmatrix}1\\0\end{pmatrix} , \qquad c = u\begin{pmatrix}0\\1\end{pmatrix} ,$$

$$b = -\frac{1}{2}\lambda_1 ia , \qquad d = -\frac{1}{2}\lambda_2 ic ,$$

then a and c are linearly independent and

$$\Omega(e_1, e_2) = a \wedge b + c \wedge d .$$

Let $Y_1, Y_2 \in T_0 U(4)/U(2)$ be the vectors with vertical parts $\omega(e_1)$, $\omega(e_2)$ (resp.) and horizontal parts (a, c), (b, d) (resp.). Then, as in the proof of Theorem 4.1,

$$(0, o) < (e_1, Y_1) < \{(e_1, Y_1), (e_2, Y_2)\}$$

is a chain of ordinary integral elements satisfying the hypothesis of the Cartan-Kähler theorem. Hence an appropriate integral submanifold exists.

The second part of Theorem 5.4 follows similarly, given Lemma 5.1. □

COROLLARY 5.5. *Every connection on a $SU(2)$-bundle over $\boldsymbol{R}^2$ comes locally from a map*

$$g: \boldsymbol{R}^2 \longrightarrow U(3)/U(2) \times U(1) .$$

Connections on $U(1)$-bundles over $\boldsymbol{R}^2$ come locally from a map

$$h: \boldsymbol{R}^2 \longrightarrow U(2)/U(1) \times U(1) .$$

6. $O(2)$ and $O(3)$-bundles over $\boldsymbol{R}^3$. Let ω be a connection on an $O(2)$-bundle over $\boldsymbol{R}^3$, with curvature Ω.

THEOREM 6.1. *Suppose* $\Omega_0 \neq 0$. *Then there exists a neighborhood* V *of* 0 *such that* ω *is induced by a map*

$$f: V \longrightarrow O(3)/O(2) \times O(1) .$$

Proof. Let

$$\Omega(e_i, e_j) = \begin{pmatrix} 0 & \alpha_{ij} \\ -\alpha_{ij} & 0 \end{pmatrix}$$

where we have chosen our basis e_1, e_2, e_3 for $\boldsymbol{R}^3$ so that $\alpha_{12} \neq 0$. Let

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$a_2 = \begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix}$$
$$a_3 = \begin{pmatrix} -\alpha_{23}/\alpha_{12} \\ \alpha_{13} \end{pmatrix} .$$

Then we have

$$\Omega(e_i, e_j) = a_i \wedge a_j .$$

Let $Y_i \in T_0O(3)/O(1)$ have vertical part $\omega(e_i)$ and horizontal part a_i. Since a_1 and a_2 are linearly independent, we have a chain of ordinary integral elements:

$$\begin{aligned}(0, 0) < (e_1, Y_1) &< \{(e_1, Y_1), (e_2, Y_2)\} \\ &< \{(e_1, Y_1), (e_2, Y_2), (e_3, Y_3)\} .\end{aligned}$$

Thus the Cartan-Kähler theorem applies. □

Define $T: \boldsymbol{R}^3 \to o(3)$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix},$$

so

$$a \wedge b = T(a \times b) , \quad a, b \in \boldsymbol{R}^3$$

where $a \times b$ is the usual cross product on $\boldsymbol{R}^3$.

LEMMA 6.2. *Suppose* $b_1, b_2, b_3 \in \boldsymbol{R}^3$ *are linearly independent. Then there exist vectors* a_1, a_2, a_3 *in* $\boldsymbol{R}^3$ *such that*

$$\begin{cases} a_1 \times a_2 = b_3 \\ a_2 \times a_3 = b_1 \\ a_3 \times a_1 = b_2 \end{cases} \tag{6.3}$$

if and only if $\det(b_1, b_2, b_3) > 0$. *In this case, a_1, a_2, a_3 must be linearly independent.*

Proof. Suppose (6.3) holds. Let $u_1, u_2, u_3 \in \boldsymbol{R}^3$ be the unique (up to sign) unit vectors such that u_1 is perpendicular to b_2 and b_3, u_2 is perpendicular to b_3 and b_1, and u_3 is perpendicular to b_1 and b_2. Then there exist unique real numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ so that

$$\begin{aligned} a_1 &= \alpha_1 u_1 \qquad & b_1 &= \beta_1 u_2 \times u_3 \\ a_2 &= \alpha_2 u_2 \qquad & b_2 &= \beta_2 u_3 \times u_1 \\ a_3 &= \alpha_3 u_3 \qquad & b_3 &= \beta_3 u_1 \times u_2 \ . \end{aligned}$$

Equation (6.3) now reduces to

$$\begin{aligned} \alpha_1\alpha_2 &= \beta_3 \\ \alpha_2\alpha_3 &= \beta_1 \\ \alpha_3\alpha_1 &= \beta_2 \ . \end{aligned}$$

These equations imply

$$\begin{aligned} \beta_2\beta_3/\beta_1 &= \alpha_1^2 \\ \beta_3\beta_1/\beta_2 &= \alpha_2^2 \\ \beta_1\beta_2/\beta_3 &= \alpha_3^2 \ , \end{aligned}$$

which can be solved if and only if $\beta_1\beta_2\beta_3 > 0$. But this holds if and only if $\det(b_1, b_2, b_3) > 0$, since

$$\det(b_1, b_2, b_3) = \beta_1\beta_2\beta_3[\det(u_1, u_2, u_3)]^2 \ . \qquad \square$$

THEOREM 6.4. *Suppose that ω is a connection on an $O(3)$-bundle over $\boldsymbol{R}^3$, with curvature Ω, and that*

$$\gamma = \det(T^{-1}\Omega(e_2, e_3),\ T^{-1}\Omega(e_3, e_1),\ T^{-1}\Omega(e_1, e_2)) \neq 0 \ .$$

Then on a neighborhood V of 0 in $\boldsymbol{R}^3$, ω is induced from a map

$$f\colon V \longrightarrow O(5)/O(2) \ .$$

In order for there to exist a neighborhood W of 0 in $\boldsymbol{R}^3$ such that ω is induced by a map

$$g\colon W \longrightarrow O(4)/O(1) \ ,$$

it is necessary and sufficient that $\gamma > 0$.

Proof. To find g, it is necessary to find vectors a_1, a_2, a_3 in $\boldsymbol{R}^3$ such that

$$\Omega(e_i, e_j) = a_i \wedge a_j \ , \quad 1 \leqq i, j \leqq 3$$

or equivalently,

$$T^{-1}\Omega(e_i, e_j) = a_i \times a_j \ , \quad 1 \leqq i, j \leqq 3 \ .$$

By Lemma 6.2, this requires that γ be positive. If γ is positive, then a_1, a_2, a_3 exist, so we can apply Cartan-Kähler as before. The linear independence of a_1, a_2, a_3 guarantees that the resulting integral element be ordinary.

If γ is negative, we need to write the curvature at 0 as

$$\Omega(e_i, e_j) = a_i \wedge a_j + b_i \wedge b_j \ .$$

Let

$$\begin{aligned} c_1 &= T^{-1}\Omega(e_2, e_3) \\ c_2 &= T^{-1}\Omega(e_3, e_1) \\ c_3 &= T^{-1}\Omega(e_1, e_2) \ , \end{aligned}$$

so we want to solve

$$\begin{cases} c_1 = a_2 \times a_3 + b_2 \times b_3 \\ c_2 = a_3 \times a_1 + b_3 \times b_1 \\ c_3 = a_1 \times a_2 + b_1 \times b_2 \end{cases} \tag{6.5}$$

given that $\det(c_1, c_2, c_3) < 0$. Let $b_3 = 0$ and let a_3 be the unique (up to sign) unit vector perpendicular to c_1 and c_2. Let a_1, a_2 be vectors satisfying

$$\begin{aligned} c_1 &= a_2 \times a_3 \\ c_2 &= a_3 \times a_1 \ . \end{aligned}$$

By Lemma 6.2, $c_3 \neq a_1 \times a_2$, so choose b_1 and b_2 with

$$b_1 \times b_2 = c_3 - a_1 \times a_2 \ . \tag{6.6}$$

Thus we can solve (6.5).

I claim that we can arrange that

(i) a_1, b_1, b_2 are linearly independent;

(ii) if $a_2 = ra_1 + sb_1 + tb_2$ then $s + rt \neq 0$.

There is only one line perpendicular to c_2 and c_3, and there exists vector v along that line such that $c_2 = a_3 \times v$. Redefine a_1 as $v + a_3$, so $c_2 = a_3 \times a_1$. Since $a_1 \neq v$, a_1 is not perpendicular to c_3. From (6.6) and the fact that a_1 is perpendicular to $a_1 \times a_2$, it follows that a_1 is not perpendicular to $b_1 \times b_2$. We know that b_1 and b_2 are linearly independent because $b_1 \times b_2 \neq 0$. If a_1 were a linear combination of b_1 and b_2, then a_1 would be perpendicular to $b_1 \times b_2$, a contradiction. Part (i) of the claim follows.

Similarly, we can arrange to have a_2, b_1, b_2 linearly independent. Suppose that

$$a_2 = ra_1 + sb_1 + tb_2 \ .$$

Then r, s, and t are unique and $r \neq 0$. Part (ii) of the claim follows if $s + rt \neq 0$.

Otherwise, assume that $s + rt = 0$. Replace b_1 with $b_1' = b_1 + \varepsilon b_2$ for sufficiently small $\varepsilon > 0$. Then (6.5) is still satisfied, and so is part (i) of the claim. Then

$$a_2 = ra_1 + sb_1' + (t - \varepsilon)b_2$$

is the unique relation of its type. This completes the proof of part (ii) of the claim since

$$s + r(t - \varepsilon) = -r\varepsilon \neq 0 .$$

Define $Y_i \in T_0O(5)/O(2)$ to have vertical part $\omega(e_i)$ and horizontal part (a_i, b_i). The existence of f now follows once we show that

$$\{(e_1, Y_1), (e_2, Y_2), (e_3, Y_3)\}$$

is an ordinary integral element. It is, by definition, an integral element.

As before, the point $(0, o)$ is regular. To show that (e_1, Y_1) is regular, we must show that

$$\{(d, d') \in \boldsymbol{R}^6 \colon a_1 \times d + b_1 \times d' = 0\}$$

has the least dimension possible, namely three. This happens precisely when a_1 and b_1 are linearly independent. But we know this to be the case by part (i) of the claim.

Finally, we must show that

$$\{(e_1, Y_1), (e_2, Y_2)\}$$

is regular. This means that if

$$a_1 \times d + b_1 \times d' = 0 \tag{6.7a}$$

$$a_2 \times d + b_2 \times d' = 0 \tag{6.7b}$$

for some $d, d' \in \boldsymbol{R}^3$, then $d = d' = 0$.

Suppose that (d, d') is a nontrivial solution to (6.7). If $d' = 0$, then d would be a multiple of a_1 and of a_2, implying that a_1 and a_2 are linearly dependent. But then c_1 and c_2 would be linearly dependent, a contradiction. Thus $d' \neq 0$. Similarly, $d \neq 0$.

It follows from part (i) of the claim that there exists a unique line l through the (a_1, b_1)-plane and the (a_2, b_2)-plane. The general solution to (6.7a) is

$$\begin{aligned} d &= \alpha_1 a_1 + \gamma_1 b_1 \\ d' &= \beta_1 b_1 + \gamma_1 a_1 \end{aligned} \tag{6.8}$$

and the general solution to (6.7b) is:

$$\begin{aligned} d &= \alpha_2 a_2 + \gamma_2 b_2 \\ d' &= \beta_2 b_2 + \gamma_2 a_2 \, . \end{aligned} \tag{6.9}$$

Hence d and d' are along l, so for some nonzero δ, $d' = \delta d$. From (6.7),

$$\begin{aligned} (a_1 + \delta b_1) \times d &= 0 \\ (a_2 + \delta b_2) \times d &= 0 \, . \end{aligned}$$

Hence for some $\mu \in \boldsymbol{R}$,

$$a_2 + \delta b_2 = \mu(a_1 + \delta b_1) \, .$$

This gives

$$a_2 = \mu a_1 + \mu \delta b_1 - \delta b_2 \, ,$$

which contradicts part (ii) of the claim. Thus $\{(e_1, Y_1), (e_2, Y_2)\}$ is a regular integral element and the hypotheses of the Cartan-Kähler theorem are satisfied. $\square$

7. $O(2)$-bundles over $\boldsymbol{R}^4$. Let Φ_0 be the 2-form on $O(3)/O(1)$ such that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi_0$ is the curvature of the canonical connection on the $O(2)$-bundle

$$O(3)/O(1) \longrightarrow O(3)/O(2) \times O(1) \, .$$

Since $\dim O(3)/O(1) = 3$, $\Phi_0 \wedge \Phi_0 = 0$. Hence if $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Omega_0$ is the curvature of the connection on the $O(2)$-bundle induced by a map

$$f\colon \boldsymbol{R}^4 \longrightarrow O(3)/O(2) \times O(1)$$

then $\Omega_0 \wedge \Omega_0 = 0$.

THEOREM 7.1. *Suppose we are given a connection ω on an $O(2)$-bundle over $\boldsymbol{R}^4$ with curvature $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Omega_0$. Assume that $\Omega_0 \neq 0$. Then locally the connection is induced by a map*

$$f\colon \boldsymbol{R}^4 \longrightarrow O(3)/O(2) \times O(1)$$

if and only if $\Omega_0 \wedge \Omega_0 = 0$.

Proof. Necessity of the condition $\Omega_0 \wedge \Omega_0 = 0$ has already been shown. We now assume that $\Omega_0 \wedge \Omega_0 = 0$. Choose coordinates with $\Omega_0(e_1, e_2) \neq 0$. Let $\alpha_{ij} = \Omega_0(e_i, e_j)$ and

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix}, \quad a_3 = \begin{pmatrix} -\alpha_{23}/\alpha_{12} \\ \alpha_{13} \end{pmatrix}, \quad a_4 = \begin{pmatrix} -\alpha_{24}/\alpha_{12} \\ \alpha_{14} \end{pmatrix}.$$

Since

$$0 = \Omega_0 \wedge \Omega_0 = (\alpha_{12}\alpha_{34} + \alpha_{23}\alpha_{14} - \alpha_{13}\alpha_{24}) dv \circ 1 ,$$

it follows that

$$\Omega(e_i, e_j) = a_i \wedge a_j .$$

Let $Y_i \in TO(3)/O(1)$ have vertical part $\omega(e_i)$ and horizontal part a_i. Using $\alpha_{12} \neq 0$, it is easy to show that

$$\{(e_1, Y_1), (e_2, Y_2), (e_3, Y_3), (e_4, Y_4)\}$$

is an ordinary integral element, so we can apply the Cartan-Kähler theorem to complete the proof. □

THEOREM 7.2. *If $\Omega_0 \wedge \Omega_0 \neq 0$ then locally the connection is induced by a map*

$$f: \boldsymbol{R}^4 \longrightarrow O(4)/O(2) \times O(2) .$$

Proof. Let

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix}, \quad a_3 = \begin{pmatrix} -\alpha_{23}/\alpha_{12} \\ \alpha_{13} \end{pmatrix}, \quad a_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} \alpha_{14} \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} \alpha_{24} \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} \alpha_{23} \\ 0 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Then

$$\Omega(e_i, e_j) = a_i \wedge a_j + b_i \wedge b_j .$$

Let Y_i have vertical part $\omega(e_i)$ and horizontal part (a_i, b_i). We can thus apply the Cartan-Kähler theorem to the integral element

$$\{(e_1, Y_1), (e_2, Y_2), (e_3, Y_3), (e_4, Y_4)\}$$

once we show it is ordinary. The least trivial step is to show that the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \alpha_{14} \\ -\alpha_{12} & 0 & 0 & \alpha_{24} \\ -\alpha_{13} & -\alpha_{23}/\alpha_{12} & 0 & \alpha_{34} \end{bmatrix}$$

has maximal rank. But this is equivalent to

$$\alpha_{12}\alpha_{34} + \alpha_{14}\alpha_{23} - \alpha_{13}\alpha_{24} \neq 0$$

which is equivalent to the hypothesis

$$\Omega_0 \wedge \Omega_0 \neq 0 .$$ □

8. The quaternionic Grassmanian. We only discuss global results. In [2], Narasimhan and Ramanan show that the canonical connection on the cannonical bundle over the real and complex Grassman manifolds is universal. The following theorem shows that their argument extends to the quaternionic case.

THEOREM 8.1. *If*

$$N \geqq n(m+1)(4mn^2+2mn+1)$$

then every connection on a principal Sp(n)*-bundle over* M^m *is induced by some map*

$$f: M^m \longrightarrow \mathrm{Sp}(N)/\mathrm{Sp}(n) \times \mathrm{Sp}(N-n) .$$

Proof. As in [2], a device using partitions of unity reduces the theorem to the following lemma. □

LEMMA 8.2. *Let* α *be a one-form on* $B^m = \{x \in \boldsymbol{R}^m : |x| \leqq 1\}$ *with values in* sp(n). *Then ther exist n-by-n quaternionic matrix valued functions* $\phi_1, \cdots, \phi_{4mn^2+2mn+1}$ *on* B^m *such that*

$$\sum_l \phi_l^* \phi_l = I$$

and

$$\sum_l \phi_l^* d\phi_l = \alpha .$$

Proof. Let $\{f_r\}$ and $\{\tilde{f}_r\}$ be sets of complex n-by-n matrices such that:

(a) Each f_r and $\tilde{f}_r$ is positive definite.

(b) Each f_r and $\tilde{f}_r$ has norm one under the usual operator norm.

(c) $\{f_r\}$ is a basis over $\boldsymbol{R}$ for the real self-adjoint matrices.

(d) $\{f_r\}$ together with $\{\tilde{f}_r\}$ form a basis over $\boldsymbol{R}$ for the complex n-by-n self-adjoint matrices.

Let g_r (resp. $\tilde{g}_r$) be the unique positive square root of f_r (resp. $\tilde{f}_r$). We imbed $\boldsymbol{C}$ into $\boldsymbol{H}$ in the usual way, i.e., $\boldsymbol{C} = \boldsymbol{R} + i\boldsymbol{R}$ and $\boldsymbol{H} = \boldsymbol{R} + i\boldsymbol{R} + j\boldsymbol{R} + k\boldsymbol{R}$. α can be written in the form

$$\alpha = i \sum a_{rs} f_r dx_s + i \sum \tilde{a}_{rs} \tilde{f}_r dx_s + j \sum b_{rs} f_r dx_s + k \sum c_{rs} f_r dx_s$$

where the functions a_{rs}, $\tilde{a}_{rs}$, b_{rs}, c_{rs} are real. Let A be a constant larger than the absolute values of these functions. Let T be the square root of $Amn(2n+1)$.

One can check that

$$I - \frac{3}{n(2n+1)} \sum f_r - \frac{1}{n(2n+1)} \sum \tilde{f}_r$$

is a nonnegative self-adjoint matrix over $\boldsymbol{C}$, so we can let h be its nonnegative square root. Let $\{\phi_l\}$ be the functions:

$$
\begin{array}{ll}
\frac{1}{T}\sqrt{\frac{A+a_{rs}}{2}}e^{iTx_s}g_r\,, & \frac{1}{T}\sqrt{\frac{A-a_{rs}}{2}}e^{-iTx_s}g_r\,, \\
\frac{1}{T}\sqrt{\frac{A+\tilde{a}_{rs}}{2}}e^{iTx_s}\tilde{g}_r\,, & \frac{1}{T}\sqrt{\frac{A-\tilde{a}_{rs}}{2}}e^{-iTx_s}\tilde{g}_r\,, \\
\frac{1}{T}\sqrt{\frac{A+b_{rs}}{2}}e^{jTx_s}g_r\,, & \frac{1}{T}\sqrt{\frac{A-b_{rs}}{2}}e^{-jTx_s}g_r\,, \\
\frac{1}{T}\sqrt{\frac{A+c_{rs}}{2}}e^{kTx_s}g_r\,, & \frac{1}{T}\sqrt{\frac{A-c_{rs}}{2}}e^{-kTx_s}g_r,\ h\,.
\end{array}
$$

It is easily checked that these functions have the required propertise. □

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