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CORONA PROBLEM FOR RIEMANN SURFACES OF PARREAU-WIDOM TYPE

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It is shown that there exists a hyperbolic regular Riemann surface R of Parreau-Widom type that is not dense in the maximal ideal space $\mathscr{M}(R)$ of the Banach algebra $H^\infty(R)$ of bounded analytic functions on R.

It has been close to twenty years since Carleson [1] positively solved the corona problem for the unit disk. Since then various subsequent developments appeared. Among them we are particularly interested in investigations on the corona problem for Riemann surfaces of positive genus. As for the positive direction, Gamelin [2], e.g., proved by using his localization principle that the corona problem can be positively answered for finite Riemann surfaces with analytic borders. As for the negative direction, B. Cole constructed a Riemann surface for which the corona problem is negatively answered (see Gamelin [3]). In connection with these results, it is an interesting and also important problem to single out the class of Riemann surfaces of general genus for which the corona problem is positively settled. One might suspect that the class of Riemann surfaces of Parreau-Widom type falls into this category since the class $H^\infty(R)$ of bounded analytic functions on a Riemann surface R of this class is known to share various nice properties with the class $H^\infty(D)$ on the unit disk D (cf. Parreau [8], Widom [10, 11], Stanton [9], Hasumi [4, 5], Hayashi [6, 7], etc). The *purpose* of this paper is to show that the above expectation is unfortunately incorrect.

Consider a *hyperbolic* Riemann surface R so that there exists the Green's function $g(z, a)$ on R with its pole at any point a in R. By the maximum principle for harmonic functions the set $R(\alpha, a) = \{z \in R;\ g(z, a) > \alpha\}$ is a subregion of R for any $\alpha > 0$ and a in R. The surface R is said to be *regular* if $R(\alpha, a)$ is relatively compact for any $\alpha > 0$ and a in R. The first Betti number $B(\alpha, a)$ of $R(\alpha, a)$ is the minimum number of generators of the first singular homology group $H_1(R(\alpha, a))$ of $R(\alpha, a)$. A hyperbolic Riemann surface R is said to be *of Parreau-Widom type* if $\int_0^\infty B(\alpha, a)d\alpha < +\infty$ for one and hence for every a in R. We denote by $\mathscr{M}(R)$ the *maximal ideal space* of $H^\infty(R)$ equipped with the Gelfand topology. We may view $\mathscr{M}(R)$ as the space $\{q\}$ of multiplicative linear functionals q on $H^\infty(R)$ with $q(1) = 1$ equipped with the weak star topology since $q \mapsto q^{-1}(0)$ is the bijective homeomorphism between $\{q\}$ and $\mathscr{M}(R)$. A point z in R corresponds to a functional q_z in $\mathscr{M}(R)$ defined by $q_z(f) = f(z)$ (point evaluation). If R is a hyperbolic Riemann surface of Parreau-Widom type, then this natural mapping $z \mapsto q_z$ gives the injective homeomorphism $R \to \mathscr{M}(R)$ and the image of R under this mapping is open in $\mathscr{M}(R)$ (see Stanton [9]) and therefore we may view R as an open subset of $\mathscr{M}(R)$. The *corona problem* asks whether R is dense in $\mathscr{M}(R)$ or not. The *main result of* this paper is the following

THEOREM. *There exists a hyperbolic regular Riemann surface R of Parreau-Widom type that is not dense in the maximal ideal space $\mathscr{M}(R)$ of $H^\infty(R)$.*

The surface R in the above theorem which we will construct is of infinite genus and infinite connectivity. It is obtained from the B. Cole example by making a minor modification. This modification is formulated as proposition in no. 1, and it is proved in nos. 2–4. The construction of R in the above theorem is carried over in nos. 5–9.

1. Consider a fixed sequence $(S_n)_1^\infty$ of interiors S_n of finite bordered Riemann surfaces $\bar{S}_n$ with analytic borders ∂S_n, two fixed sequences $(b_n)_1^\infty$ and $(c_n)_1^\infty$ of real numbers b_n and c_n with $0 < c_n < b_n$, and a variable sequence $(\eta_n)_1^\infty$ of real numbers η_n with $0 < \eta_n \leqq \min(c_n, b_n - c_n)$. By using these sequences we will construct a Riemann surface as follows.

Let X_n be a rectangular strip $\{0 \leqq \operatorname{Re} z \leqq 2, 0 < \operatorname{Im} z < b_n\}$ and X'_n a rectangular strip X_n less two vertical slits $\sigma'_n = \{\operatorname{Re} z = 1, 0 \leqq \operatorname{Im} z \leqq c_n - \eta_n\}$ and $\sigma''_n = \{\operatorname{Re} z = 1, c_n + \eta_n \leqq \operatorname{Im} z \leqq b_n\}$, i.e., $X'_n = X_n - \sigma'_n \cup \sigma''_n$, for each n. Observe that $\tau_n = \{\operatorname{Re} z = 1, |\operatorname{Im} z - c_n| < \eta_n\}$ is a cross-cut of X'_n with the length $2\eta_n$ for each n. The left and right vertical sides of X'_n (and hence of X_n) will be denoted by α_n and β_n respectively.

Weld X'_n to S_n and S_{n+1} by identifying the side α_n of X'_n with an open arc in ∂S_n and the side β_n of X'_n with an open arc in ∂S_{n+1} for each n. The resulting surface $\bigcup_{n=1}^\infty (S_n \cup X'_n)$ will be denoted by

$$R = R((\eta_n)) = R((S_n), (X_n), (\eta_n)) \,. \tag{1}$$

Here it is assumed that $\bar{S}_n \cap \bar{S}_m = \phi$ $(n \neq m)$, $\bar{X}'_n \cap \bar{X}'_m = \phi$ $(n \neq m)$, and $\bar{X}'_n \cap \bar{S}_k = \phi$ $(k \neq n, n+1)$ in R. By using X_n instead of X'_n we construct the Riemann surface $\bigcup_{n=1}^\infty (S_n \cup X_n)$ in the same fashion as $R((S_n), (X_n), (\eta_n)) = \bigcup_{n=1}^{\infty} (S_n \cup X_n')$. The resulting surface will be denoted by

$$(2) \qquad R = R((S_n), (X_n)) .$$

Hence $R((S_n), (X_n), (\eta_n)) = R((S_n), (X_n)) - \bigcup_{n=1}^{\infty} (\sigma_n' \cup \sigma_n'')$. Clearly the surfaces R in (1) or (2) can be embedded in a larger Riemann surface W such that $W - \bar{R} \neq \phi$. Therefore the surfaces R given by (1) or by (2) are *hyperbolic*. We will prove the following

PROPOSITION. *If the sequence* $(\eta_n)_1^{\infty}$ *converges to zero sufficiently rapidly, then* $R = R((S_n), (X_n), (\eta_n))$ *is a hyperbolic regular Riemann surface of Parreau-Widom type.*

It can happen that $R = R((S_n), (X_n))$ is neither regular nor of Parreau-Widom type. In such a case $R = R((S_n), (X_n), (\eta_n))$ is of the same sort if $(\eta_n)_1^{\infty}$ converges to zero not so rapidly. The proof of the proposition will be given in nos. 2–4.

2. We denote by $R_n = \bigcup_{k=1}^{n} (S_k \cup X_k') - \beta_n$ the *initial part* of $R = R((S_n), (X_n), (\eta_n))$ and by $R_n' = R - R_n \cup \beta_n$ the *terminal part* of R. Recall that the first Betti number B of a finite Riemann surface W with border ∂W is given by $B = 1 - \chi = 2g + m - 1$ where χ is the Euler characteristic of W, g the genus of W, and m the number of components of ∂W. Let B_n be the first Betti number of R_n. Observe that B_n is finite since R_n is the interior of the finite bordered surface $\bar{R}_n$ and that B_n does not depend on the choice of (η_n) since $R((S_n), (X_n), (\eta_n))$ are homeomorphic to each other for all choices of (η_n). Then fix a sequence $(\varepsilon_n)_1^{\infty}$ of positive numbers ε_n such that $\varepsilon_n > \varepsilon_{n+1}$ $(n=1, 2, \cdots)$, $\lim_n \varepsilon_n = 0$, and $\sum_{n=2}^{\infty} B_n(\varepsilon_{n-1} - \varepsilon_n) < +\infty$.

Fix a point a in S_1 and let $g(z, a)$ be the Green's function of $R = R((S_n), (X_n), (\eta_n))$ with its pole at a. We also denote by $\hat{g}(z, a)$ the Green's function of $\hat{R} = R((S_n), (X_n), (\eta_n'))$ with $\eta_n' = \min(c_n, b_n - c_n)$. Then every R is a subsurface of $\hat{R}$ and therefore $g(z, a) \leqq \hat{g}(z, a)$ on R. Choose and fix $M > 0$ so large that $\hat{U} = \{z; \hat{g}(z, a) > M\}$ is contained in S_1 and simply connected. Then $U = \{z; g(z, a) > M\}$ is a subset of $\hat{U}$ and also simply connected. Hence the Betti number B_0 of U is zero.

Let Y_n be the part $\{1 < \operatorname{Re} z \leqq 2, 0 < \operatorname{Im} z < b_n\}$ of X_n, $\hat{R}_n'$ the terminal part of $\hat{R}$, and w_n the harmonic measure of τ_n with respect to the region $Y_n \cup \hat{R}_n'$. Then

$$g(z, a) \leqq M w_n(z) \quad (z \in Y_n \cup R_n')$$

and, in particular, $\sup_{\beta_n} g(\cdot, a) \leqq M \sup_{\beta_n} w_n$. It is clear that $\lim_{\eta_n \to 0} w_n = 0$ uniformly on $Y_n \cup R'_n$ less any neighborhood of the left vertical side of Y_n. We can thus choose a sequence $(\eta_n)_1^\infty$ converging to zero enough rapidly so that

$$(3) \qquad \sup_{\beta_n} g(\cdot, a) \leqq \varepsilon_n \quad (n = 1, 2, \cdots) .$$

3. We pause here to observe the following. Let W_0 be a subsurface of a Riemann surface W. Consider the first singular homology groups $H_1(W_0)$ and $H_1(W)$ of W_0 and W, respectively. A cycle γ on W_0 is automatically a cycle on W and this gives a natural group homomorphism $\gamma \mapsto \gamma$ of $H_1(W_0)$ to $H_1(W)$. Suppose that W_0 satisfies the condition (N): Any connected component of $W - W_0$ is not compact. Under this condition (N), the above natural homomorphism $H_1(W_0) \to H_1(W)$ is *injective*. To see this, let γ be a cycle on W_0 which is homologous to zero on W, i.e., $\gamma \sim 0$ on W. We have to show that γ is homologous to zero already on W_0, i.e., $\gamma \sim 0$ on W_0. Since $\gamma \sim 0$ on W and $\gamma \in H_1(W_0)$, we can express γ as $\gamma \sim \sum_{j=1}^{l} \partial\sigma_j^2$ where each σ_j^2 is a *simplicial* 2-simplex *on* W with $\partial\sigma_j^2 \subset W_0$. If $\sigma_j^2 \not\subset W_0$ for some j, then σ_j^2 must contain a component of $W - W_0$ since $\partial\sigma_j^2 \subset W_0$. This contradicts the condition (N). We have then $\sigma_j^2 \subset W_0$ and therefore $\gamma \sim 0$ on W_0.

4. We have already remarked that $R(\alpha, a) = \{z; g(z, a) > \alpha\}$ is a region for any $\alpha > 0$ as a consequence of the maximum principle for harmonic functions. In view of (3), $R(\alpha, a) \subset R_n$ for $\alpha > \varepsilon_n$. Since $g(\cdot, a) \leqq \varepsilon_n$ on ∂R_n, we see that $R(\alpha, a)$ is relatively compact for every $\alpha > 0$. This proves that R is *regular*.

Again by the maximum principle, it is readily seen that $R - R(\alpha, a)$ has no compact component. In particular, if $\alpha > \alpha'$, then $R(\alpha, a)$ is a subsurface of $R(\alpha', a)$ satisfying (N) with respect to $R(\alpha', a)$. Therefore the natural homomorphism $H_1(R(\alpha, a)) \to H_1(R(\alpha', a))$ is injective and a fortiori the minimum number of generators of $H_1(R(\alpha, a))$, which is the first Betti number $B(\alpha, a)$ of $R(\alpha, a)$, is less than or equal to that of $H_1(R(\alpha', a))$, which is $B(\alpha', a)$. Therefore we have

$$(4) \qquad B(\alpha, a) \leqq B(\alpha', a) \quad (\alpha \geqq \alpha') .$$

Similarly $R(\alpha, a)$ is a subsurface of R_n for $\alpha > \varepsilon_n$ with the property (N) and the natural homomorphism $H_1(R(\alpha, a)) \to H_1(R_n)$ is injective. Recall that the first Betti number of R_n is B_n and hence we have

$$(5) \qquad B(\alpha, a) \leqq B_n \quad (\alpha \geqq \varepsilon_n) .$$

For $\alpha \geqq M$, the region $R(\alpha, a)$ is contained in U and simply connected. Therefore $B(\alpha, a)=0$ for $\alpha \geqq M$ and hence $\int_M^\infty B(\alpha, a)d\alpha = 0$. By (5) we see that $\int_{\varepsilon_1}^M B(\alpha, a)d\alpha \leqq B_1(M-\varepsilon_1)$. Again by (5), $\int_{\varepsilon_n}^{\varepsilon_{n-1}} B(\alpha, a)d\alpha \leqq B_n(\varepsilon_{n-1}-\varepsilon_n)$ for every $n \geqq 2$. Hence we have

$$\int_0^\infty B(\alpha, a)d\alpha \leqq B_1(M-\varepsilon_1) + \sum_{n=2}^\infty B_n(\varepsilon_{n-1}-\varepsilon_n) < +\infty ,$$

which proves that $R = R((S_n), (X_n), (\eta_n))$ is *of Parreau-Widom type.*

5. We proceed to the construction of R in our theorem. For the purpose we will briefly describe here how the *B. Cole example* W is constructed that is not dense in the maximal ideal space $\mathscr{M}(W)$ of $H^\infty(W)$. The construction is done based upon two crucial steps. The first is the following *Existence of a malformed finite surface*: Let δ be an arbitrary real number with $0<\delta<1$ and m an arbitrary positive integer. Then there exists a finite bordered Riemann surface $\bar{W}_m$ with its interior W_m and with an analytic border ∂W_m and a pair f_m, g_m of functions in $H^\infty(W_m)$ with $|f_m|$, $|g_m| \leqq 1$ and $|f_m|+|g_m| \geqq \delta$ on W_m such that whenever $f_m\phi + g_m\psi = 1$ is satisfied on W_m for a pair ϕ and ψ of functions in $H^\infty(W_m)$, we have $\sup_{W_m}|\phi| + \sup_{W_m}|\psi| \geqq m$. For its proof see Gamelin [3, pp. 47–49].

6. Another important device for the construction is the following *Approximation theorem* which is easily deduced by a standard successive approximation procedure from the Bishop generalization to Riemann surfaces of the Mergelyan approximation theorem: Let $(K_n)_1^\infty$ be a sequence of compact subsets K_n of a Riemann surface R such that $K_n \cap K_m = \phi$ $(n \neq m)$, γ_n a curve in $R - \bigcup_{k=1}^\infty K_k$ except its end points connecting a boundary point of K_n to a boundary point in K_{n+1} such that $\gamma_n \cap \gamma_m = \phi$ $(n \neq m)$. Assume that $R-F$ has no relatively compact component where $F = \bigcup_{n=1}^\infty (K_n \cup \gamma_n)$. To each function ϕ continuous on F and analytic in the interior of F and each positive number ε there exists an analytic function Φ on R such that $\sup_{K_n \cup \gamma_n}|\Phi - \phi| < \varepsilon/n$ $(n=1, 2, \cdots)$.

7. Let W_m be as in no. 5 for each m and Z_m be a finite Riemann surface obtained from W_m by attaching an annulus to each boundary component of W_m. By using a sequence $(L_m)_1^\infty$ of rectangular strips L_m we construct a Riemann surface $R((Z_m), (L_m))$ defined in (2). Let γ_m be a curve in $R((Z_n), (L_n)) - \bigcup_1^\infty \bar{W}_n$ except its end points connecting a boundary point of $\bar{W}_m$ and a boundary point of $\bar{W}_{m+1}$ for each m such that $\gamma_n \cap \gamma_m = \phi$ $(n \neq m)$. Then $F = \bigcup_1^\infty (\bar{W}_n \cup \gamma_n)$

is qualified to be an F in no. 6 with respect to the Riemann surface $R((Z_n), (L_n))$. Consider continuous functions f_0 and g_0 on F such that $f_0 | W_m = f_m$, $g_0 | W_m = g_m$ $(m = 1, 2 \cdots)$, $|f_0|$, $|g_0| \leqq 1$ and $|f_0| + |g_0| \geqq \delta$ on F. By no. 6 there exist analytic functions f and g on $R((Z_n), (L_n))$ such that

$$(6) \qquad \sup_{\bar{W}_m \cup \gamma_m} | f - f_0 | + \sup_{\bar{W}_m \cup \gamma_m} | g - g_0 | < \delta/4m \quad (m = 1, 2, \cdots) .$$

8. Let W be a connected neighborhood of F in $R((Z_n), (L_n))$ such that $|f|$, $|g| \leqq 2$, $|f| + |g| \geqq \delta/2$ on W. The surface W is the B. Cole example (see Gamelin [3, pp. 49–52]). For each m, let S_m be a finite surface with an analytic border ∂S_m such that $\bar{W}_m \subset S_m \subset \bar{S}_m \subset W \cap Z_m$ and Y_m a rectangular strip $\{0 \leqq \operatorname{Re} z \leqq 2, 0 < \operatorname{Im} z < b_m\}$ in the sense of conformal equivalence such that the left side of Y_m is a part of ∂S_m, the right side of Y_m is a part of ∂S_{m+1}, $Y_m \cap (\bigcup_1^\infty S_n) = \phi$, $\bigcup_1^\infty (\bar{S}_m \cup \bar{Y}_m) \subset W$, and $\bigcup_1^\infty (S_n \cup Y_n) \supset F$. Consider the surface $R((S_n), (Y_n))$ defined in (2), which may also be called the *B. Cole example* since it is a neighborhood of F contained in W. Let σ_n' and σ_n'' be as in no. 1 with $c_n = b_n/2$, for example. We finally consider

$$(7) \qquad R = R((S_n), (Y_n), (\eta_n)) = R((S_n), (Y_n)) - \bigcup_{n=1}^{\infty} (\sigma_n' \cup \sigma_n'')$$

defined in (1). By proposition in no. 1, $R = R((S_n), (Y_n), (\eta_n))$ can be made to a *hyperbolic regular Riemann surface of Parreau-Widom type* if the sequence $(\eta_n)_1^\infty$ is so chosen that it converges to zero sufficiently rapidly.

9. Since the surface R given by (7) now so made that it is of Parreau-Widom type, it has many nice properties concerning the class $H^\infty(R)$. For example, $H^\infty(R)$ separates points in R and the natural injection $R \to \mathscr{M}(R)$ of R into the maximal ideal space $\mathscr{M}(R)$ of $H^\infty(R)$ is bicontinuous and the image in $\mathscr{M}(R)$, identified with R, is open in $\mathscr{M}(R)$ (Stanton [9]). The proof of our theorem is over if we show that R is not dense in $\mathscr{M}(R)$.

Observe that functions f and g in no. 3 may be viewed as in $H^\infty(R)$ and, by (6), satisfy $|f|$, $|g| \leqq 2$, $|f| + |g| \geqq \delta/2$ on R, and

$$(8) \qquad \sup_{\bar{W}_m} |f - f_m| + \sup_{\bar{W}_m} |g - g_m| < 1/m \quad (m = 1, 2, \cdots) ,$$

where $\bigcup_1^\infty \bar{W}_n \subset R$. Suppose that the indefinite equation $f\phi + g\psi = 1$ on R has solutions ϕ and ψ in $H^\infty(R)$. Set $f_m\phi + g_m\psi = \lambda_m$ on $\bar{W}_m$. Observe that $\sup_{\bar{W}_m} |1 - \lambda_m| < (\sup_{\bar{W}_m} |\phi| + \sup_{\bar{W}_m} |\psi|)/m$ as a consequence of (8). Therefore $f_m(\phi/\lambda_m) + g_m(\psi/\lambda_m) = 1$ on W_m with ϕ/λ_m and ψ/λ_m

in $H^\infty(W_m)$. By no. 5, we then have $\sup_{\overline{W}_m}|\phi/\lambda_m| + \sup_{\overline{W}_m}|\psi/\lambda_m| \geqq m$ $(m = 1, 2, \cdots)$, which contradicts $\sup_{\overline{W}_m}|1 - \lambda_m| \to 0$ $(m \to \infty)$. This shows that there exists a maximal ideal M_0 in $\mathscr{M}(R)$ containing f and g. Then $f(M_0) = g(M_0) = 0$ and the assumption $\inf(|f|+|g|) \geqq \delta/2 > 0$ imply that M_0 is not in the closure of R in $\mathscr{M}(R)$ so that R is not dense in $\mathscr{M}(R)$.

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