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ISOPERIMETRIC EIGENVALUE PROBLEM OF EVEN ORDER DIFFERENTIAL EQUATIONS

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This paper is concerned with the following eigenvalue problem

$$(1)\qquad \begin{cases} x^{(2n)} + (-1)^{n+1}\lambda p(t)x = 0 \\ x^{(2k)}(0) = 0 = x^{(2k)}(1)\,, \quad k = 0, 1, \cdots, n-1\,, \end{cases}$$

where $p(t)$ is assumed to be positive and continuous in $[0, 1]$. For the class of functions $q(t)$ which are equimeasurable to $p(t)$, we shall show that the rearrangement of $p(t)$ in symmetrically increasing order maximizes the least positive eigenvalue of (1), while the rearrangement of $p(t)$ in symmetrically decreasing order minimizes it.

Rearrangements of sets of numbers and functions are defined and investigated in detail in the book by Hardy, Littlewood and Pólya [11, Chapter X] and the book by Pólya and Szegö [18]. Using these notions, classes of nonhomogeneous strings, membranes, rods and plates with equimeasurable densities are considered in [3, 4, 5, 10] and the extremum of the principal frequencies are found for these classes. In particular, the above assertion has been proven by Beesack and Schwarz [5] and Fink [10] for $n = 1$. For $n = 2$, the proof is given by Banks [3]. Our proof will differ from those given for the special cases in that we will rely on some of the results in the theory of positive operators [12, 13, 14, 15, 16, 17] and certain rearrangement inequalities [18, 19]. All the required results will be explicitly stated in the sequel; the explanations of which, however, will be brief.

2. Rearrangement inequalities. Let h be a real function defined on a subset S of R^n, we shall denote the level set

$$\{t \in S : h(t) \geqq c\}$$

by $L(h, c)$. Two real functions $f(t)$ and $g(t)$ defined on $[0, 1]$ are called similarly ordered if, for each pair of points t_1, t_2 of $[0, 1]$, we have

$$[f(t_1) - f(t_2)][g(t_1) - g(t_2)] \geqq 0\ ;$$

f and g are called oppositely ordered if f and $-g$ are similarly ordered. If for each $c \in R$, the measure of $L(f, c)$ is equal to that of $L(g, c)$, then we say that f and g are equimeasurable. Let f, $\check{f}$ and $\hat{f}$ be equimeasurable, and in addition let $\check{f}(t)$ and $(2t-1)^2$ be similarly ordered, and $\hat{f}(t)$ and $(2t-1)^2$ be oppositely ordered. The uniquely defined and continuous functions $\check{f}(t)$ and $\hat{f}(t)$ are called the rearrangement of $f(t)$ in symmetrically increasing, respectively decreasing order (for detail of these statements and their validity, see [11, Chapter X]).

LEMMA 1. ([11, *Theorem* 378 *and* 18, p. 153]). *Suppose f, f_1, f_2, g, g_1 and g_2 are real continuous functions defined on* [0, 1], *f_1 and g_1 are similarly ordered, f_2 and g_2 are oppositely orcered, f, f_1 and f_2 are equimeasurable, and also g, g_1 and g_2 are equimeasurable, then*

$$\int_0^1 f_2 g_2 \leqq \int_0^1 fg \leqq \int_0^1 f_1 g_1 .$$

Call a real function h defined on a convex subset S of R^n *quasiconcave* if each of its level sets $L(h, c)$ is convex [2, p. 145]. The following is a slightly modified version of a result of Vollman [19, Theorem 2.1].

LEMMA 2. *Let $K(t, s)$ be a continuous, nonnegative, quasiconcave function defined on* $[0, 1] \times [0, 1]$ *which satisfies $K(t, s) = K(1-t, 1-s)$. Let p, q be nonnegative, continuous functions defined on* [0, 1] *with $\hat{p}$, $\hat{q}$ their rearrangements in symmetrically decreasing order. Then*

$$\int_0^1\int_0^1 K(t, s)p(s)q(s)q(t)dsdt \leqq \int_0^1\int_0^1 K(t, s)\hat{p}(s)\hat{q}(s)\hat{q}(t)dsdt .$$

We remark that under the same assumptions in Lemma 2, the original version only asserts that

$$\int_0^1\int_0^1 K(t, s)p(s)q(t)dsdt \leqq \int_0^1\int_0^1 K(t, s)\hat{p}(s)\hat{p}(t)dsdt .$$

We can, however, first strengthen the conclusion of Lemma 2.4 in [19] to

$$\int_{L_c(K)} p(x)q(x)q(t)dA \leqq \int_{L_c(K)} \hat{p}(x)\hat{q}(x)\hat{q}(t)dA ,$$

and then prove Lemma 2 in a way similar to the one used in the proof of the original version. Since the modifications are slight, the proof is thus omitted.

3. Positive operators. Let B be a real Banach space. A closed subset K of B is a cone if the following conditions are satisfied:

(i) If $x \in K$ and $y \in K$, then $x + y \in K$.

(ii) If $x \in K$ and $t \geqq 0$, then $tx \in K$.

(iii) If $x \in K$ and $x \neq 0$, then $-x \notin K$.

A cone is said to be solid if it contains interior elements. An operator T defined on B is said to be positive (with respec to K) if it leaves the cone K invariant and u_0-positive if nonzero u_0 exists in K so that for every nonzero u in K, positive numbers s, t and positive integer p can be found satisfying $su_0 \leqq T^p u \leqq tu_0$ where we write $x \leqq y$ if $y-x \in K$ and we write $x \prec y$ if $y-x \in K$ and $y-x \neq 0$.

LEMMA 3. ([13, 14, 15, 16]). *Let T be a linear, u_0-positive and completely continuous operator defined on a real Banach space B with solid cone K. Then T has exactly one (normalized) eigenvector in K and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Let B' denote the dual space of continuous linear functionals on B, and let K' denote the dual cone of all elements of B' that are nonnegative on K, i.e.,

$$K' = \{x' \in B' \colon \langle x, x' \rangle \geqq 0 \text{ for all } x \in K\} ,$$

where $\langle x, x' \rangle$ denotes the number $x'(x)$. If T is a linear operator defined on B, we shall denote its special radius by $r(T)$, i.e.,

$$r(T) = \sup \{|\lambda| \colon \lambda \in \sigma(T)\} .$$

LEMMA 4. ([17, *Lemma* 3.3]). *Let T be a linear, positive and completely continuous operator defined on a real Banach space B with cone K. For $x \neq 0$, let*

$$S = \{\lambda \in R \colon \lambda\langle x, x' \rangle \leqq \langle Tx, x' \rangle, x' \in K'\} . \tag{2}$$

Let

$$r_x(T) = \begin{cases} \sup S & \text{if } S \neq \varnothing \\ -\infty & \text{if } S = \varnothing . \end{cases} \tag{3}$$

Then $r_x(T) \leqq r(T)$.

The set of $2n$-times continuously differentiable real functions $C^{(2n)}[0, 1]$ equipped with the norm

$$\|f\| = \max_{1 \leqq j \leqq 2n} \{\sup_{0 \leqq t \leqq 1} |f^{(j)}(t)|\}$$

is a Banach space. In the sequel, we shall denote the subset

$$\{f \in C^{(2n)}[0, 1] \colon f^{(2k)}(0) = 0 = f^{(2k)}(1) \text{ for } k = 0, 1, \cdots, n-1 , \\ \text{and } (-1)^k f^{(2k)}(t) \geqq 0 \text{ for } 0 \leqq k \leqq n-1 \text{ and } 0 \leqq t \leqq 1\}$$

of $C^{(2n)}[0, 1]$ by K_n. K_n is a solid cone of $C^{(2n)}[0, 1]$ as may be verified directly.

4. The Green's functions associated with (1). Let the function $G_1(t, s)$ and its successive iterates be defined as follows

$$G_1(t, s) = \begin{cases} t(1-s) & \text{if } 0 \leqq t \leqq s \\ s(1-t) & \text{if } s \leqq t \leqq 1 , \end{cases} \tag{4}$$

$$G_n(t, s) = \int_0^1 G_1(t, r)G_{n-1}(r, s)dr \quad (n = 2, 3, \cdots) . \tag{5}$$

If $g(t)$ is any function continuous in the interval [0, 1], then it is easily verified that the unique solution of the differential system

$$\begin{aligned} &(-1)^n x^{(2n)}(t) = g(t) \\ &\quad x^{(2k)}(0) = 0 = x^{(2k)}(1) , \qquad k = 0, 1, \cdots, n-1 \end{aligned}$$

is

$$x(t) = \int_0^1 G_n(t, s)g(s)ds .$$

In fact $G_n(t, s)$ is the familiar Green's function of the system. Consequently, system (1) can be transformed into an integral equation of the form

$$\lambda T_n x = x . \tag{6}$$

Where $T_n: C^{(2n)}[0, 1] \to C^{(2n)}[0, 1]$ is defined by

$$(T_n x) = \int_0^1 G_n(t, s)p(s)x(s)ds . \tag{7}$$

T_n is clearly linear, furthermore, since $G_n(t, s)$ and $p(s)$ are continuous, T_n is also compact.

LEMMA 5. *For each positive integer m, $G_m(t, s)$ is positive in the interior of* $[0, 1] \times [0, 1]$ *and zero on the boundary.*

LEMMA 6. $G_n(t, s) = G_n(s, t) = G_n(1-s, 1-t) = G_n(1-t, 1-s)$, $G_n(1-t, s) = G_n(1-s, t)$ *and*

$$\int_0^1 G_1(t, s)ds = t(1-t)/2 .$$

LEMMA 7. *Let y be a continuous, nonnegative function which does not vanish identically in* [0, 1], *then positive α can be found such that for $t \in [0, 1]$*

$$\alpha t(1-t) \leqq \int_0^1 G_1(t, s)y(s)ds \ . \tag{8}$$

Lemma 5 follows directly from the definition of $G_m(t, s)$. Lemma 7 is a result in [14, p. 283]. Lemma 6 is a result of Cheng [6, Corollary 4.6] which also follows from direct verification. Note that Lemma 6 implies that $G_n(t, s)$ takes on the same value at the corners of any parallelogram lying in the square $[0, 1] \times [0, 1]$ and having sides parallelled to the diagonals of $[0, 1] \times [0, 1]$.

LEMMA 8. *$G_n(t, s)$ is quasiconcave on $[0, 1] \times [0, 1]$.*

Proof. We start by defining a sequence of polynomials $f_1, f_2, f_3, \cdots$ by means of the conditions

$$\begin{aligned} &f_1(x) = x/2 \\ &f_n'(x) = f_{n-1}(x) && n > 1 \\ &f_{2n-1}(-1) = 0 && n > 1 \\ &f_{2n}(x) = f_{2n}(-x) && n \geqq 1 \ . \end{aligned}$$

Denote the points $(-1, -1)$, $(0, 0)$, $(1, -1)$ and $(0, -2)$ by A, B, C and D respectively. Let $H_n(u, v)$ be the function

$$H_n(u, v) = \begin{cases} (-1)^n[f_{2n}(u) - f_{2n}(v)] & \text{if } (u, v) \in \Delta ABC \\ (-1)^n[f_{2n}(u) - f_{2n}(-v-2)] & \text{if } (u, v) \in \Delta ADC \ . \end{cases}$$

Under the change of variables

$$\begin{aligned} t &= (u - v)/2 \ , \qquad & s &= (u + v + 2)/2 \\ u &= t + s - 1 \ , \qquad & v &= s - t - 1 \end{aligned}$$

it is easily seen that the square with vertices A, B, C and D is transformed into $[0, 1] \times [0, 1]$. We assert that

$$G_n(t, s) = H_n(t + s - 1, s - t - 1) \ , \qquad (t, s) \in [0, 1] \times [0, 1] \ .$$

Indeed, if we set $G_n'(t, s) = H_n(t + s - 1, s - t - 1)$, we may verify directly that $G_n'(t, s)$, when regarded as a function of t with s fixed, satisfies the following conditions:

(i) Together with its first $2n - 2$ derivatives, it is continuous on $[0, 1]$. At the point $t = s$, the $(2n - 1)$th derivative has an upward jump $(-1)^n$.

(ii) Its $2n$th derivative is identically zero.

(iii) It satisfies the boundary conditions in (1).

Since the Green's function is the only function with the above properties $G_n(t, s) = G_n'(t, s)$.

Since for each $m \leqq n$, $G_m(t, s) > 0$ in the interior of $[0, 1] \times [0, 1]$, it is clear that $H_m(u, v) > 0$ for $-1 \leqq v < u \leqq 0$. Hence, $(-1)^m(f_{2m}(u) - f_{2m}(v)) > 0$ for $-1 \leqq v < u \leqq 0$, that is, $(-1)^m f_{2m}$ is strictly increasing in $[-1, 0]$. Since $f_{2m-1}(-1) = 0$, $(-1)^m f'_{2m} = (-1)^m f_{2m-1} > 0$ and $(-1)^m f''_{2m-1} = (-1)^m f_{2m-3} = (-1)(-1)^{m-1} f_{2m-3} < 0$ over $(-1, 0]$. We therefore conclude that $(-1)^m f_{2m-1}$ is positive and concave over $(-1, 0]$.

To show that for every $c > 0$, $L(G_n, c)$ is a convex set, it is sufficient to show that $L(H_n, c)$ is bounded on one side of the line $v = -1$ by a concave curve, and on the other side by a convex curve. But in view of Lemma 6 (and the statements following Lemma 7), it suffices to show that the part of $L(H_n, c)$ contained in the triangle $-1 \leqq v < u \leqq 0$ is bounded by a concave curve. For this purpose, we implicitly differentiate $H_n(u, v) = c$ to obtain [8, p. 223]

$$\frac{dv}{du} = -\frac{(-1)^n f_{2n}(u)}{(-1)^{n+1} f'_{2n}(v)} = \frac{f_{2n-1}(u)}{f_{2n-1}(v)} \neq 0$$

and

$$\begin{aligned} \frac{d^2v}{du^2} &= -\frac{[f'_{2n}(v)]^2(-1)^n f''_{2n}(u) + [f'_{2n}(u)]^2(-1)^{n+1} f''_{2n}(v)}{(-1)^{n+1}[f'_{2n}(v)]^3} \\ &= \frac{[f_{2n}(u)]^2}{f'_{2n}(v)}\left\{\frac{f''_{2n}(u)}{[f'_{2n}(u)]^2} - \frac{f''_{2n}(v)}{[f'_{2n}(v)]^2}\right\} \\ &= \frac{[f_{2n-1}(u)]^2}{f_{2n-1}(v)}\left[\frac{1}{f_{2n-1}(v)} - \frac{1}{f_{2n-1}(u)}\right]' \end{aligned}$$

for $-1 < v < u \leqq 0$. But since $(-1)^n f_{2n-1}$ is positive and concave over $(-1, 0]$, thus $1/(-1)^n f_{2n-1}$ is convex over $(-1, 0]$ (see [2, p. 156]), so that $(1/(-1)^n f_{2n-1})'$ is increasing in $(-1, 0]$. Consequently,

$$\frac{[f_{2n-1}(u)]^2}{(-1)^n f_{2n-1}(v)}\left[\frac{1}{(-1)^n f_{2n-1}(v)} - \frac{1}{(-1)^n f_{2n-1}(u)}\right]' \leqq 0$$

for $-1 < v < u \leqq 0$. This shows that $d^2v/du^2 \leqq 0$ for $-1 < u \leqq 0$ so that the part of $L(H_n, c)$ contained in the triangle $-1 \leqq v < u \leqq 0$ is indeed bounded above by a concave curve. The proof is complete.

5. Existence of eigenvalues. It is known (see for instance [7, pp. 228–230, and 9, 1]) that the selfadjoint and positive definite eigenvalue problem (1) has a smallest positive eigenvalue which is simple and the corresponding eigenfunctions have no zeros in $(0, 1)$. Here, we shall give an alternate proof which also shows that the corresponding eigenfunctions belong to K_n. For this purpose, we first show that the operator T_n defined in the last section is u_0-positive with respect to K_n.

Let x be an arbitrary nonzero element of K_n. Recall that for each positive integer m, $T_m x$ is the unique solution of

$$(-1)^m y^{(2m)} = px$$
$$y^{(2k)}(0) = 0 = y^{(2k)}(1)\,, \qquad k = 0, 1, \cdots, m-1\,.$$

In view of this and (7),

$$(T_m x)'' = -T_{m-1}x \qquad \text{if} \quad m > 1\,; \tag{9}$$

furthermore, by Lemma 5, $T_m x \in K_m$ for each $m \leqq n$. Let

$$u_0 = T_{n-1}u^*\,, \tag{10}$$

where $u^*(t) = t(1-t)$. Since $u^* \in K_j$ for any $j \geqq 1$, $u_0 \in K_m$ for any $m \geqq 1$, and in particular, $u_0 \in K_n$. We assert that positive numbers α and β can be found such that

$$\alpha u_0 \leqq T_n x \leqq \beta u_0\,. \tag{11}$$

First recall from Lemma 7 that positive number α can be found such that

$$\alpha u^*(t) \leqq (T_1 x)(t)\,, \qquad 0 \leqq t \leqq 1\,.$$

Thus

$$\alpha u^*(t) \leqq (T_1 x)(t) \leqq \beta u^*(t)\,, \qquad 0 \leqq t \leqq 1$$

where $\beta = \max\{p(t)x(t): 0 \leqq t \leqq 1\}$. Consequently, by (9) and induction

$$\begin{aligned}(-1)^{n-1}(T_n x - \alpha u_0)^{(2n-2)}(t) &= (T_1 x - \alpha u^*)(t) \geqq 0\\ &\ \vdots\\ (-1)(T_n x - \alpha u_0)''(t) &= (T_{n-1}x - \alpha T_{n-2}u^*)(t) \geqq 0\end{aligned}$$

for $0 \leqq t \leqq 1$. In other words, we have shown that $T_n x - \alpha u_0 \in K_n$. Similarly, we can show that $\beta u_0 - T_n x \in K_n$.

We conclude that T_n is u_0-positive so that according to Lemma 3, T_n has exactly one (normalized) eigenvector in K_n and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue. In view of (6), we have thus shown the following

THEOREM 1. *The eigenvalue problem* (1) *has exactly one (normalized) eigenvector in* K_n *and the corresponding eigenvalue is simple, positive, and smaller than the absolute value of any other eigenvalue.*

In the sequel, we shall denote the smallest eigenvalue of (1) by $\lambda(p)$.

COROLLARY 1. *Let $x(t)$ be an eigenfunction of* (1) *corresponding to $\lambda(p)$, then $x(t) \neq 0$ for any $t \in [0, 1]$.*

Proof. Since $\lambda(p)$ is simple, we may assume $x(t) \geqq 0$ for $0 \leqq t \leqq 1$. If $n = 1$, then

$$x'' = -px \leqq 0$$

on $[0, 1]$. If $n > 1$, then by Theorem 1, $x'' \leqq 0$ on $[0, 1]$ also. Thus x is a nonnegative and concave function. Since $x(0) = 0 = x(1)$, $x(t)$ cannot vanish in $(0, 1)$.

COROLLARY 2. *If $p(t)$ in symmetric in* $[0, 1]$ (i.e., $p(t) = p(1 - t)$ *for* $t \in [0, 1]$), *and if $x(t)$ is an eigenfunction corresponding to $\lambda(p)$, then $x(t) = x(1 - t)$ for $t \in [0, 1]$.*

Proof. We may verify by direct substitution into (1) that $x(1 - t)$ is also an eigenfunction corresponding to $\lambda(p)$. Consequently, $x(t) = \alpha x(1 - t)$ for some nonzero number α. But since $x(1/2) \neq 0$, thus $\alpha = 1$ as required.

COROLLARY 3. *The spectral radius $r(T_n)$ is equal to $\lambda^{-1}(p)$.*

6. Isoperimetric inequalities. In this section, we shall prove the following result as asserted in §1.

THEOREM 2. *Let $p(t)$ be a positive and continuous function defined on* $[0, 1]$, *and let $\check{p}(t)$ and $\hat{p}(t)$ be respectively the rearrangements of $p(t)$ in symmetrically increasing and decreasing order. Consider the three eigenvalue problems* (1) *and*

$$\begin{aligned} &u^{(2n)} + (1-)^n \check{p}(t) u = 0 \\ &u^{(2k)}(0) = 0 = u^{(2k)}(1)\,, \qquad k = 0, 1, \cdots, n-1 \end{aligned} \tag{12}$$

$$\begin{aligned} &v^{(2n)} + (-1)^n \hat{p}(t) v = 0 \\ &v^{(2k)}(0) = 0 = v^{(2k)}(1)\,, \qquad k = 0, 1, \cdots, n-1\,. \end{aligned} \tag{13}$$

Denote their least positive eigenvalues by $\lambda(p)$, $\lambda(\check{p})$ and $\lambda(\hat{p})$ respectively. Then

$$\lambda(\hat{p}) \leqq \lambda(p) \leqq \lambda(\check{p})\,.$$

We first show that $\lambda(p) \leqq \lambda(\check{p})$. We recall that [7, p. 239 and

1] the least positive eigenvalue of (1) is equal to

$$\min\left\{\int_0^1[x^{(n)}]^2\Big/\int_0^1 px^2\right\}$$

where the minimum is taken over functions $x \in C^{(2n)}[0, 1]$ that satisfy the boundary conditions in (1) and for which the denominator is positive. Furthermore, no function other than the corresponding eigenfunction yields the minimum.

Let $u(t)$ be a nonnegative eigenfunction of (12) corresponding to $\lambda(\check{p})$. Since $\check{p}(t) = \check{p}(1-t)$ for $t \in [0, 1]$, by Corollaries 1 and 2, $u(t)$ is symmetric in $[0, 1]$, positive for $0 < t < 1$ and concave on $[0, 1]$. Consequently, $u^2(t)$ is together with $u(t)$, symmetrically decreasing so that $\check{p}(t)$ and $u^2(t)$ are oppositely ordered. But then by Lemma 1,

$$\begin{aligned}\lambda(\check{p}) &= \int_0^1[u^{(n)}]^2\Big/\int_0^1 \check{p}u^2 \geqq \int_0^1[u^{(n)}]^2\Big/\int_0^1 pu^2\\ &\geqq \min\left\{\int_0^1[x^{(n)}]^2\Big/\int_0^1 px^2\right\} = \lambda(p)\end{aligned}$$

as required.

Next we show that $\lambda(\hat{p}) \leqq \lambda(p)$. For this purpose, we need the following

THEOREM 3. *The least positive eigenvalue of* (1) *satisfies*

$$\lambda^{-1}(p) = \max \frac{\int_0^1\int_0^1 G_n(t, s)p(s)u(s)u(t)dsdt}{\int_0^1 u^2(s)ds}$$

where the maximum is taken over nonzero elements in K_n. Furthermore, the unique function, except for a constant multiple, which yields the maximum is the eigenfunction corresponding to $\lambda(p)$.

Proof. According to Lemma 4 and Corollary 3, for any nonzero x in $C^{(2n)}[0, 1]$,

$$r_x(T_n) \leqq r(T) = \lambda^{-1}(p)\ ,$$

so that

$$\sup_{\substack{x \in K_n\\ x\neq 0}} r_x(T_n) \leqq \lambda^{-1}(p)\ .$$

Now for each nonzero u in K_n, define the positive linear functional $u' \in K_n'$ by

$$\langle x, u'\rangle = \int_0^1 x(s)u(s)ds$$

for all $x \in K_n$. Then for each $x \in K_n$, we have that

$$\sup\{\lambda \in R: \lambda\langle x, u'\rangle \leqq \langle T_n x, u'\rangle\} \leqq r_x(T_n)$$

and consequently, that

$$\sup\{\lambda \in R: \lambda\langle u, u'\rangle \leqq \langle T_n u, u'\rangle\} \leqq r_u(T_n) \leqq \lambda^{-1}(p)\ ,$$

and

$$\sup_{\substack{u \in K_n \\ u \neq 0}} \frac{\int_0^1 (T_n u)(s)u(s)ds}{\int_0^1 u^2(s)ds} \leqq r_u(T_n) \leqq \lambda^{-1}(p)\ .$$

Since we have equality when u is equal to a constant multiple of the eigenfunction corresponding to $\lambda(p)$, the first part of the theorem is proven.

To prove the remainder of the theorem, let $v \in K_n$ be such that $\lambda^{-1}(p) = \langle T_n v, v\rangle/\langle v, v\rangle$. Then

$$\frac{\langle T_n v, v\rangle}{\langle v, v\rangle} \leqq r_v(T_n) \leqq \lambda^{-1}(p) = \frac{\langle T_n v, v\rangle}{\langle v, v\rangle}$$

shows that $r_v(T_n) = \langle T_n v, v\rangle/\langle v, v\rangle$. It follows that $\langle T_n v - r_v(T_n)v, x'\rangle \geqq 0$ for all $x' \in K_n'$, and consequently, by the Krein-Rutman theorem [15, Theorem 1.1], that $T_n v - r_v(T_n)v \in K_n$. We assert that v is an eigenfunction corresponding to $\lambda(p)$. If not, there would exist a positive number α and a positive integer m such that

$$T_n^m(T_n v - r_v(T_n)v) = T_n(T_n^m v) - r_v(T_n)(T_n^m v) \succ \alpha u_0$$

where u_0 is given by (10). Let $z = T_n^m v$. Since $z \in K_n$, there exists a positive number β (as can be seen from (11)) such that $z \succ \beta u_0$. Hence, for sufficiently small $\varepsilon > 0$,

$$T_n z - r_v(T_n)z - \varepsilon z \succ (\alpha - \varepsilon\beta)u_0$$

where $(\alpha - \varepsilon\beta) > 0$. Consequently,

$$\frac{\langle T_n z, x'\rangle}{\langle z, x'\rangle} \geqq r_z(T_n) \geqq r_z(T_n) + \varepsilon\ ,$$

which contradicts the fact that $r_z(T_n) \leqq \lambda^{-1}(p) = r_v(T_n)$. The proof is complete.

We remark that the proof given above is similar to that of Theorem 3.1 in [12]. However we feel that there are enough differences to include it here.

Now let u be the normalized eigenfunction corresponding to $\lambda(p)$. Then

$$\lambda^{-1}(p) = \frac{\int_0^1\int_0^1 G_n(t, s)p(s)u(s)u(t)dsdt}{\int_0^1 u^2(s)ds} .$$

Let $\hat{u}$ be the rearrangement of u in symmetrically decreasing order, then by Lemmas 2 and 8,

$$\int_0^1\int_0^1 G_n(t, s)p(s)u(s)u(t)dsdt \leqq \int_0^1\int_0^1 G_n(t, s)\hat{p}(s)\hat{u}(s)\hat{u}(t)dsdt .$$

Thus

$$\begin{aligned}\lambda^{-1}(p) &\leqq \frac{\int_0^1\int_0^1 G_n(t, s)\hat{p}(s)\hat{u}(s)\hat{u}(t)dsdt}{\int_0^1 \hat{u}^2(s)ds} \\ &\leqq \max_{\substack{v\in K_n \\ v\neq 0^n}} \frac{\int_0^1\int_0^1 G_n(t, s)\hat{p}(s)v(s)v(t)dsdt}{\int_0^1 v^2(s)ds} \\ &= \lambda^{-1}(\hat{p}) .\end{aligned}$$

Consequently, $\lambda(\hat{p}) \leqq \lambda(p)$ as required. The proof of Theorem 2 is complete.

7. **Conclusion remarks.** We remark that in Theorem 2, $\lambda(\hat{p}) = \lambda(p)$ only if $p \equiv \hat{p}$. Indeed, if $\lambda(\hat{p}) = \lambda(p)$, then by Theorem 3, an eigenfunction u corresponding to $\lambda(\hat{p})$ is also an eigenfunction corresponding to $\lambda(p)$. Substitute u into (1) and (12) respectively, we see that

$$u^{(2n)} + (-1)^n p(t)u = u^{(2n)} + (-1)^n \hat{p}(t)u$$

for $0 < t < 1$. Consequently, $p(t) = \hat{p}(t)$ for $0 < t < 1$ and by continuity $p(t) = \hat{p}(t)$ for $0 \leqq t \leqq 1$. Similarly, we can also show that $\lambda(\check{p}) = \lambda(p)$ only if $p \equiv \check{p}$.

We have mentioned that Beesack and Schwarz [5] and Banks [3] proved $\lambda(\hat{p}) \leqq \lambda(p)$ for $n = 1$ and 2 respectively. However, a close examination of their proofs reveals the fact that in order to establish by similar arguments the more general result, we shall run into the difficulty in constructing from a nonnegative function u (satisfying the boundary conditions in (1)) two functions $\hat{u}$ and v, where $\hat{u}$ is the rearrangement of u in symmetrically decreasing order and v is symmetric in [0, 1] such that

$$\int_0^1 u^{(n)} = \int_0^1 v^{(n)}$$

and $\hat{u}(t) \leqq v(t)$ for $0 \leqq t \leqq 1$. This difficulty we have avoided by employing an extremal characterization (which is essentially a minimax principle) of $\lambda^{-1}(p)$ and a rearrangement inequality. In view of the fact that a large body of minimax principles exists for positive operators [12, 17], our approach indicates that other isoperimetric eigenvalue problems (e.g., fixed end-points problems [3]) can similarly be solved, provided, of course, that Vollman's inequality can be applied. Moreover, since the rearrangement inequality of Vollman clearly depends on the quasiconcavity of the kernel $K(t, s)$, our approach also indicates a close connection between the quasiconcavity of Green's function and the optimality of eigenvalues depending on equimeasurable densities.

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