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STRONG RESULT FOR REAL ZEROS OF RANDOM POLYNOMIALS

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Let N_n be the number of real zeros of $\sum_{r=0}^{n} a_r X_r x^r = 0$ where X_r's are independent random variables identically distributed belonging to the domain of attraction of normal law; $a_0, a_1, a_2 \cdots a_n$ are nonzero real numbers such that $(k_n/t_n) = o(\log n)$ where $k_n = \max_{0 \le r \le n} |a_r|$ and $t_n = \min_{0 \le r \le n} |a_r|$. Further we suppose that the coefficients have zero means and $P\{X_r \ne 0\} > 0$. Then there exists a positive integer n_0 such that

$$P\{\sup_{n>n_0} (N_n/D_n) < \mu\} > 1 - \mu' \{\log ((k_{n_0}/t_{n_0}) \log\log n)/\log n_0\}^{1-\varepsilon/2}$$

for $n > n_0$ and $1 > \varepsilon > 0$ where $D_n = (\log n/\log (k_n/t_n) \log\log n)^{(1-\varepsilon)/2}$.

1. Let N_n be the number of real roots of a random algebraic equation

$$\sum_{r=0}^{n} X_r x^r = 0 ,$$

where X_r's are independent, identically distributed random variables. The problem of finding the lower bound of N_n has been considered by various authors. Considering the coefficients as normally distributed or uniformly distributed in $[-1, 1]$, assuming the values $+1$ or -1 with equal probability Littlehood and Offord [8] have shown that $N_n > \mu \log n/\log\log n$ except for a set of measure at most $\mu'/\log n$, n being sufficiently large. Evans [4] has studied the strong version of Littlehood and Offord and has shown that in case of Gaussian distributed coefficients N_n is greater than $\mu \log n/\log\log n$ except for a set of measure at most μ' $(\log\log n_0/\log n_0)$ for $n > n_0$. The above result is strong in the following sense.

Theorem of Littlehood and Offord is of the form

$$P\{(N_n/D'_n) < \mu\} \longrightarrow 1 \text{ as } n \longrightarrow \infty ,$$

where $D'_n = \log n/\log\log n$. But the theorem of Evans is of the form

$$P\{\sup_{n>n_0} (N_n/D'_n) < \mu\} \longrightarrow 1 \text{ as } n_0 \longrightarrow \infty .$$

Considering the coefficients of $\sum_{r=0}^{n} a_r X_r x^r = 0$ as symmetric stable variables Samal and Mishra [13] have shown that

$$P\{(N_n/D_n^*) < \mu\} > 1 - \frac{\mu'}{\{\log(k_n/t_n)\log n\}(\log n)^{\alpha-1}} \text{ if } 1 \leqq \alpha < 2$$

and

$$> 1 - \frac{\mu' \log(k_n/t_n)\log n}{\log n} \text{ if } \alpha = 2 ,$$

where $k_n = \max_{0 \leqq r \leqq n} |a_r|$, $t_n = \min_{0 \leqq r \leqq n} |a_r|$ and $D_n^* = (\log n / \log ((k_n/t_n)\log n))$. Samal and Mishra [13] have studied the strong version of the above theorem and have shown that $P\{\sup_{n>n_0}(N_n/D_n^*) < \mu\}$

$$> 1 - \frac{\mu}{\{\log(\log n_0/\log(k_{n_0}/t_{n_0})\log n_0)\}^{\alpha-1}} \text{ where } \alpha > 1 .$$

Mishra and Nayak [9] have proved that

$$P\{(N_n/D_n^*) < \mu\} > 1 - \frac{\mu'}{\{\log((k_n/t_n)\log n)\}(\log n)^{1-\varepsilon}}$$

for every positive $\varepsilon < 1$, when the coefficients belong to the domain of attraction of the normal law.

Object of this paper is to show that

$$P\{\sup_{n>n_0}(N_n/D_n) < \mu\} > 1 - \mu'\left\{\frac{\log((k_{n_0}/t_{n_0})\log\log n_0)}{\log n_0}\right\}^{1/2}$$

for $0 < \varepsilon < 1$, when the coefficients belong to the domain of attraction of the normal law. Therefore it is a strong result of Mishra and Nayak.

Throughout this paper we shall denote μ's for positive constants which may assume different values in different occurences and $V(\cdot)$ for the variance of a random variable.

2. In the sequel we shall need the following definition, and theorem due to Karamata, (cf. Ibragimov and Linnik [6] p. 394), for the proof of our main result.

DEFINITION. A function H: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ is called a slowly varying function if

$$\lim_{x\to\infty} \frac{H(\gamma x)}{H(x)} = 1, \ (\gamma > 0) . \tag{2.1}$$

We have a few characterization of the slowly varying functions due to Karamata.

By writting $H(1/t) = h(t)$, we may define a slowly varying function $h\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ with the property that

$$\lim_{x\to 0} \frac{h(\gamma x)}{h(x)} = 1,\ (\gamma > 0)\ . \tag{2.2}$$

With this the Karamata theorem, (cf. Ibragimov and Linnik [6], p. 394), may be stated as follows.

THEOREM 1. *A slowly varying function h with the property (2.2) which is integrable on any finite interval may be represented in the form*

$$h(x) = c(x)\exp\left(-\int_a^x \frac{\bar{\varepsilon}(u)}{u}du\right),$$

where

$$\lim_{x\to 0} c(x) = c \neq 0,\quad \lim_{x\to 0}\bar{\varepsilon}(x) = 0 \text{ and } a > 0\ .$$

We establish the following formulae which will be necessary for the proof of the main theorem.

Let a sequence of independent and identically distributed random variables $\{X_r\}$ with mean zero belong to the domain of attraction of the normal law. Then their common characteristic function $\phi(t)$ is given by (cf. Ibragimov and Linnik [6], p. 91),

$$\log\phi(t) = -\frac{t^2}{2}H(|t|^{-1})(1 + o(1))\ , \tag{2.3}$$

where $H(t)$ is a slowly varying function as $t \to \infty$ and is given by the formula

$$H(x) = -\int_0^x u^2 d\psi(x) = \int_{-x}^x u^2 dG(u)\ , \tag{2.4}$$

where $\Psi(x) = 1 - G(x) + G(x)$ and $G(x)$ is the common distribution function.

Also

$$|\phi(t)| \sim \exp\left\{-\frac{t^2}{2}H(|t|^{-1})\right\}\ . \tag{2.5}$$

If we put $H(1/t) = L(t)$, then $L(t)$ is slowly varying as $t \to 0$. Then (2.3) and (2.5) will take the forms

$$\log\phi(t) = -\frac{t^2}{2}L(|t|)(1 + o(1))$$

and

$$|\phi(t)| \sim \exp\left\{-\frac{t^2}{2}L(|t|)\right\}$$

respectively. Since $L(|t|)$ is positive we can write the characteristic function ϕ as

$$\phi(t) = \exp\left\{-\frac{t^2}{2}h(t)\right\} \tag{2.6}$$

where $h(t) = L(|t|)(1 + o(1))$ with the property

$$h(t) = \operatorname{Re} h(t)(1 + o(1)) , \tag{2.7}$$

as

$$\operatorname{Re} h(t) = L(|t|)\ (1 + o(1)) .$$

Now $h(t)$ is slowly varying as $t \to 0$, since for $\gamma > 0$,

$$\lim_{t\to 0}\frac{h(\gamma(t))}{h(t)} = \lim_{t\to 0}\frac{L(\gamma|t|)(1 + o(1))}{L(|t|)(1 + o(1))} = 1 .$$

Consider the function $h_1(t)$ determined by

$$h_1(t) = \begin{cases}\operatorname{Re} h(t) \text{ if } V(X_r) = \infty , \\ \sigma^2 \text{ if } V(X_r) = \sigma^2 < \infty .\end{cases}$$

Clearly $h_1(t)$ is slowly varying in a neighborhood of the origin. By (2.7),

$$h(t) = h_1(t)\ (1 + o(1)), \text{ in both cases as } t \longrightarrow 0 . \tag{2.8}$$

Since expectation is zero, by virtue of (2.4), we have

$$\lim_{x\to\infty} H(x) = \int_{-\infty}^{\infty} u^2 dG(u) = \sigma^2 .$$

Therefore when variance is infinite, $\lim_{x\to\infty} H(x) = \infty$, so that $\lim_{t\to 0} L(t) = \infty$. Thus we have for infinite variance,

$$\lim_{t\to 0} h_1(t) = \infty . \tag{2.9}$$

THEOREM 2. *Let*

$$f(x) = \sum_{r=0}^{n} a_r X_r x^r$$

be a polynomial of degree n, where X_r's are independent and identically distributed random variables which belong to the domain of attraction of the normal law, have zero means and $P\{X_r \neq 0\} > 0$. Let $a_0, a_1, a_2 \cdots a_n$ be nonzero real number such that $(k_n/t_n) = o\,(\log n)$

where $k_n = \max_{0 \leq r \leq n} |a_r|$ *and* $t_n = \min_{0 \leq r \leq n} |a_r|$. *Then there exists a positive* n_0 *such that the number of real roots of* $f(x) = 0$ *is at least* $\mu\{\log n / \log ((k_n/t_n) \log \log n)\}^{1/2}$ *outside a set of measure at most* $\mu'\{\log ((k_{n_0}/t_{n_0}) \log \log n_0)/\log n_0\}^{(1-\varepsilon)/2}$ *for* $n > n_0$ *and* $1 > \varepsilon > 0$.

3. Proof of the Theorem 2.

Take constants A and D such that

$$0 < D < 1 \quad \text{and} \quad A > 1 . \tag{3.1}$$

Let

$$\lambda_m = m \log \log n . \tag{3.2}$$

Let

$$M_n = [d^2(\log \log n)^2(k_n/t_n)^2(\sqrt{2} + 1)^2(Ae/D)] + 1 , \tag{3.3}$$

where b is a positive constant greater than one whose choice will be made later and $[x]$ denotes the greatest integer not exceeding x.

It follows from (3.3) that

$$\mu_1\left(\frac{k_n}{t_n} \log \log n\right)^2 \leqq M_n \leqq \mu_2 \left(\frac{k_n}{t_n} \log \log n\right)^2 . \tag{3.4}$$

We define

$$\phi(x) = x^{[\log x]+x} . \tag{3.5}$$

Let k be the integer determined by

$$\phi(8k + 7)M_n^{8k+7} \leqq n < \phi(8k + 11)M_n^{8k+11}. \tag{3.6}$$

The first inequality of (3.5) gives

$$\log \phi(8k + 7) + (8k + 7) \log M_n \leqq \log n ,$$

or

$$(8k + 7) \log M_n < \log n ,$$

which by help of (3.4) yields

$$k < \frac{\mu \log n}{\log \left(\frac{k_n}{t_n} \log \log n\right)} .$$

Again the right hand side inequality of (3.4) gives

$$\begin{aligned}
\log n &< \log \phi(8k + 11) + (8k + 11) \log M_n \\
&= (\log (8k + 11) + 8k + 11) \log (8k + 11) + (8k + 11) \log M_n \\
&< 2(8k + 11)^2 + (8k + 11) \log M_n < \mu_3 k^2 \log M_n ,
\end{aligned}$$

whence by (3.4), we have

$$\mu_0\Big(\frac{\log n}{\log (k_n/t_n \log\log n)}\Big)^{1/2} < k\,.$$

Therefore

$$(3.7)\qquad \mu_0\Big(\frac{\log n}{\log (k_n/t_n \log\log n)}\Big)^{1/2} < k < \mu\frac{\log n}{\log (k_n/t_n \log\log n)}\,.$$

Since $(k_n/t_n) = o(\log n)$ by hypothesis, it follows from (3.7), that $k \to \infty$ as $n \to \infty$.

We have $f(x_m) = U_m + R_m$ at the points

$$(3.8)\qquad x_m = \Big(1 - \frac{1}{\phi(4m+1)M_n^{4m}}\Big)^{1/2}$$

for $m = [k/2] + 1,\ [k/2] + 2 \cdots k$, where

$$U_m = \sum_1 a_r X_r x_m^r$$

and

$$R_m = (\sum_2 + \sum_3) a_r X_r x_m^r\ ,$$

the index r ranging from $\phi(4m-1)M_n^{4m-1} + 1$ to $\phi(4m+3)M_n^{4m+3}$ in $\sum_1$, from 0 to $\phi(4m-1)M_n^{4m-1}$ in $\sum_2$ and from $\phi(4m+3)M_n^{4m+3} + 1$ to n in $\sum_3$. (We shall use the notations $\sum_1$, $\sum_2$ and $\sum_3$ to carry the above meaning throughout this paper.)

We have also

$$(3.9)\qquad f(x_{2m}) = U_{2m} + R_{2m},\ f(x_{2m+1}) = U_{2m+1} + R_{2m+1}\,.$$

By (3.7), we have $2k + 1 < n$ for large n. Also the maximum index in U_{2m+1} for $m = k$ is $\phi(8k+7)M_n^{8k+7}$, which by (3.6) is consistent with (3.9).

We define normalizing constants V_m starting from the relation

$$(3.10)\qquad (1/V_m^2)\sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / V_m)\ ,$$

where θ is a small positive number whose final choice will be dealt with later. Such normalizing constants V_m always exist when θ is sufficiently small. (Cf. Ibragimov and Maslova [7], p. 232.)

Now if $V(X_r) = \infty$, we have

$$V_m^2 = \sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / V_m) > \sum_1 a_r^2 x_m^{2r}$$

(by (2.9), since θ is small),

$$> t_n^2 \sum_{\phi(4m-1)M_n^{4m-1}+1}^{\phi(4m+1)M_n^{4m}} x_m^{2r}$$

$$> t_n^2\{\phi(4m+1)M_n^{4m} - \phi(4m-1)M_n^{4m-1}\}\left\{1 - \frac{1}{\phi(4m+1)M_n^{4m}}\right\}^{\phi(4m+1)M_n^{4m}}$$

$$> t_n^2\phi(4m+1)M_n^{4m}(D/Ae) .$$

Or

$$M_n^{2m} < (Ae/D\,\phi(4m+1))^{1/2}(V_m/t_n) . \tag{3.11}$$

Again if $V(X_r) = \sigma^2 < \infty$, then

$$V_m^2 = \sigma^2 \sum_1 a_r^2 x_m^{2r}$$

$$> \sigma^2\phi(4m+1)M_n^{4m}(D/Ae) .$$

Or

$$M_n^{2m} < (Ae/D\,\phi(4m+1))^{1/2}(V_m/\sigma t_n) . \tag{3.12}$$

The following lemmas are necessary for the proof of the theorem.

LEMMA 1.

$$|\sum_2 a_r X_r x_m^r| < \lambda_m W_m ,$$

except for a set of measure at most $\mu/\lambda_m^{2-\varepsilon}$ *for* $\varepsilon > 0$, *where*

$$W_m^2 = \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta/W_m) . \tag{3.13}$$

Proof. The characteristic function of $(1/W_m)\sum_2 a_r X_r x_m^r$ is given by

$$\phi_m(t) = \exp\left(-\frac{t^2}{2}h_m(t)\right)$$

where

$$h_m(t) = (1/W_m^2)\sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta/W_m) . \tag{3.14}$$

We have by Theorem 1 for $|t| < \theta$,

$$h_1(a_r x_m^r t/W_m)(h_1(a_r x_m^r\theta/W_m))^{-1} = \frac{L(|a_r x_m^r t/W_m|)(1+o(1))}{L(|a_r x_m^r \theta/W_m|)(1+o(1))}$$

$$= \frac{c(|a_r x_m^r t/W_m|)(1+o(1))}{c(|a_r x_m^r\theta/W_m|)(1+o(1))} \exp\left\{\int_{|a_r x_m^r t/W_m|}^{|a_r x_m^r \theta/W_m|} \frac{\bar{\varepsilon}(u)}{u}\,du\right\} ,$$

where $\lim_{x\to 0} c(x) = c \neq 0$, $\lim_{x\to 0}\bar{\varepsilon}(x) = 0$. Again since $\lim_{u\to 0}\bar{\varepsilon}(u) = 0$, there exists a positive t_0 such that for $|t| < \theta < t_0^{-1}$ and $\varepsilon > 0$, $|\bar{\varepsilon}(u)| < \varepsilon$. Thus we have

$$h_1(a_r x_m^r t/W_m) \leqq \left|\frac{t}{\theta}\right|^{-\varepsilon} h_1(a_r x_m^r \theta/W_m) .$$

Now

$$\begin{aligned}\operatorname{Re} h_m(t) &= (1/W_m^2) \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r t/W_m) \\ &\leqq |t/\theta|^{-\varepsilon}(1/W_m^2) \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta/W_m) \leqq |t/\theta|^{-\varepsilon} \\ &\quad \text{(by (3.13))} .\end{aligned}$$

But by (2.7), $h_m(t) = \operatorname{Re} h_m(t)(1 + o(1))$ as $t \to 0$. Therefore for $|t| < t_0^{-1}$ and $\varepsilon > 0$, we have

$$|h_m(t) < \mu_1 |t|^{-\varepsilon} .$$

Thus in a neighborhood of zero,

$$|\phi_m(t) - 1| = \left|\exp\left\{-\frac{t^2}{2} h_m(t)\right\} - 1\right| \leqq \mu_1 |t|^{2-\varepsilon} . \tag{3.15}$$

By Gnedenko and Kolmogorov [5],

$$\begin{aligned}P\{|\sum_2 a_r X_r x_m^r| > \lambda_m W_m\} &< 2 - \left|(\lambda_m/2)\int_{-2/\lambda_m}^{2/\lambda_m} \phi_m(t)dt\right| \\ &\leqq (\lambda_m/2)\int_{-2/\lambda_m}^{2/\lambda_m} |\phi_m(t) - 1|dt \leqq \lambda_m \mu_1 \int_0^{2/\lambda_m} |t|^{2-\varepsilon} dt \quad \text{(by (3.15))} , \\ &\leqq \mu/\lambda_m^{2-\varepsilon} .\end{aligned}$$

Hence the result.

Adopting the above procedure we can also prove the following lemma.

LEMMA 2.

$$|\sum_3 a_r X_r x_m^r| < \lambda_m Z_m ,$$

except for a set of measure at most $\mu/\lambda_m^{2-\varepsilon}$ *where*

$$Z_m^2 = \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta/Z_m) .$$

Now we proceed to estimate R_m. By virtue of Lemma 1 and Lemma 2, we have

$$|R_m| < \lambda_m(W_m + Z_m) ,$$

for sufficiently large value of m.

Now if $V(X_r) = \infty$, we have

$$|R_m| < \lambda_m k_n d\{(\sum_2 x_m^{2r})^{1/2} + (\sum_3 x_m^{2r})^{1/2}\} , \tag{3.16}$$

where

$$d = \max_{0 \leqq r \leqq n} \{(h_1(a_r x_m^r \theta / W_m))^{1/2},\ (h_1(a_r x_m^r \theta / Z_m))^{1/2}\} .$$

We have

$$\begin{aligned} \frac{\phi(4m+3)}{\phi(4m+1)} &= \frac{(4m)^{[\log(4m+3)]+4m+3}(1+3/4m)^{[\log(4m+3)]+4m+3}}{(4m)^{[\log(4m+1)]+4m+1}(1+1/4m)^{[\log(4m+1)]+4m+1}} \\ &> (4m)^{\log(4m+3/4m+1)+2} = 16m^2(4m)^{\log(4m+3/4m+1)} > m^2 . \end{aligned}$$

Therefore

$$\phi(4m+3) > m^2\phi(4m+1) \tag{3.17}$$

and similarly

$$\phi(4m+1) > m^2\phi(4m-1) . \tag{3.18}$$

Now

$$\begin{aligned} \sum_2 x_m^{2r} &< 1 + \phi(4m-1)M_n^{4m-1} < 2\phi(4m-1)M_n^{4m-1} \\ &< (2/m^2)\phi(4m+1)M_n^{4m-1} \text{ (by (3.18))} , \end{aligned} \tag{3.19}$$

and

$$(\sum_3 x_m^{2r}) < (\sum_{m^2\phi(4m+1)M_n^{4m+1}} x_m^{2r})$$

$$\text{(since by (3.17), } m^2\phi(4m+1) < m^2\phi(4m+3)) ,$$

$$\begin{aligned} &= \phi(4m+1)M_n^{4m}\left\{M_n 1 - \frac{1}{\phi(4m+1)M_n^{4m}}\right\}^{m2\phi(4m+1)M_n^{4m+1}} \\ &< \phi(4m+1)M_n^{4m}e^{-m^2M_n} < \phi(4m+1)M_n^{4m}(m^2M_n)^{-1} \text{ (since } e^{-x}<x^{-1}) , \end{aligned}$$

$$= (1/m^2)\phi(4m+1)M_n^{4m-1} . \tag{3.20}$$

Hence by (3.19) and (3.20) we have from (3.16),

$$\begin{aligned} |R_m| &< d\lambda_m \frac{(\sqrt{2}+1)}{m}\{\phi(4m+1)\}^{1/2}(M_n^{2m}/M_n^{1/2}) \\ &< \frac{d(\sqrt{2}+1)(Ae/D)^{1/2}(k_n/t_n)\log\log n\, V_m}{M_n^{1/2}} \end{aligned}$$

$$\text{(by (3.2) and (3.11))} ,\quad < V_m \text{ (by (3.3))}.$$

Again if $V(X_r) = \sigma^2 < \infty$, then

$$|R_m| < \lambda_m \sigma\{\sum_2 x_m^{2r})^{1/2} + (\sum_3 x_m^{2r})^{1/2}\}$$
$$< \frac{\log\log n(\sqrt{2}+1)(D/Ae)^{1/2}(k_n/t_n)V_m}{M_n^{1/2}}$$

(by (3.2) and (3.12)),

$$< \frac{d(\sqrt{2}+1)(k_n/t_n)\log\log n\, V_m}{M_n^{1/2}} . \quad \text{(since } d > 1.) \quad < V_m .$$

Since $k \to \infty$ as $n \to \infty$, it follows that when n is sufficiently large

$$|R_m| < V_m ,$$

for $m = [k/2]+1, [k/2]+2, \cdots, k$, except for a set of measure at most

(3.21) $$(\mu/\lambda_m^{2-\varepsilon}) .$$

Thus we have $|R_{2m}| < V_{2m}$ and $|R_{2m+1}| < V_{2m+1}$ for $m = m_0, m_0+1, \cdots, k$, where $m_0 = [k/2]+1$.

The measure of the exceptional set is at most

(3.22) $$(\mu'/\lambda_{2m}^{2-\varepsilon}) + (\mu'/\lambda_{2m+1}^{2-\varepsilon}) < (\mu'/\lambda_m^{2-\varepsilon}) .$$

Again we proceed to estimate

$$\begin{aligned} P^* &= P\{U_{2m} > V_{2m}, U_{2m+1} < -V_{2m+1}\} \cup \{U_{2m} < -V_{2m}, U_{2m+1} > V_{2m+1}\} \\ &= P\{U_{2m} > V_{2m}\}P\{U_{2m+1} < -V_{2m+1}\} \\ &\quad + P\{U_{2m} < -V_{2m}\}P\{U_{2m+1} > V_{2m+1}\} . \end{aligned}$$

Let $G_m(x)$ and $g_m(t)$ be the distribution function and the characteristic function of (U_m/V_m) respectively. Then

$$g_m(t) = \exp\left\{\frac{t^2}{2}\frac{1}{V_m^2}\sum_1 a_r^2 x_m^{2r} h(a_r x_m^r t/V_m)\right\} .$$

Let

(3.23) $$F(x) = \int_{-\infty}^{x} \exp(-u^2/2)du .$$

It follows from (3.11) and (3.12) that $V_m \to \infty$ as $m \to \infty$ and then $(a_r x_m^r t/V_m) \to 0$. Therefore when $m \to \infty$ we have by (2.8),

$$h(a_r x_m^r t/V_m) = h_1(a_r x_m^r t/V_m)\ (1 + o(1))$$

and by Theorem 1, it can be shown that

$$h_1(a_r x_m^r t/V_m) = \|\theta/t\|^{o(1)} h_1(a_r x_m^r \theta/V_m)\ (1 + o(1))$$

and as such

$$g_m(t) = \exp\left\{-\frac{t^2}{2}\frac{1}{V_m^2}\sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / V_m)\left|\frac{\theta}{t}\right|^{o(1)}(1+o(1))(1+o(1))\right\}$$

$$= \exp\left\{\frac{|t|^{2-o(1)}}{2}\left|\frac{\theta}{t}\right|^{o(1)}(1+o(1))\right\} \text{ (by (3.10))} .$$

Therefore as $m \to \infty$, $g_m(t) \to \exp(-t^2/2)$ uniformly in any bounded interval of t-values. Hence

$$\sup_x |G_m(x) - F(x)| = o(1) . \tag{3.24}$$

Then we have for $\varepsilon > 0$,

$$|G_m(-1) - F(-1)| < \varepsilon \tag{3.25}$$

and

$$|G_{2m+1}(-1) - F(-1)| < \varepsilon . \tag{3.26}$$

By (3.25) and (3.26), we have

$$P\{U_{2m} < -V_{2m}\} > F(-1) - \varepsilon$$

and

$$P\{U_{2m+1} < -V_{2m+1}\} > F(-1) - \varepsilon .$$

In the similar way using (3.24) we can show that

$$P\{U_{2m} > V_{2m}\} > 1 - F(1) - \varepsilon$$

and

$$P\{U_{2m+1} > V_{2m+1}\} > 1 - F(1) - \varepsilon .$$

Therefore $P^* > 2(F(1) - \varepsilon)(1 - F(1) - \varepsilon)$. Thus P^* is greater than a quantity which tends to $2F(-1)(1 - F(1))$ as $m \to \infty$ with n. This limit being positive we conclude that

$$P^* > \delta > 0 \text{ for all large } m . \tag{3.27}$$

Now we define events E_m and F_m as follows:

$$E_m = \{U_{2m} > V_{2m},\ U_{2m+1} < -V_{2m+1}\} ,$$
$$F_m = \{U_{2m} < -V_{2m},\ U_{2m+1} > V_{2m+1}\} .$$

By (3.27), we have

$$P\{E_m \cup F_m\} > \delta > 0 .$$

Let $P\{E_m \cup F_m\} = \delta_m$, so that $\delta_m > \delta > 0$.

Let y_m be the random variable such that it takes value 1 on $E_m \cup F_m$ and 0 elsewhere. In otherwords,

$$y_m = \begin{cases} 1 \text{ with probability } \delta_m, \\ 0 \text{ with probability } 1 - \delta_m . \end{cases}$$

The y_m's are thus independent random variables with $E(y_m) = 0$ and $V(y_m) = \delta_m - \delta_m^2 < 1$. We write

$$z_m = \begin{cases} 0 \text{ if } |R_{2m}| < V_{2m} \text{ and } |R_{2m+1}| < V_{2m+1} \\ 1 \text{ otherwise} . \end{cases}$$

Moreover, we have $f(x_{2m}) = U_{2m} + R_{2m}$ and $f(x_{2m+1}) = U_{2m+1} + R_{2m+1}$. Let $\alpha_m = y_m - y_m z_m$. Now $\alpha_m = 1$ only if $y_m = 1$ and $z_m = 0$, which implies the occurrence of one of the events

$$\begin{array}{ll} \text{(i)} & U_{2m} > V_{2m},\ |R_{2m}| < V_{2m} ; \\ & U_{2m+1} < -V_{2m+1},\ |R_{2m+1}| < V_{2m+1} , \\ \text{(ii)} & U_{2m} < -V_{2m},\ |R_{2m}| < V_{2m} ; \\ & U_{2m+1} > V_{2m+1},\ |R_{2m+1}| < V_{2m+1} . \end{array}$$

It is obvious that (i) implies $f(x_{2m}) > 0$ and $f(x_{2m+1}) < 0$, and (ii) implies that $f(x_{?m}) < 0$ and $f(x_{2m+1}) > 0$. Thus if $\alpha_m = 1$, there is a root of the polynomial in the interval (x_{2m}, x_{2m+1}). Hence the number of roots in (x_{2m_0}, x_{2k+1}) must exceed $\sum_{m=m_0}^{k} \alpha_m$.

We appeal to the strong law of large numbers in the following form. The technique has been earlier used by Evans [4], Samal and Mishra [12] and [13].

Let $y_1, y_2, \cdots$, be a sequence of independent random variables identically distributed with $V(y_i) < 1$ for all i, then for each $\varepsilon > 0$,

$$P\left\{ \sup_{k \geqq k_0} \left| \frac{1}{k} \sum_{i=1}^{k} (y_i - E(y_i)) \right| > \varepsilon \right\} < B/\varepsilon^2 k_0 , \tag{3.28}$$

where B is a positive constant.

In the present case,

$$\begin{aligned} \left| \sum_{m=m_0}^{k} (\alpha_m - E(y_m)) \right| &\leqq \left| \sum_{m=m_0}^{k} (y_m - E(y_m)) \right| + \left| \sum_{m=m_0}^{k} y_m z_m \right| \\ &\leqq \left| \sum_{m=m_0}^{k} (y_m - E(y_m)) \right| + \left| \sum_{m=m_0}^{k} z_m \right| \quad (\text{since } y_m \leqq 1) . \end{aligned} \tag{3.29}$$

Since $E(z_m) = 1 \cdot P\{z_m = 1\} < P\{|R_m| > V_m\}$ we have from (3.21),

$$E(z_m) < \mu/\lambda_m^{2-\varepsilon} . \tag{3.30}$$

Now we have

$$P\left\{ \sum_{m=m_0}^{k} z_m \geqq (k - m_0 + 1)\varepsilon_1 \right\} < \mu/\lambda_{m_0}^{2-\varepsilon} .$$

Hence outside an exceptional set of measure at most

$$\sum_{(k-m_0+1)\geqq k_0} (\mu/\lambda_{m_0}^{2-\varepsilon}) ,$$

we have

$$\sup_{(k-m_0+1)\geqq k_0} (1/(k - m_0 + 1)) \sum_{m=m_0}^{k} z_m < \varepsilon_1 ;$$

and therefore,

$$\sup_{(k-m_0+1)k\geqq k_0} (1/(k - m_0 + 1))\left| \sum_{m=m_0}^{k} (\alpha_m - E(y_m))\right|$$
$$\leqq \sup_{(k-m_0+1)\geqq k_0} (1/(k - m_0 + 1))\left| \sum_{m=m_0}^{k} (y_m - E(y_m))\right| + \varepsilon_1 .$$

Now by using strong law of large numbers,

$$P\left\{ \sup_{(k-m_0+1)\geqq k_0} \left|(1/(k - m_0 + 1)) \sum_{m=m_0}^{k} (\alpha_m - E(y_m))\right| > \varepsilon\right\}$$
$$< B/(\varepsilon - \varepsilon_1)^2 k_0 = \mu/k_0 .$$

By (3.1)

$$\lambda_{m_0}^{2-\varepsilon} = (m_0 \log\log n)^{2-\varepsilon} .$$

For large n, $m_0 \log\log n > m_0$, and therefore

$$\sum (\mu/\lambda_{m_0}^{2-\varepsilon}) < \sum (\mu/m_0^{2-\varepsilon}) .$$

Hence outside a set S_{k_0}, where

$$P(S_{k_0}) < \mu/k_0 + \sum_{(k-m_0+1)\geqq k_0} (\mu/m_0^{2-\varepsilon}) , \tag{3.31}$$

we have

$$(1/(k - m_0 + 1))\left| \sum_{m=m_0}^{k} (\alpha_m - E(y_m))\right| < \varepsilon . \tag{3.32}$$

Also

$$E(y_m) = \delta_m > \delta .$$

Therefore,

$$N_n > \sum_{m=m_0}^{k} \alpha_m > \sum_{m=m_0}^{k} \delta - (k - m_0 + 1)\varepsilon > (k - [k/2])$$
$$> \mu(\log n/\log ((k_n/t_n) \log\log n))^{1/2} \text{ (by (3.7))} ,$$

for all k such that $k - m_0 + 1 > k_0$, or in otherwords for all $n > n_0$. Now

$$P(S_{k_0}) < (\mu/k_0) + \mu \sum_{k\geqq(2k_0-1)} (1/m_0)^{2-\varepsilon}$$
$$= \frac{\mu}{k_0} + \mu \left\{\frac{1}{k_0^{2-\varepsilon}} + 2\left(\frac{1}{k_0^{2-\varepsilon} + 1} + \frac{1}{k_0^{2-\varepsilon} + 2} + \cdots\right)\right\}$$
$$< (\mu/k_0) + 2\mu \sum_{k\geqq k_0} (1/k^{2-\varepsilon}) .$$

It can be easily shown that for $0 < \varepsilon < 1$,

$$\sum_{k \geqq k_0} (1/k^{2-\varepsilon}) < (1/(1-\varepsilon)k_0^{1-\varepsilon}) .$$

Hence

$$P(S_{k_0}) < (\mu/k_0) + (1/(1-\varepsilon)k_0^{1-\varepsilon}) < \mu_1/k_0^{1-\varepsilon}$$

(since by hypothesis $0 < \varepsilon < 1$, $k_0 > k_0^{(1-\varepsilon)/2}$),

$$< \mu'\{\log((k_{n_0}/t_{n_0}) \log\log n_0)/\log n_0\}^{1-\varepsilon} \text{ (by (3.7))} .$$

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