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REMARKS ON NONLINEAR CONTRACTIONS

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Throughout this paper, we assume that K is strongly normal, that $P = \{d(x, y); x, y \in X\}$, that $\bar{P}$ denotes the weak closure of P, and that $P_1 = \{z; z \in \bar{P}$ and $z \neq \mathcal{O}\}$. The main result of this paper is the following.

Let (X, d) be a nonempty K-complete metric space, and let S, T be mappings of X into itself satisfying (1) and (2).

$$(1) \qquad \phi(d(Sx, Ty)) \leqq d(x, y)\,, \quad x \neq y \in X\,,$$

$$(2) \qquad \phi(t) > t \quad \textbf{for any} \quad t \in P_1\,,$$

where $\phi: P_1 \to K$ is lower semicontinuous on P_1.

Then exactly one of the following three statements holds:

(a) S and T have a common fixed point, which is the only periodic point for both S and T;

(b) There exist a point $x_0 \in X$ and an integer $p > 1$ such that $Sx_0 = x_0 = T^p x_0$ and $Tx_0 \neq x_0$;

(c) There exist a point $y_0 \in X$ and an integer $q > 1$ such that $S^q y_0 = y_0 = Ty_0$ and $Sy_0 \neq y_0$.

Recently, J. Eisenfeld and V. Lakshmikantham [6, 7, 8], J. C. Bolen and B. B. Williams [1], S. Heikkila and S. Seikkala [9, 10], K. J. Chung [3, 4], M. Kwapisz [12] J. Wazewski [16] proved some fixed point theorems in abstract cones which extend and generalize many known results. In this paper, we extend some main results of A. Meir and E. Keeler [14] and C. L. Yen and K. J. Chung [17] to cone-valued metric spaces.

(I). **Definitions.** Let E be a normed space. A set $K \subset E$ is said to be a cone if (i) K is closed (ii) if $u, v \in K$ then $\alpha u + \tau v \in K$ for all $\alpha, \tau \geqq 0$, (iii) $K \cap (-K) = \{\mathcal{O}\}$ where $\mathcal{O}$ is the zero of the space E, and (iv) $K^0 \neq \phi$ where K^0 is the interior of K. We say $u \geqq v$ if and only if $u - v \in K$, and $u > v$ if and only if $u - v \in K$ and $u \neq v$. The cone K is said to be strongly normal if there is a $\delta > 0$ such that if $z = \sum_{i=1}^n b_i x_i$, $x_i \in K$, $\|x_i\| = 1$, $b_i \geqq 0$, $\sum_{i=1}^n b_i = 1$, implies $\|z\| > \delta$. The cone K is said to be normal if there is a $\delta > 0$ such that $\|f_1 + f_2\| > \delta$ for $f_1, f_2 \in K$ and $\|f_1\| = \|f_2\| = 1$. The norm in E is said to be semimonotone if there is a numerical constant M such that $\mathcal{O} \leqq x \leqq y$ implies $\|x\| \leqq M\|y\|$ (where the constant M does not depend on x and y).

Let X be a set and K a cone. A function $d: X \times X \to K$ is said to be a K-metric on X if and only if (i) $d(x, y) = d(y, x)$, (ii) $d(x, y) = \mathcal{O}$ if and only if $x = y$, and (iii) $d(x, y) \leqq d(x, z) + d(z, y)$.

A sequence $\{x_n\}$ in a K-metric space X is said to converge to x_0 in X if and only if for each $u \in K^0$ there exists a positive integer N such that $d(x_n, x_0) \leqq u$ for all $n \geqq N$. A sequence $\{x_n\}$ in X is Cauchy if and only if for each $u \in K^0$ there exists a positive integer N such that $d(x_n, x_m) \leqq u$ for all $n, m \geqq N$. The K-metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges.

Throughout the rest of this paper we assume that K is strongly normal, that E is a reflexive Banach space, that (X, d) is a complete K-metric space, that $P = \{d(x, y); x, y \in X\}$, that $\bar{P}$ denotes that weak closure of P, and that $P_1 = \{z; z \in \bar{P} \text{ and } z \neq \mathcal{O}\}$.

(II). **Preliminary results.** In this section we list Mazur lemma and needed properties of cone K and the related K-metric space which will be used in our theorem.

(a) "Strongly normal" is normal.

(b) A necessary and sufficient condition for the cone K to be normal is that the norm be semimonotone (cf. [11]).

(c) If the sequence $\{u_n\}$ in E converges (in norm) to u, the sequence $\{v_n\}$ in E converges (in norm) to v and $u_n \leqq v_n$ for each n, then $u \leqq v$.

(d) If $\{x_n\}$ is a sequence in the K-metric space X that has a limit in X, then the limit is unique.

(e) If $u \in K^0$, then there exists a positive number c such that if $v \in \{p; \|p\| < c\} \cap K$ then $v \leqq u$.

(f) If h is an element in the Banach space E, $h_n \in K$ for each n, $h \leqq h_n$ for each n and $\{h_n\}$ converges (in norm) to $\mathcal{O}$ in E, then $-h \in K$.

(g) If $u \in K^0$ and $\{h_n\}$ is a sequence in K which converges (in norm) to $\mathcal{O}$ in E, then there exists a positive integer N such that $h_n \leqq u$ for $n \geqq N$.

(h) If $\{x_n\}$ is a sequence in the K-metric space X that is convergent to x in X then $\{d(x_n, x)\}$ converges (in norm) to $\mathcal{O}$ in E.

(i) Mazur lemma [5, 13]. Let E be a normed space and $\{u_n\}$ a sequence in E converging weakly to u. Then there is a sequence of convex combinations $\{v_n\}$ such that $v_n = \sum_{i=n}^{N} b_i u_i$ where $\sum_{i=n}^{N} b_i = 1$, and $b_i = b_i(n) \geqq 0$, $n \leqq i \leqq N = N(n)$ which converges to u in norm.

(j) Let the sequence $\{u_n\}$ in E be weakly convergent to v, if $u_n \geqq \mathcal{O}$ for each $n \geqq 1$ then $v \geqq \mathcal{O}$.

(III). **Examples and main results.**

EXAMPLE 1. Let $E = R$ (all real numbers) and $K = R^+$ (all non-negative real numbers), then K is strongly normal and semimonotone, and K satisfies the law of trichotomy.

EXAMPLE 2. Let $E = R^2$ and $K = \{z \in R^2; 0 < a \leqq \operatorname{Arg} z \leqq b < \pi/2\} \cup \{\mathcal{O}\}$, where the symbol $\operatorname{Arg} z$ denotes the argument of the complex number z. Although K is strongly normal, semimonotone, K doesn't satisfy the law of trichotomy.

The mapping $\phi: P_1 \to K$ is said to be lower semicontinuous if $\{u_n\}$ and $\{\phi u_n\}$ are both weakly convergent, then $\lim \phi u_n \geqq \phi(\lim u_n)$.

The property of the law of trichotomy of the set R has been used in the proof of [14] and [17] but it can not be used in our Theorem 1 (cf. Example 2). The proof of Theorem 1 differs from that of theorem [14] and theorem [17].

THEOREM 1. *Let (X, d) be a nonempty complete K-metric space, and let S, T be mappings of X into itself satisfying* (1) *and* (2).

$$(1) \qquad \phi(d(Sx, Ty)) \leqq d(x, y)\ , \quad x \neq y \in X\ ,$$

$$(2) \qquad \phi(t) > t \quad \textit{for any} \quad t \in P_1\ ,$$

where $\phi: P_1 \to K$ is lower semicontinuous on P_1.

Then exactly one of the following three statements holds:

(a) S and T have a common fixed point, which is the only periodic point for both S and T;

(b) There exist a point $x_0 \in X$ and an integer $p > 1$ such that $Sx_0 = x_0 = T^p x_0$ and $Tx_0 \neq x_0$;

(c) There exist a point $y_0 \in X$ and an integer $q > 1$ such that $S^q y_0 = y_0 = Ty_0$ and $Sy_0 \neq y_0$.

(IV). Lemmas and proofs.

LEMMA 1. *For each $x_0 \in X$, we define a sequence $\{x_n\}$ recursively as follows:*

$$x_1 = Sx_0,\ x_2 = Tx_1,\ \cdots,\ x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ \cdots\ .$$

Then the sequence $\{d(x_n, x_{n+1})\}$ weakly converges to $\mathcal{O}$ if $d(x_n, x_{n+1}) > \mathcal{O}$ for all $n \geqq 1$.

Proof. Suppose that $d(x_n, x_{n+1}) > \mathcal{O}$ for all $n \geqq 1$. Let $d_n = d(x_n, x_{n+1})$. It follows, by (1) and (2), that, for each positive integer n,

$$(3) \qquad \begin{aligned} d_{2n+1} &= d(Sx_{2n}, Tx_{2n+1}) < \phi(d_{2n+1}) \leqq d_{2n}\ , \\ d_{2n} &= d(Sx_{2n}, Tx_{2n-1}) < \phi(d_{2n}) \leqq d_{2n-1}\ . \end{aligned}$$

Therefore $\{d_n\}$ is decreasing and bounded. Let $\{d_{n(i)}\}$ be a subsequence of $\{d_n\}$. Since $\{d_n\}$ is bounded, there exists a subsequence $\{d_{m(i)}\}$ of $\{d_{n(i)}\}$ such that $\{d_{m(i)}\}$ weakly converges to $z \in K$ and $\{d_{m(i)-1}\}$ to $t \in K$.

From the fact that $d_{m(i)-1} \geqq d_{m(i)} \geqq d_{m(i+1)-1}$, we see that $z = t$. Because $\mathscr{O} \leqq \phi(d_{m(i)}) \leqq d_{m(i)-1}$, we see that $\{\phi(d_{m(i)})\}$ is bounded. For convenience, we can assume that $\{\phi(d_{m(i)})\}$ has a weak limit. By the lower semi-continuity, we have $\phi(z) \leqq z$. Therefore $z = \mathscr{O}$ and $\{d_n\}$ weakly converges to $\mathscr{O}$.

LEMMA 2. *If y is a fixed point for S, then for each $x \in X$, $x \neq y$, either there exists a positive integer p such that $T^p x = y$ or else $\{d(T^n x, y)\}$ weakly converges to $\mathscr{O}$. Moreover, if $\{d(T^n x, y)\}$ weakly converges to $\mathscr{O}$, then $Ty = y$; and if $Ty \neq y$, then $T^p y = y$ for some $p > 1$.*

Proof. Suppose that $d(T^n x, y) > \mathscr{O}$. By (1), we have

$$d(y, T^{n+1}x) = d(Sy, T^{n+1}x) < \phi(d(Sy, T^{n+1}x)) \leqq d(y, T^n x)$$

for all $n = 1, 2, \cdots$. As in Lemma 1, we see $\{d(y, T^n x)\}$ weakly converges to $\mathscr{O}$.

Since

$$\begin{aligned} d(T^n x, Ty) &\leqq d(T(T^{n-1}x), S(T^n x)) + d(S(T^n x), Ty) \\ &\leqq \phi(d(T(T^{n-1}x), S(T^n x))) + \phi(d(S(T^n x), Ty)) \\ &\leqq d(T^{n-1}x, T^n x) + d(y, T^n x) \\ &\leqq 3d(y, T^{n-1}x) , \end{aligned}$$

and

$$d(y, Ty) \leqq d(y, T^n x) + d(T^n x, Ty) ,$$

we have, as $n \to \infty$, $y = Ty$.

LEMMA 3. *If S, T have fixed points x_1, x_2 respectively in X, then $x_1 = x_2$ and x_1 is the unique periodic point for S and T.*

Proof. If $x_1 \neq x_2$, then $d(x_1, x_2) < \phi(d(Sx_1, Tx_2)) \leqq d(x_1, x_2)$, a contradiction. Moreover, if $T^q x = x$, then, by Lemma 2, there is a positive integer p such that $T^p x = x_1$, and therefore $T^r x_1 = x$ for some integer $r > 0$. But $Tx_1 = x_1$, so that $x_1 = x$; and by the same argument, if $S^q x = x$, then $x = x_1$, which completes the proof.

Proof of Theorem 1. For a fixed $x_0 \in X$, we define $\{x_n\}$ recursively $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$, $n = 0, 1, 2, \cdots$, as in Lemma 1.

Case 1. Suppose $d(x_n, x_{n+1}) = \mathscr{O}$ for some even integer $n \geqq 1$. Then $x_n = x_{n+1} = Sx_n$ is a fixed point of S, so that by Lemma 2, either x_n is a fixed point of T or else $Tx_n \neq x_n$ and there is a positive integer $p > 1$ such that $T^p x_n = x_n$.

Case 2. Suppose $d(x_n, x_{n+1}) = \mathcal{O}$ for some odd integer $n \geqq 1$. Then by the same argument, we have either $Sx_n = Tx_n = x_n$ or else $Sx_n \neq x_n$ and $S^q x_n = x_n$ for some integer $q > 1$.

Case 3. Suppose $d(x_n, x_{n+1}) \neq \mathcal{O}$ for all $n = 1, 2, \cdots$. Then $\{d(x_n, x_{n+1})\}$ weakly converges to $\mathcal{O}$. We wish to show that $\{x_n\}$ is a Cauchy sequence. Suppose not. Then there is an $\varepsilon \in K^0$ such that for every integer, there exist integers $n(i)$ and $m(i)$ with $i \leqq n(i) < m(i)$ such that

$$d(x_{n(i)}, x_{m(i)}) \nleqq \varepsilon \,. \tag{4}$$

Let, for each integer i, $m(i)$ be the least integer exceeding $n(i)$ satisfying (4); that is,

$$d(x_{n(i)}, x_{m(i)}) \nleqq \varepsilon \quad \text{and} \quad d(x_{n(i)}, x_{m(i)-1}) \leqq \varepsilon \,. \tag{5}$$

Since K is semimonotone, the sequence $\{d(x_{n(i)}, x_{m(i)-1})\}$ is bounded. Consequently the sequence $\{d(x_{n(i)}, x_{m(i)})\}$ is bounded.

Because E is a reflexive Banach space, for convenience, we let

$$\begin{cases} \{d(x_{n(i)}, x_{m(i)})\} & \text{be weakly convergent to } z_1 \,, \\ \{d(x_{n(i)}, x_{m(i)-1})\} & \text{be weakly convergent to } z_2 \,, \\ \{d(x_{n(i)-1}, x_{m(i)-1})\} & \text{be weakly convergent to } z_3 \,, \end{cases} \tag{A}$$

where z_1, z_3 and z_2 are in K. According to the triangular inequality, we have

$$d(x_{n(i)}, x_{m(i)-1}) + d(x_{n(i)}, x_{n(i)-1}) \geqq d(x_{n(i)-1}, x_{m(i)-1}) \,, \tag{6}$$

$$d(x_{n(i)-1}, x_{m(i)-1}) + d(x_{n(i)-1}, x_{n(i)}) \geqq d(x_{n(i)}, x_{m(i)-1}) \,, \tag{7}$$

$$d(x_{n(i)}, x_{m(i)}) + d(x_{m(i)}, x_{m(i)-1}) \geqq d(x_{n(i)}, x_{m(i)-1}) \,, \tag{8}$$

$$d(x_{n(i)}, x_{m(i)-1}) + d(x_{m(i)-1}, x_{m(i)}) \geqq d(x_{n(i)}, x_{m(i)}) \,. \tag{9}$$

From (6), (7), (8), (9) and Lemma 1, we see that $z_1 \geqq z_2$, $z_2 \geqq z_1$, $z_2 \geqq z_3$, $z_3 \geqq z_2$ and $z_1 = z_2 = z_3 = z$ (say). For convenience, we assume that $n(i) + m(i)$ is odd. We see that

$$\phi(d(x_{n(i)}, x_{m(i)})) \leqq d(x_{n(i)-1}, x_{m(i)-1}) \,. \tag{10}$$

Let $\{\phi(d(x_{n(i)}, x_{m(i)}))\}$ have a weak limit. Therefore we have $\phi(z) \leqq z$, we obtain that $z = \mathcal{O}$. (If $n(i) + m(i)$ is even, we shall consider putting the sequence $\{d(x_{n(i)+1}, x_{m(i)})\}$, instead of $\{d(x_{n(i)}, x_{m(i)})\}$, into (10).) By (4) and (g), there exist a positive number s and a subsequence $\{d(x_{p(i)}, x_{q(i)})\}$ of $\{d(x_{n(i)}, x_{m(i)})\}$ such that the sequence $\{d(x_{p(i)}, x_{q(i)})\}$ doesn't converge to $\mathcal{O}$ (in norm) and $\lim_{i\to\infty} \|d(x_{p(i)}, x_{q(i)})\| = s > 0$. Since the sequence $\{d(x_{p(i)}, x_{q(i)})\}$ weakly converges to $\mathcal{O}$, by

Mazur lemma, then there is a sequence of convex combinations $\{v_n\}$ such that

$$v_n = \sum_{j=n}^{N} b_j u_j ,$$

where $\sum_{j=n}^{N} b_j = 1$, $b_j = b_j(n) \geqq 0$, $n \leqq j \leqq N = N(n)$ and $u_j = d(x_{p(j)}, x_{q(j)})$, which converges to $\mathcal{O}$ (in norm). For convenience, we can assume $s = 1$. Since K is strongly normal, then there exists a $\delta > 0$ such that $\|v_n\| > \delta$, when n is sufficiently large. Because $\{v_n\}$ converges to $\mathcal{O}$ (in norm), this is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. By completeness, there is a $u \in X$ such that $\{x_n\}$ converges to u in X. We see that

$$d(Tu, u) \leqq d(Tu, Tx_{2n+1}) + d(x_{2n+2}, u) .$$

Let $\{y_n\} \subset X$ converge to y with $y_n \neq y_{n+1}$ and $y_n \neq y$ for all $n \geqq 1$. Then

$$\begin{aligned} d(Ty_n, Ty) &\leqq d(Ty_n, Sy_{n+1}) + d(Sy_{n+1}, Ty) \\ &\leqq d(y_n, y_{n+1}) + d(y_{n+1}, y) . \end{aligned}$$

We have, as $n \to \infty$, $Tu = u$. Similarly we have $Su = u$. These three cases show that at least one of (a), (b), (c) in Theorem 1 holds; and therefore, by Lemma 3, exactly one of (a), (b), (c) in Theorem 1 holds.

If E is the set of all real numbers and if K is the set of all nonnegative reals, then, from (4), (10) and Lemma 1, Theorem 1 may now be restated in the following form.

THEOREM 2. *Let (X, d) be a nonempty complete metric space, and let S, T be mappings of X into itself satisfying* (1) *and* (2).

(1) $\phi(d(Sx, Ty)) \leqq d(x, y)$, $x \neq y \in X$,

(2) $\phi(t) > t$ *for any* $t \in P_1$,

where ϕ is lower semicontinuous from the right on P_1.

Then exactly one of (a), (b) and (c) as in Theorem 1 holds.

Utilizing the way of the proof of Theorem 1 [15], we have the following result.

THEOREM 3. *Let S, T be mappings on a nonempty complete metric space (X, d). Then the following conditions are equivalent:*

(i) *For any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$d(Sx, Ty) < \varepsilon \quad \textit{whenever} \quad \varepsilon \leqq d(x, y) < \varepsilon + \delta(\varepsilon) ,$$

(ii) *There exists a self mapping ϕ of $[0, \infty)$ into $[0, \infty]$ such* *that* $\phi(s) > s$ *for all* $s > 0$, ϕ *is lower semicontinuous from the right on* $(0, \infty)$ *and*

$$\phi(d(Sx, Ty)) \leqq d(x, y)\ , \quad x \neq y \in X\ .$$

From Theorem 3, we have the following result.

THEOREM 4. *Let* (X, d) *be a complete metric space, and let* S, T *be mappings of* X *into itself satisfying condition* (i) *in Theorem* 3; *then exactly one of* (a), (b) *and* (c) *as in Theorem* 1 *holds.*

Theorem 4 was proved in [17] by Chi-Lin Yen and Kun-Jen Chung, but it is a special case of our Theorem 1.

REMARK 1. If $S = T = F$ in Theorem 4, any one of (a), (b) and (c) implies that F has a fixed point, that is, that S and T have a common fixed point. Hence (a) holds; namely T has a unique fixed point. This result was proved by A. Meir and E. Keeler [14].

REMARK 2. The condition that two mappings T and S satisfy (i) in Theorem 3 does not imply $S = T$ (cf. [17]).

The author would welcome an example of a strongly normal cone K in a reflexive infinite dimensional Banach space.

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