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EVENLY DISTRIBUTED SUBSETS OF S^n AND A COMBINATORIAL APPLICATION

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A family $\mathscr{F}$ of nonempty subsets of the n-sphere S^n is said to be evenly distributed if every open hemisphere contains at least one set of $\mathscr{F}$. This paper first proves an antipodal theorem for evenly distributed families of nonempty closed subsets of S^n, and then applies it to improve a recent combinatorial result of Kneser-Lovász-Bárány.

For a positive integer n, let S^n denote the n-sphere $\{x \in \boldsymbol{R}^{n+1}: \|x\| = 1\}$ in the Euclidean $(n+1)$-space $\boldsymbol{R}^{n+1}$. For a subset A of S^n, $-A$ denotes the antipodal set of A: $-A = \{-x: x \in A\}$. For each $x \in S^n$, let $H(x)$ be the open hemisphere $H(x) = \{y \in S^n: (x, y) > 0\}$, where (x, y) is the inner product of x and y. Following Gale [5], we say that a family $\mathscr{F}$ of nonempty subsets of S^n is *evenly distributed*, if for every $x \in S^n$, the open hemisphere $H(x)$ contains at least one set of $\mathscr{F}$.

THEOREM 1. *Let n, m be two positive integers. Let $\mathscr{F}$ be an evenly distributed family of nonempty closed subsets of S^n. Let $\mathscr{F}$ be partitioned into m subfamilies $\mathscr{F} = \bigcup_{i=1}^m \mathscr{F}_i$ such that for each i and for any two subsets A', A'' in the same subfamily $\mathscr{F}_i$, $A' \cup (-A'')$ is not contained in any open hemisphere. Then m is necessarily $\geqq n+2$. Furthermore, there exist $n+2$ indices $1 \leqq \nu_1 < \nu_2 < \cdots < \nu_{n+2} \leqq m$ and $n+2$ sets $A_j \in \mathscr{F}_{\nu_j}$ $(1 \leqq j \leqq n+2)$ such that the union $\bigcup_{j=1}^{n+2} (-1)^j A_j$ is contained in an open hemisphere.*

Proof. For each $i = 1, 2, \cdots, m$, let G_i be the set of those points $x \in S^n$ for which the open hemisphere $H(x)$ contains at least one set of $\mathscr{F}_i$. Clearly G_i is open in S^n. As $\mathscr{F} = \bigcup_{i=1}^m \mathscr{F}_i$ is evenly distributed, we have $S^n = \bigcup_{i=1}^m G_i$. Furthermore, G_i contains no pair of antipodal points. In fact, $x \in G_i$ and $-x \in G_i$ would mean the existence of $A' \in \mathscr{F}_i$ and $A'' \in \mathscr{F}_i$ such that $A' \subset H(x)$ and $A'' \subset H(-x)$. Then we would have $A' \cup (-A'') \subset H(x)$, against our hypothesis.

The open covering $S^n = \bigcup_{i=1}^m G_i$ can be shrunken to a closed covering, i.e., we can find closed sets $F_i \subset G_i$ $(1 \leqq i \leqq m)$ such that $S^n = \bigcup_{i=1}^m F_i$. Then none of the F_i's contains a pair of antipodal points. By the classical antipodal theorem of Lusternik-Schnirelmann-Borsuk [2], [3], [8], m is necessarily $\geqq n+2$. Moreover, by a result in our paper [4], which asserts slightly more than the

Lusternik-Schnirelmann-Borsuk theorem, there exist $n+2$ indices $1 \leqq \nu_1 < \nu_2 < \cdots < \nu_{n+2} \leqq m$ such that $\bigcap_{j=1}^{n+2} (-1)^j F_{\nu_j} \neq \emptyset$. Then for any point z in this intersection, we have $-z \in \bigcap_{j \text{ odd}} F_{\nu_j} \subset \bigcap_{j \text{ odd}} G_{\nu_j}$ and $z \in \bigcap_{j \text{ even}} F_{\nu_j} \subset \bigcap_{j \text{ even}} G_{\nu_j}$. Hence there exist $n+2$ sets $A_j \in \mathscr{F}_{\nu_j}$ $(1 \leqq j \leqq n+2)$ such that $A_j \subset H(-z)$ for odd j, and $A_j \subset H(z)$ for even j. In other words, the union $\bigcup_{j=1}^{n+2} (-1)^j A_j$ is contained in the open hemisphere $H(z)$. This completes the proof.

As an application of Theorem 1, we have the following combinatorial result.

THEOREM 2. *Let k, n, m be three positive integers. Let E be a finite set with at least $2k+n$ elements, and let $\mathscr{F}$ denote the family of those subsets of E which have exactly k elements. If $\mathscr{F}$ is partitioned into m subfamilies $\mathscr{F} = \bigcup_{i=1}^{m} \mathscr{F}_i$ such that for each i, no two subsets in the same subfamily $\mathscr{F}_i$ are disjoint, then $m \geqq n+2$. Furthermore, there exist $n+2$ indices $1 \leqq \nu_1 < \nu_2 < \cdots < \nu_{n+2} \leqq m$ and $n+2$ sets $A_j \in \mathscr{F}_{\nu_j}$ $(1 \leqq j \leqq n+2)$ such that the union $\bigcup_{j \text{ odd}} A_j$ is disjoint from the union $\bigcup_{j \text{ even}} A_j$.*

Proof. According to a theorem of Gale [5], there exist $2k+n$ points on S^n such that every open hemisphere contains at least k of these points. As E has at least $2k+n$ elements, E can be regarded as a subset of S^n such that the family $\mathscr{F}$ (of all subsets of E with k elements) is evenly distributed. For each i and for any two subsets A', A'' in the same subfamily $\mathscr{F}_i$, we have $A' \cap A'' \neq \emptyset$ and therefore $A' \cup (-A'')$ is not contained in any open hemisphere. By Theorem 1, m is necessarily $\geqq n+2$. Furthermore, there exist $n+2$ indices $1 \leqq \nu_1 < \nu_2 < \cdots < \nu_{n+2} \leqq m$ and $n+2$ sets $A_j \in \mathscr{F}_{\nu_j}$ $(1 \leqq j \leqq n+2)$ such that $\bigcup_{j=1}^{n+2} (-1)^j A_j$ is contained in an open hemisphere $H(z)$. Then $\bigcup_{j \text{ odd}} A_j$ and $\bigcup_{j \text{ even}} A_j$ are contained in $H(-z)$ and $H(z)$ respectively, and therefore are disjoint.

Obviously Theorem 2 can be interpreted as a result on coloring (with m colors) of the $(k-1)$-dimensional faces of a simplex of dimension $\geqq 2k+n-1$ such that no two $(k-1)$-dimensional faces of the same color are disjoint.

The partial conclusion $m \geqq n+2$ in Theorem 2 was conjectured by Kneser [6] in 1955, and proved recently by Lovász [7] and Bárány [1]. In proving $m \geqq n+2$, both these authors use the Lusternik-Schnirelmann-Borsuk theorem. Bárány's proof depends also on Gale's theorem.

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