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ON THE IMAGE OF THE GENERALIZED GAUSS MAP OF A COMPLETE MINIMAL SURFACE IN R^4

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The generalized Gauss map of an immersed oriented surface M in R^4 is the map which associates to each point of M its oriented tangent plane in $G_{2,4}$, the Grassmannian of oriented planes in R^4. The Grassmannian $G_{2,4}$ is naturally identified with Q_2, the complex hyperquadric

$$\left\{[z_1, z_2, z_3, z_4] \,\middle|\, \sum_{k=1}^{4} z_k^2 = 0\right\} \quad \text{in} \quad P^3(C)\,.$$

The normalized Fubini-Study metric on $P^3(C)$ with holomorphic curvature 2 induces an invariant metric on $Q_2 \cong G_{2,4}$, which corresponds exactly to the metric on the canonical representation of $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$ in R^6 as $\{X \in R^6 \mid x_1^2 + x_2^2 + x_3^2 = (1/2), x_4^2 + x_5^2 + x_6^2 = (1/2)\}$. The product representation above allows us to associate with any map g in Q_2 two canonical projections g_1, g_2. In the case where g is complex analytic map defined on some Riemann surface S_0, the projections g_1, g_2 are complex analytic also. Detailed treatment can be found in the recent work of Hoffman and Osserman.

The study of the image of the Gauss map of a complete minimal surface in R^3 was motivated in one way to generalize a classical theorem of S. Bernstein [1], and was initiated by Osserman [7, 8, 9]. And the value distribution of the generalized Gauss map of a complete minimal surface in R^4, due to the product representation of Q_2, can therefore be studied in a similar manner. In fact, results treating the case in R^3 have been extended to that in R^4 by Chern [3], Osserman [9], Hoffman and Osserman [5]. Very recently, Xavier [10] has made a remarkable improvement in the study of the case in R^3. Therefore it's quite natural to extend it to the case in R^4, which will be shown in the following theorem.

THEOREM 1. *Let S be a complete minimal surface in R^4 with g its generalized Gauss map and g_1, g_2, the corresponding projections. Then S must be a plane if*

(i) *both g_1 and g_2 omit more than 6 points, or*

(ii) *one projection is constant and the other omits more than 4 points.*

Proof. Let S be given by

$$X: S_0 \longrightarrow R^4 \tag{1}$$

where S_0 is a Riemann surface. Its generalized Gauss map can be expressed by

$$g=[\phi_1(\zeta),\ \phi_2(\zeta),\ \phi_3(\zeta),\ \phi_4(\zeta)] \tag{2}$$

where

$$\phi_k(\zeta) = 2\frac{\partial x_k}{\partial \zeta} \tag{3}$$

with ζ a local complex parameter. And the projection g_1, g_2 are expressed by

$$g_1 = \frac{\phi_3 + i\phi_4}{\phi_1 - i\phi_2}\,, \quad g_2 = \frac{\phi_3 - i\phi_4}{\phi_1 - i\phi_2}\,. \tag{4}$$

The induced metric is given by

$$ds^2 = \frac{1}{4}|f|^2(1 + |g_1|^2)(1 + |g_2|^2)|d\zeta|^2 \tag{5}$$

where $f(\zeta) = \phi_1 - i\phi_2$. For detailed explanation, see Osserman [9].

Without loss of generality, we may assume S_0 to be simply connected. Combining our hypothese in (i), (ii) with the Koebe uniformization theorem and the Picard's theorem, we may assume further that S_0 is the unit disk $D = \{\zeta \in \boldsymbol{C} \mid |\zeta| < 1\}$.

A crucial lemma used by Xavier [10] can be adapted easily in our case as:

LEMMA. *Let $g_1\colon D \to \boldsymbol{C} - \{0, a\}(a \neq 0)$, $g_2 = D \longrightarrow \boldsymbol{C} - \{0, b\}(b \neq 0)$ be holomorphic functions. Then*

$$\int_D \prod_{j=1}^{2}\left[\frac{|g_j'|}{|g_i|^{\alpha} + |g_j|^{2-\alpha}}\right]^p d\xi dn < \infty$$

for any $\alpha = 1 - 1/k$, $k \in \boldsymbol{Z}^+$ and $0 \leqq p < 1/2$, where $\zeta = \xi + i\eta$.

Now we proceed our proof. Suppose S is not a plane. Under the hypothese in (i) or (ii), we may assume that both g_1 and g_2 are holomorphic.

For the case (i), suppose g_1 omits $a_1, \cdots, a_6$ in $\boldsymbol{C}$ and g_2 omits $b_1, \cdots, b_6$ in $\boldsymbol{C}$. Consider the function

$$h = g_1' g_2' f^{-2/p} \prod_{i=1}^{6}(g_1 - a_i)^{-\alpha} \prod_{j=1}^{6}(g_2 - b_j)^{-\alpha}\,, \tag{6}$$

where $\alpha = 1 - 1/k$ with $10/11 \leqq \alpha < 1$ and $p = 5/12\alpha$.

For the case (ii), suppose g_1 constant and g_2 omits $b_1, \cdots, b_4$ in $\boldsymbol{C}$. And consider the function

$$h = g_2' f^{-2/p} \prod_{j=1}^{4} (g_2 - b_j)^{-\alpha}, \tag{7}$$

where $\alpha = 1 - 1/k$ with $10/11 \leqq \alpha < 1$ and $p = 3/4\alpha$.

In both cases, using the same arguments in [10], we can see that from one hand, essentially due to a theorem of Yau [11, Th. 1].

$$|h| \notin L^p(S_0) \tag{8}$$

and from the other hand, by direct calculation, we get

$$|h| \in L^p(S_0) \tag{9}$$

which is impossible. □

Next we shall extend a theorem of Gackstatter [6] on complete abelian minimal surfaces in $\boldsymbol{R}^3$ to those in $\boldsymbol{R}^4$.

THEOREM 2. *Let S be a complete abelian minimal surface in $\boldsymbol{R}^4$, and g its generalized Gauss map. Then S must be plane if either*

(a) *one projection, say g_1, omits more than 4 points and the other projection g_2 omits more than 3 points, or*

(b) *g_1 is constant and g_2 omits more than 3 points.*

Proof. By a complete abelian minimal surface S in $\boldsymbol{R}^4$. We mean that S can be constructed out of a meromorphic differential $fd\zeta$ and two meromorphic functions g_1, g_2 on a compact Riemann surface $\bar{M}$ with the metric

$$ds^2 = \frac{|f|^2}{4}(1 + |g_1|^2)(1 + |g_2|^2)|d\zeta|^2$$

which never vanishes. And the construction is made in the sense of L. Bers [2] such that the immersion is given by the formula

$$x = \mathrm{Re}\int \frac{f}{2}(1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2))d\zeta \tag{10}$$

on a covering space M over $\bar{M} - \{p \mid ds^2(p) = \infty\}$ as long as (10) is well-defined. The boudary points to the metric ds^2 are those finitely many points $p_1, \cdots, p_r$ in $\bar{M}$ where $ds^2 = \infty$.

By a rotation of S, we may assume that

(i) both g_1 and g_2 have only simple poles, and they don't have poles in common,

(ii) the poles of g_1, g_2 don't fall into the boundary points $p_1, \cdots, p_r$, and hence,

(iii) at each pole of g_1 or g_2, f must have a simple zero,
(iv) f has no other zeros, and
(v) at each p_j, f must have a pole of order $m_j \geqq 1$.

Now suppose g_1 is an N_1-sheet and g_2 is an N_2-sheet branching covering. Then by the Riemann relation for the differential $fd\zeta$, we have

$$(N_1 + N_2) - \sum_{j=1}^{r} m_j = 2\gamma - 2 \tag{11}$$

where γ is the genus of $\bar{M}$.

And by the Riemann relation for g_1 and g_2, in case of nonconstant, we have

$$\sum_{g_1} (l_1 - 1) - 2N_1 = 2\gamma - 2 \tag{12}$$

$$\sum_{g_2} (l_2 - 1) - 2N_2 = 2\gamma - 2 \tag{13}$$

where $\sum_{g_1}(l_1 - 1)$ and $\sum_{g_2}(l_2 - 1)$ are the total branching orders of g_1, g_2, respectively.

Now suppose S is nonflat, i.e., g_1, g_2 can't both be constant, and that

(a) g_1 omits 5 values $a_1, \cdots, a_5$, g_2 omits 4 values $b_1, \cdots, b_4$ and neither one is constant. Then clearly

$$g_1^{-1}\{a_\nu \mid 1 \leqq \nu \leqq 5\} \subset \{p_1, \cdots, p_r\}\ , \tag{14}$$

$$g_2^{-1}\{b_\mu \mid 1 \leqq \mu \leqq 4\} \subset \{p_1, \cdots, p_r\}\ . \tag{15}$$

And (12), (13) can be written as

$$\sum_{g_1 \neq a_\nu} (l_1 - 1) + 3N_1 = 2\gamma - 2 + \sum_{g_1 = a_\nu} 1\ , \tag{16}$$

$$\sum_{g_2 \neq b_\mu} (l_2 - 1) + 2N_2 = 2\gamma - 2 + \sum_{g_2 = b_\mu} 1\ . \tag{17}$$

Comparing with (11), we get

$$2\sum_{j=1}^{r} m_j < \sum_{g_1 = a_\nu} 1 + \sum_{g_2 = b_\mu} 1 \tag{18}$$

which contradicts (14) and (15).

(b) g_1 constant and g_2 omits 4 points $b_1, \cdots, b_4$. Clearly (15) and (17) still hold with $N_1 = 0$, $N_2 > 0$. From (11), (17), we have

$$\sum_{j=1}^{r} m_j < \sum_{g_2 = b_\mu}^{r} 1 \tag{19}$$

which contradicts (15). □

Corollary. *If* (a) g_1 *omits exactly* 4 *points and* g_2 *omits exactly*

4 points, or

(b) *g_1 constant and g_2 omits exactly 3 points, then* (a) $r = 4$, $m_j = 1$ *or* (b) $r = 3$, $m_j = 1$, *respectively. Further, in neither case S can have flat points.*

Proof. Note that p is a flat point of S if and only if $g_1'(p) = 0$ and $g_2'(p) = 0$. In case (a) comparing (11) with

$$\sum_{g_1 \neq a_\nu} (l_1 - 1) + 2N_1 = 2\gamma - 2 + \sum_{g_1 = a_\nu} 1$$

and (17), we get $r = 4$, $m_j = 1$, $l_1 \equiv 1$, $l_2 \equiv 1$.

And in case (b) comparing (11) with

$$\sum_{g_2 \neq b_\mu} (l_2 - 1) + N_2 = 2\gamma - 2 + \sum_{g_2 = b_\mu} 1$$

and $N_1 = 0$, we get $r = 3$, $m_j = 1$, $l_2 \equiv 1$. □

For complete minimal surface with finite total curvature, it's known [4] that $M = \bar{M} - \{p_1, \cdots, p_r\}$ and $m_j \geqq 2$. Thus, Theorem 2 and corollary together give an alternative proof of

THEOREM 3 (*Hoffman-Osserman* [5]). *Let S be a complete minimal surface in $\boldsymbol{R}^4$ with finite total curvature. Then S must be a plane if*

(a) *both g_1 and g_2 omit more than 3 points, or*

(b) *g_1 constant and g_2 omits more than 2 points.*

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