PACIFIC JOURNAL OF MATHEMATICS
Vol. 105, No 1, 1983

INDICATOR FUNCTIONS WITH LARGE FOURIER TRANSFORMS

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We consider the question of when the function

$$t \mapsto t\hat{1}_F(t)$$

is bounded, where 1_F is the indicator function of a compact set F in R and "$\hat{}$" denotes the Fourier transform.

We are concerned in this note with a question of P. R. Masani about the rate of decrease of certain Fourier transforms on the real line **R**. Throughout, all unexplained notation is as in [**1**]. For $f \in \mathfrak{L}_1(\mathbf{R})$, we write

$$(1) \qquad \hat{f}(t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(x)\exp(itx)\,dx \qquad (t \in \mathbf{R}).$$

(It is convenient to use $\exp(itx)$ in the integral in (1) in place of the equally common $\exp(-itx)$.)

Masani has asked whether or not there exist compact subsets F of **R** with Lebesgue measure $\lambda(F) > 0$ such that the function

$$(2) \qquad t \mapsto t\hat{1}_F(t)$$

is unbounded. By the Cantor-Bendixson theorem, we may suppose that F is perfect. For a bounded closed interval $[a, b] \subset \mathbf{R}$, the function $t\hat{1}_{[a,b]}(t)$ is

$$(3) \qquad -i(2\pi)^{-1/2}(\exp(ibt) - \exp(iat)),$$

which is trivially bounded. For $a = \inf F$ and $b = \sup F$, write $U = [a, b] \setminus F$ and get

$$(4) \qquad t\hat{1}_F(t) + t\hat{1}_U(t) = -i(2\pi)^{-1/2}(\exp(ibt) - \exp(iat)),$$

so that the function (2) is bounded if and only if the function

$$(5) \qquad t \mapsto t\hat{1}_U(t) = h_U(t)$$

is bounded. Thus Masani's problem is equivalent to the problem of finding bounded open subsets U of **R** whose complements contain no isolated points and for which the function h_U is unbounded.

We note a simple case in which h_U is bounded. Suppose that

$$(6) \qquad \lambda([\inf U, \sup U] \setminus U) = 0,$$

as happens for example if U is the union of the complementary intervals in [0, 1] of Cantor's ternary set. Then (4) and (6) give

$$(7)\quad t\cdot 0 + t\hat{1}_U(t) = -i(2\pi)^{-1/2}(\exp(i(\sup U)t) - \exp(i(\inf U)t)).$$

The same holds if U is the union of a finite family of open sets for each of which (6) holds.

We have no complete classification of the open subsets U of $\mathbf{R}$ for which the function h_U is bounded. However, there is one special case where the answer is clear, as a consequence of a theorem of L. H. Loomis [3].

Given a closed subset F of $\mathbf{R}$, let $P(F)$ be the set of all condensation points x of F (every neighborhood of x contains an uncountable subset of F). As is well known, $P(F)$ is perfect or void and $F\setminus P(F)$ is countable.

THEOREM A. *Suppose that the bounded open subset U of $\mathbf{R}$ is the union of a countably infinite family of non-abutting open intervals $\{]a_j, b_j[\}_{j=1}^{\infty}$ and that the boundary $\partial U = U^-\setminus U$ has an accumulation point outside of the perfect set $P(\partial U)$.*[1] *Then the function h_U is unbounded.*

Proof. For convenience we will use $\mathfrak{S}$, the usual space of rapidly decreasing complex-valued C^∞ functions on $\mathbf{R}$. The Fourier transformation (1) maps $\mathfrak{S}$ onto the corresponding space of functions on the dual line. The identity

$$(8)\qquad (g')\hat{}(t) = -it\hat{g}(t) \qquad (g\in\mathfrak{S})$$

is standard.

We now assume that h_U is bounded. We will ultimately obtain a contradiction. For all real-valued $g\in\mathfrak{S}$, (8) and Parseval's identity give

$$\begin{aligned}(9)\quad ih_U * \hat{g}(0) &= i(2\pi)^{-1/2}\int_{-\infty}^{\infty} t\hat{1}_U(t)\hat{g}(-t)\,dt\\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{1}_U(t)\overline{(-it\hat{g}(t))}dt\\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{1}_U(t)\overline{(g')\hat{}(t)}dt\\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} 1_U(x)g'(x)\,dx\\ &= (2\pi)^{-1/2}\sum_{j=1}^{\infty}\int_{a_j}^{b_j} g'(x)\,dx = (2\pi)^{-1/2}\sum_{j=1}^{\infty}\big(g(b_j)-g(a_j)\big).\end{aligned}$$

[1] Note that $P(\partial U)$ is void if and only if ∂U is countable. In this case any accumulation point of ∂U will serve our purpose.

Let f be any real-valued function in $\mathfrak{S}$ and let s be a fixed real number. Replace g in (9) by the function

$$t \to \hat{f} * \hat{g}(s+t).$$

The identity (9) becomes

$$(10)\quad ih_U * \hat{f} * \hat{g}(s)$$
$$= (2\pi)^{-1/2} \sum_{j=1}^{\infty} \left[f(b_j)g(b_j)\exp(ib_j s) - f(a_j)g(a_j)\exp(ia_j s)\right].$$

Now consider a point of accumulation x_0 of ∂U that does not lie in $P(\partial U)$. There is a real valued function f in $\mathfrak{S}$ such that $f(x_0) = 1$ and f vanishes in an open neighborhood V of the set $P(\partial U)$. We choose and fix such a function f. Let g be a real-valued function in $\mathfrak{S}$ that vanishes on $\partial U \setminus V$. For such a function g, the function fg vanishes at all of the points a_j and b_j, as a moment's thought shows. Thus the identity (10) shows that

$$h_U * \hat{f} * \hat{g} = 0.$$

For each x not in $\partial U \setminus V$, we can define the real-valued function g in $\mathfrak{S}$ so that $g(x) = 1$ and so that g vanishes on $\partial U \setminus V$. Therefore the spectrum of the function $h_U * \hat{f}$ is contained in the countable closed set $\partial U \setminus V$, which is contained in $\partial U \setminus P(\partial U)$. Loomis ([**3**], Theorem 4) has shown that a bounded measurable function on a locally compact Abelian group G whose spectrum is compact and contains no nonvoid perfect subset is almost periodic. (For the present case, $G = \mathbf{R}$, these are exactly the functions in $\mathfrak{L}_\infty(\mathbf{R})$ with bounded countable spectrum.) Therefore the function $h_U * \hat{f}$ is continuous and almost periodic for all functions f of the form described above.

Now let

$$t \to \sum_{k=1}^{n} \mu_k \exp(ic_k t) = p_f(t)$$

be a trigonometric polynomial on $\mathbf{R}$ such that

$$(11)\qquad \left\| h_U * \hat{f} - p_f \right\|_\infty < \tfrac{1}{4}.$$

Computing a convolution at 0, we use (11) and (10) to infer that

$$(12)\qquad \tfrac{1}{4}\|g\|_1 \geq \left| \left(h_U * \hat{f} - p_f \right) * \hat{g}(0) \right|$$
$$= \left| \sum_{j=1}^{\infty} \left[f(b_j)g(b_j) - f(a_j)g(a_j)\right] - \sum_{k=1}^{n} \mu_k g(c_k) \right|.$$

Since $f(x_0) = 1$, there is an open neighborhood W of x_0 with compact closure such that $|f(x)| \geq \frac{3}{4}$ for all $x \in W^-$. Plainly W^- and $P(\partial U)$ are disjoint. Since $W \cap (\partial U)$ is (countably) infinite and disjoint from $P(\partial U)$, it contains a point x_1 of ∂U that is isolated in ∂U and is different from all of the points $c_1, c_2, \ldots, c_n$. Note that x_1 cannot be x_0 and that the only possible isolated points of ∂U are endpoints a_j and b_j of the component intervals of U.

Suppose that we have a real-valued function g in $\mathfrak{S}$ such that $g(x_1) = 1$, g vanishes in a neighborhood of the compact set $(\partial U \setminus \{x_1\}) \cup \{c_1, c_2, \ldots, c_n\}$, and $\|\hat{g}\|_1 = 1$. Put this g into formula (12). Since f vanishes on $P(\partial U)$ and g vanishes on ∂U except at x_1, the only surviving term in the second line of (12) is $\pm f(x_1)g(x_1)$. Since $|f(x_1)| \geq \frac{3}{4}$ by construction, (12) yields

$$\frac{1}{4} \geq |f(x_1)g(x_1)| \geq \frac{3}{4}|g(x_1)| = \frac{3}{4}, \tag{13}$$

a contradiction. Therefore the function h_U is unbounded.

To finish the proof, we need only to find a function with the properties ascribed to g in the preceding paragraph. This is standard save for the requirement that g be in $\mathfrak{S}$. Imitating the standard construction, we suppose first that $x_1 = 0$. Let δ be any positive real number, and take ψ to be an even nonnegative C^∞ function with support $[-\frac{1}{2}\delta, \frac{1}{2}\delta]$ for which

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} \psi^2(x)\,dx = 1.$$

Define g as the convolution $\psi * \psi$. Plainly g is in $\mathfrak{S}$ and has support $[-\delta, \delta]$. Since ψ is real-valued, we have

$$g(0) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \psi(x)\psi(-x)\,dx = 1$$

and

$$\begin{aligned} \|\hat{g}\|_1 &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \hat{g}(t)\,dt = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \hat{\psi}(t)^2\,dt \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \psi^2(x)\,dx = 1. \end{aligned}$$

For $x_1 \neq 0$, use the translated function $x \to g(-x_1 + x)$, whose support is $[x_1 - \delta, x_1 + \delta]$ and whose Fourier transform at t is $\exp(ix_1t)\hat{g}(t)$. □

REMARKS. Let $(\gamma_j)_{j=1}^{\infty}$ be any bounded sequence of complex numbers such that $(|\gamma_j|)_{j=1}^{\infty}$ is bounded away from zero. Consider the function

$$\varphi = \sum_{j=1}^{\infty} \gamma_j 1_{]a_j, b_j[}, \tag{14}$$

where the open set $U = \bigcup_{j=1}^{\infty}]a_j, b_j[$ satisfies the hypotheses of Theorem A. The proof of Theorem A can be repeated with an obvious modification in (11) to prove that the function

$$t \to t\hat{\varphi}(t)$$

is unbounded. If f is a continuous function on $\mathbf{R}$ such that f' exists except possibly at a countable set of points and if both f and f' are in $\mathfrak{L}_1(\mathbf{R})$, then f is absolutely continuous and

$$(f')\hat{}(t) = -i\hat{f}(t)$$

for all $t \in \mathbf{R}$. Thus the function

$$t \to t\hat{f}(t)$$

is not only bounded but is $o(1)$. Adding to such f any function φ of the form (14), we get more functions g in $\mathfrak{L}_1(\mathbf{R})$ for which the function

$$t \to t\hat{g}(t)$$

is unbounded.

EXAMPLE A. Let $\{]a_j, b_j[\}_{j=1}^{\infty}$ be a countably infinite family of non-void, non-abutting open intervals in $\mathbf{R}$ and as above write U for the set $\bigcup_{j=1}^{\infty}]a_j, b_j[$. Suppose that U is bounded. It is easy to see that ∂U is the closure of the countable set $H = \{a_1, a_2, \ldots, a_n, \ldots\} \cup \{b_1, b_2, \ldots, b_n, \ldots\}$. If H^- is countable, then the open set U satisfies the hypotheses of Theorem A, since the perfect set $P(\partial U) = P(H^-)$ is void. A continuum of such open sets exist and can be constructed *ad libitum*. Thus open sets U for which h_U is unbounded exist in profusion.

EXAMPLE B. We now present a construction that is roughly the antithesis of Example A, in that the set H consists solely of isolated points, while the set $P(H^-)$ is equal to $H^- \setminus H$ and is homeomorphic to Cantor's ternary set. At the same time the function h_U is unbounded for this set U. Thus we will show that the hypotheses of Theorem A are not necessary in order for the function h_U to be unbounded.

For every positive integer n, let E_n be the set of all sequences $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ where each entry ε_j is either 1 or -1. Let C_n be the subset of E_n consisting of all $\boldsymbol{\varepsilon}$ with $\varepsilon_1 = 1$. For each $\boldsymbol{\varepsilon}$ in E_n, let $I(n, \boldsymbol{\varepsilon})$ be the open interval

$$\left] \sum_{j=1}^{n} \varepsilon_j 4^{-j} - \tfrac{1}{2}4^{-n-1}, \sum_{j=1}^{n} \varepsilon_j 4^{-j} + \tfrac{1}{2}4^{-n-1} \right[. \tag{15}$$

Let U be the union of all of the intervals $I(n, \boldsymbol{\varepsilon})$ as $\boldsymbol{\varepsilon}$ runs through all of the 2^n elements of E_n and n runs through the set of all positive integers.

We find that

$$I(n,\boldsymbol{\varepsilon}) \cap I(n',\boldsymbol{\varepsilon}') = \varnothing \tag{16}$$

unless $n = n'$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}'$. As in Example A, write H for the set of all endpoints of all of the intervals $I(n, \boldsymbol{\varepsilon})$. Let D be the set of all numbers of the form

$$\sum_{j=1}^{\infty} \beta_j 4^{-j},$$

where each β_j is either 1 or -1. We find that

$$D = H^{-} \setminus H = \partial U. \tag{17}$$

The details of proving (16) and (17) are simple enough but are also somewhat tedious, and we omit them. Note that

$$\sup U = \tfrac{1}{3}, \quad \inf U = -\tfrac{1}{3}, \quad \text{and} \quad \lambda(U) = \tfrac{1}{4}. \tag{18}$$

We now compute the function h_U.

Given an interval $]c-\gamma, c+\gamma[$ ($c \in \mathbf{R}$, $\gamma > 0$), we have

$$\exp(i(c+\gamma)t) - \exp(i(c-\gamma)t) = 2i\sin(\gamma t)\exp(ict). \tag{19}$$

For every positive integer n, (3) and (19) show that

$$\begin{aligned} \sum_{\boldsymbol{\varepsilon}\in E_n} t\hat{1}_{I(n,\boldsymbol{\varepsilon})}(t) &= \sum_{\boldsymbol{\varepsilon}\in E_n} 2\sin\left(\tfrac{1}{2}4^{-n-1}t\right)\exp\left(i\left(\sum_{j=1}^{n}\varepsilon_j 4^{-j}\right)t\right) \\ &= \sum_{\boldsymbol{\varepsilon}\in C_n} 2\sin\left(\tfrac{1}{2}4^{-n-1}t\right)\left[\exp\left(i\left(\sum_{j=1}^{n}\varepsilon_j 4^{-j}\right)t\right)\right. \\ &\qquad \left. + \exp\left(i\left(-\sum_{j=1}^{n}\varepsilon_j 4^{-j}\right)t\right)\right] \\ &= 2\sin\left(\tfrac{1}{2}4^{-n-1}t\right)\prod_{r=1}^{n}\left[\exp(i4^{-r}t) + \exp(-i4^{-r}t)\right] \\ &= 2^{n+1}\sin\left(\tfrac{1}{2}4^{-n-1}t\right)\prod_{r=1}^{n}\cos(4^{-r}t). \end{aligned} \tag{20}$$

Add (20) over all positive integers n to obtain

$$h_U(t) = \sum_{n=1}^{\infty} 2^{n+1}\sin\left(\tfrac{1}{2}4^{-n-1}t\right)\prod_{r=1}^{n}\cos(4^{-r}t). \tag{21}$$

For a given positive integer p, let us compute (21) for $t = 2\pi 4^{p}$. For $n = 1, 2, \ldots, p-1$, we have

$$\sin\left(\tfrac{1}{2}4^{-n-1}2\pi 4^{p}\right) = \sin(\pi 4^{p-n-1}) = 0. \tag{22}$$

For $n = p$, we have

$$(23) \qquad \sin\left(\tfrac{1}{2}4^{-p-1}2\pi 4^{p}\right) = \sin\left(\tfrac{1}{2}\pi\right) = 2^{-1/2}.$$

Also for $n = p$, we have

$$(24) \qquad \prod_{r=1}^{p} \cos(4^{-r}2\pi 4^{p}) = \prod_{r=1}^{p} \cos(2\pi 4^{p-r}) = 1.$$

For $n \geq p + 1$, we have

$$(25) \qquad \prod_{r=1}^{n} \cos(2\pi 4^{p-r}) = 0,$$

since

$$\cos(2\pi 4^{p-p-1}) = \cos\left(\tfrac{1}{2}\pi\right) = 0.$$

Combining (21)–(25), we see that

$$(26) \qquad h_U(2\pi 4^{p}) = 2^{p+1/2},$$

so that $h_U(t)$ is unbounded.

It is of some interest to examine the rate of growth of the function $h_U(t)$ for U's as in Theorem A.

EXAMPLE C. Let φ be any continuous nondecreasing function on $[1, \infty[$ such that $\lim_{t\to\infty} \varphi(t) = \infty$. We can find a bounded open set U such that $h_U(t)$ is unbounded and

$$h_U(t) = O(\varphi(|t|)).$$

To find such a set U, let $\psi = \psi(u)$ be the function defined on $[\varphi(1), \infty[$ such that: if φ assumes the value u at exactly one point t, then $\psi(u) = t$; if φ assumes the value u exactly in an interval $[a, b]$ with $a < b$, then $\psi(u) = b$. That is, ψ is as close to the inverse function of φ as one can get. It is plain that $\lim_{u\to\infty} \psi(u) = \infty$ and that ψ is strictly increasing.

It is easy to construct an infinite series $\sum_{n=1}^{\infty} r_n$ of positive terms such that

$$(27) \qquad \sum_{n=N+1}^{\infty} r_n = \frac{1}{\psi(N+1)}$$

for all positive integers N. Let $\{]a_n, b_n[\}_{n=1}^{\infty}$ be a set of open intervals with the following properties for all n:

$$a_n < b_n; \quad b_n - a_n = r_n; \quad b_{n+1} < a_n;$$

and

$$\lim_{n\to\infty} a_n = 0.$$

It is plain that $P(\partial U) = \varnothing$, and so by Theorem A the function $h_U(t)$ is unbounded. For every positive integer N, we have

$$(28)\qquad |h_U(t)| \leq \left| \sum_{n=1}^{N} (\exp(ib_n t) - \exp(ia_n t)) \right| + \left| \sum_{n=N+1}^{\infty} (\exp(ib_n t) - \exp(ia_n t)) \right| \leq 2N + |t| \sum_{n=N+1}^{\infty} (b_n - a_n) = 2N + |t| \frac{1}{\psi(N+1)}.$$

Given a real number t of absolute value at least 1, let N be the integer such that

$$N \leq \varphi(|t|) < N + 1.$$

This gives us

$$\psi(N) \leq \psi(\varphi(|t|)) < \psi(N+1).$$

By our definition of ψ, we have

$$|t| \leq \psi(\varphi(|t|)),$$

$$|h_U(t)| \leq 2\varphi(|t|) + \psi(N+1)\frac{1}{\psi(N+1)} = O(\varphi(|t|)).$$

Thus the function $h_U(t)$ can go to infinity arbitrarily slowly.

Finally we compute the exact rate of growth of the function $h_U(t)$ for the open set U of Example B. The equality (26) shows that

$$(29)\qquad |h_U(t)| \geq Ct^{1/2}$$

for arbitrarily large positive values of t. On the other hand, consider all of the intervals $I(n, \varepsilon)$ for $n \leq N$, N being an arbitrary positive integer. There are exactly $2^{N+1} - 1$ such intervals. The sum of the measures of all of the intervals $I(n, \varepsilon)$ for $n \geq N + 1$ is 2^{-N-2}. Accordingly, (28) shows that

$$(30)\qquad |h_U(t)| \leq 2(2^{N+1} - 1) + |t| 2^{-N-2}.$$

For a given t of absolute value at least 4, define N by

$$2^{2N+2} \leq |t| < 2^{2N+3}.$$

From (30) we get

$$|h_U(t)| \leq 2|t|^{1/2} + 2^{-1/2}|t|^{1/2},$$

so that

$$|h_U(t)| = O(|t|^{1/2}). \tag{31}$$

The estimates (29) and (31) show that $|h_U(t)| = O(|t|^\alpha)$ for $\alpha = \frac{1}{2}$ but for no smaller exponent α.

We are indebted to Professor Masani for the following remarks on the origin of his problem.

Question. Let $\mathfrak{X}$ be a complex Banach space, and let $\{(U(t)\colon t \in \mathbf{R}\}$ be a strongly continuous group of linear isometries of $\mathfrak{X}$ onto $\mathfrak{X}$ with infinitesimal generator A. For what bounded Borel subsets S of $\mathbf{R}$ is it the case that

$$\text{Range} \int_S U(t)\, dt \subset \text{Dom } A? \tag{32}$$

This question arises naturally in the theory of $\mathfrak{X}$-valued stationary measures over $\mathbf{R}$. See [**4**], page 303, Theorem 3.6. The inclusion (32) holds provided that S is a closed interval. This is proved in [**2**], §10.3, page 307. Thus (32) holds if S is a union of finitely many closed intervals.

Now suppose that $\mathfrak{X}$ is a Hilbert space. The problem of the inclusion (32) reduces to the problem of Masani stated in the second paragraph of this note. To see this, write

$$U(t) = \int_{\mathbf{R}} \exp(itx)\, d(E(x)), \quad \text{so that} \quad A = \int_{\mathbf{R}} ix\, d(E(x)).$$

It is then easy to see that

$$\int_S U(t)\, dt = \int_{\mathbf{R}} \hat{1}_S(x) d(E(x))$$

and that

$$A \int_S U(t)\, dt \subset \int_{\mathbf{R}} ix\hat{1}_S(x)\, d(E(x)). \tag{33}$$

Now (32) holds if and only if the operator on the left side of (33) is continuous on $\mathfrak{X}$, that is, if and only if the function $x \mapsto x\hat{1}_S(x)$ is E-essentially bounded on $\mathbf{R}$. It is also easy to see that a bounded Borel set S satisfies (32) for *all* $U(\cdot)$ if and only if the function $x \mapsto x\hat{1}_S(x)$ is bounded on $\mathbf{R}$. Thus finding the bounded Borel sets satisfying (32) yields the problem stated in the second paragraph of this note.

Finally we remark that Masani [**4**], page 304, Proposition 3.8, has proved a special case of Example A.

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Received July 7, 1981 and in revised form January 19, 1982. The research of all three authors was supported in part by the National Science Foundation, USA.

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