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HAUSDORFF MEASURE, BMO, AND ANALYTIC FUNCTIONS

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Besicovitch's theorem on removable singularities is extended to function of class BMO. The extende d theorem admits a converse.

Let W be an open set in the plane, and the class BMO(W) be defined as follows: a measurable function f on W is BMO if to each ball B contained in W there is a constant $c=c(B)$ so that $\iint_B |f(x, y)-c(B)|dxdy \leqq Am(B)$, with a constant $A = A(f)$. For any complex function f on $W, S(f)$ is the set of points at which f fails to admit a complex derivative; $S(f)$ in general is neither open nor closed, but is in fact a Borel set.

THEOREM (a). *Let $f \in$ BMO(W) and suppose that $S(f)$ has 1-dimensional Hausdorff measure 0. Then there is a function f_1, holomorphic on W, equal to f on $W - S(f)$.*

(b) *Let S be a compact set of positive 1-measure. Then there is a function g, analytic off S, of class BMO(R^2) with Taylor expansion $g(z) = z^{-1} + \cdots$ at infinity.*

Proof of (a). This relies on Theorem 1 of [3] and the following variant form of Vitali's covering theorem: if a sequence of open balls $(B(a_i, r_i))_1^\infty$ covers a bounded set E, then it contains a disjoint collection $(B(a_j, r_j))_1^\infty$ such that $\bigcup_j B(a_j, 3r_j)$ covers E. Let V be a bounded subset of W and $\varepsilon>0$; we construct coverings of $V\cap S(f)$ and $V - S(f)$ separately. Inasmuch as $S(f)$ has 1-measure 0 we can cover it with balls $B(a_i, r_i)$ such that $B(a_i, 2r_i) \subsetneqq W$ and $\sum r_i < \varepsilon$. For each point z in $V - S(f)$ we can find a number $r(z) > 0$ so that $B(z, 6r(z)) \subsetneqq W$ and $|f(w) - f(z) - (z - w)f'(z)| < \epsilon r(z)$ when $w \in B(z, 6r(z))$. The collection $B(z, r(z))$ contains a disjoint sequence $B(z_j, r(z_j))$ such that $\cup B(z_j, 3\pi(z_j)) \supsetneqq V - S(f)$; by the disjointness, $\sum 9\pi\lambda^2(z_j) \leqq 9m(V)$. Using the fact that constants are analytic, we see that the conditions of [3, p. 108] are fulfilled, so that $f = f_1$ a.e., for some function f_1 holomorphic on W. Then $f - f_1$ is differentiable on $S - S(f)$, and so $f = f_1$ there.

An easy improvement can be obtained from [3], namely, the constant $c(B)$ in the definition of BMO(W) can be replaced by a polynomial (depending on B).

Proof of (b). By a theorem of Frostman [2, p. 7], S carries a probability measure μ, such that $\mu(B(z, r)) \leqq cr$ for every ball B of radius $r > 0$. Let

$$g(z) = \int (z - \zeta)^{-1} \mu(d\zeta)$$

so that g is analytic off S and $g(z) = z^{-1} + \cdots$. To prove that $g \in \mathrm{BMO}(R^2)$ we choose a ball $B = B(w, r)$, set $B^* = B(w, 2r)$ and $C = R^2 - B^*$. Let

$$g_1(z) = \iint_{B*} (z - \zeta)^{-1} \mu(d\zeta)$$
$$g_2(z) = g(z) - g_1(z) .$$

Now

$$\iint_B |g_1(z)| dxdy \leqq \mu(B^*) \sup \iint_B |z - \zeta|^{-1} dxdy$$
$$= 2\pi r \mu(B^*) \leqq 4\pi c r^2 .$$

Further

$$\iint_B |g_2(z) - g_2(w)| dxdy \leqq \int_C \int_B |(z - \zeta)^{-1} - (w - \zeta)^{-1}| dxdy \mu(d\zeta) .$$

From the inequality $|\zeta - w| > 2r$ $(\zeta \in C)$, we find that the inner integral doesn't exceed $(4\pi/3) \cdot r^3 |\zeta - w|^{-2}$, and the entire integral is at most

$$(4\pi/3) \cdot r^3 \int_C |w - \zeta|^{-2} \mu(d\zeta)$$
$$\leqq (4\pi/3) \cdot r^3 c (2r)^{-1} = 2\pi/3 \cdot cr^2 .$$

Hence

$$\iint_B |g(z) - g_2(w)| dxdy \leqq 3cm(B) .$$

REMARKS. (i) A Borel set S of positive 1-dimensional measure contains a compact set S_0 of the same kind (Besicovitch) [2, p. 11].

(ii) Besicovitch proved (a) for bounded functions; for continuous f_1 he proved the sufficiency of the hypothesis that $S(f)$ have σ-finite 1-dimensional measure. Combining his method for this variant, with the one presented above for BMO, we can replace continuity of f by VMO (vanishing mean oscillation), that is $\iint_B |f(x, y) - c(B)| dxdy \leqq m(B) \varepsilon(m(B))$, where $\varepsilon(0+) = 0$.

(iii) The variant just mentioned also admits a converse; to explain this we observe that if the probability measure μ figuring in the proof of (b) has the stronger property that $\mu(B(z, r)) \leqq r\varepsilon(r)$ with $\varepsilon(0+) = 0$, then the function g is $\mathrm{VMO}(R^2)$. We use the following theorem [4]; a Borel set S, not of σ-finite 1-dimensional measure, contains a compact set S_0, with positive Hausdorff measure for a measure function $h(u) = u\varepsilon(u)$; by Frostman's theorem S_0 then carries a probability measure μ with the stronger property needed to improve VMO to BMO.

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