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REGULAR EMBEDDINGS OF A GRAPH

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In this paper we study embeddings of a graph G in Euclidean space R^n that are 'regular' in the following sense: given any two distinct vertices u and v of G, the distance between the corresponding points in R^n equals α if u and v are adjacent, and equals β otherwise. It is shown that for any given value of $s = (\beta^2 - \alpha^2)/\beta^2$, the minimum dimension of a Euclidean space in which G is regularly embeddable is determined by the characteristic polynomials of G and $\bar{G}$.

1. Introduction. To embed a graph in Euclidean spaces with various restrictions, and to find the minimum dimension of the space for these embeddings, are interesting problems [**1**], [**4**], [**5**]. In this paper we consider a regular embedding of a graph.

An embedding of a graph G in a Euclidean space R^n is called a *regular embedding* of G provided that, for any two distinct vertices u and v of G, the distance between the corresponding points in R^n equals α if u and v are adjacent, and equals β otherwise. The vertices of G are mapped onto distinct points of R^n, but there is no restriction on the crossing of edges. The value $s = (\beta^2 - \alpha^2)/\beta^2$ is called the *parameter* of the regular embedding. Let $\dim(G, s)$ denote the minimum number n such that G can be regularly embedded in R^n with parameter s.

Consider, for example, the circuit graph C_5. For every regular embedding of C_5, it is seen that

$$\tfrac{1}{2}\left(-\sqrt{5} - 1\right) \le s \le \tfrac{1}{2}\left(\sqrt{5} - 1\right)$$

and

$$\dim(C_5, s) = \begin{cases} 2 & \text{if } s = \frac{1}{2}\left(\pm\sqrt{5} - 1\right), \\ 4 & \text{otherwise.} \end{cases}$$

The 'critical' embeddings of C_5 in R^2 with $s = \frac{1}{2}(\pm\sqrt{5} - 1)$ are illustrated in Fig. 1.

Let $\phi(G; x)$ denote the characteristic polynomial of a graph G (that is, $\phi(G; x) = |x\mathbf{I} - \mathbf{A}(G)|$), and put

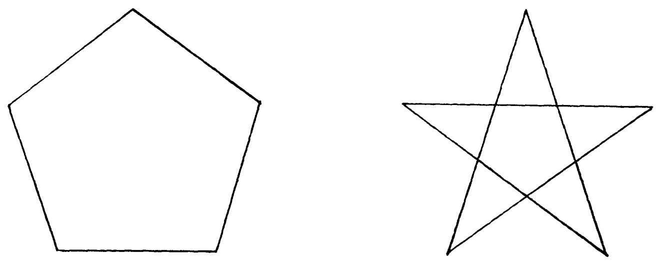
$$\Phi(G; x) = \phi(G; -x) - (-1)^g \phi(\bar{G}; x - 1),$$

where g is the number of vertices of G, and $\bar{G}$ is the complement of G. Let x^- and x^+ be, respectively, the minimum root and the maximum root of the polynomial $\Phi(G;\ x)$. Suppose that $x^- < 0$, and $1 < x^+$. Then our results are stated as follows.

For every regular embedding of G, $1/x^- \leq s \leq 1/x^+$ *and*

$$\dim(G, 1/x^*) = g - 1 - (\textit{the multiplicity of the root } x^*),$$

where $x^* = x^-$ *or* x^+. *For other values of* s, $\dim(G, s) = g - 1$.



$s = \frac{1}{2}(+\sqrt{5} - 1)$ $s = \frac{1}{2}(-\sqrt{5} - 1)$

FIGURE 1

2. A theorem for isometric embeddings. We shall recall a theorem in distance geometry ([2], Ch. IV). Let $S = \{p_0, \ldots, p_k\}$ be a finite semimetric space with distance function d. The determinant

$$\begin{vmatrix} 0 & 1 & \cdot & \cdots & 1 \\ 1 & 0 & d_{01} & \cdots & d_{0k} \\ \cdot & d_{10} & 0 & & \cdot \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & d_{k0} & \cdot & \cdots & 0 \end{vmatrix} \qquad d_{ij} := d(p_i, p_j)^2$$

is called the *Cayley-Menger determinant* of the semimetric space (S, d), and is denoted by $D(S)$ or by $D(p_0 \cdots p_k)$. Note that the value of the determinant does not depend on a labeling (ordering) $p_0, \ldots, p_k$ of the points of S.

If $S = \{p_0, \ldots, p_k\} \subset R^n$, $n \geq k$, then we denote by $\mathrm{Vol}(S)$ the k-dimensional volume of the simplex (perhaps degenerate) spanned by S. In this case, $\mathrm{Vol}(S)$ and the Cayley-Menger determinant of S are related as follows:

$$\mathrm{Vol}(S)^2 = \frac{(-1)^{k+1}}{2^k (k!)^2} D(S).$$

For details, see Blumenthal [2], p. 98.

A semimetric space S is said to be irreducibly embeddable in R^n provided that it is isometric to a subset of R^n but not isometric to any subset of R^{n-1}.

THEOREM (*Blumenthal* [2]). *A semimetric space S is irreducibly embeddable in R^n if and only if*

(i) *S contains an $(n+1)$-tuple $p_0,\dots,p_n$ such that*

$$\operatorname{sign} D(p_0 \cdots p_j) = (-1)^{j+1} \qquad (j = 1,\dots,n);$$

(ii) *for every pair x, y of points of S,*

$$D(p_0 \cdots p_n, x) = D(p_0 \cdots p_n, y) = D(p_0 \cdots p_n, x, y) = 0.$$

3. The Cayley-Menger polynomial of a graph. A regular embedding of a graph G with parameter s is called, briefly, an *s-embedding* of G. To apply Blumenthal's theorem let us define a distance function d_s on the vertex set $V(G)$ of G by

$$d_s(u, v) = \begin{cases} 0 & \text{if } u = v, \\ (1-s)^{1/2} & \text{if } u \text{ and } v \text{ are adjacent,} \\ 1 & \text{otherwise.} \end{cases}$$

Then the Cayley-Menger determinant of the semimetric space $(V(G), d_s)$ is a polynomial in s, which we shall call the *Cayley-Menger polynomial* of G and denote by $\mathrm{CM}(G; s)$. For example, $\mathrm{CM}(K_n; s) = (-1)^n n(1-s)^{n-1}$, and $\mathrm{CM}(\overline{K}_n; s) = (-1)^n n$, where K_n denotes the complete graph of order n.

Since there is a 0-embedding of G in a Euclidean space as a regular simplex of side-length 1, we can restate Blumenthal's theorem in the following way. For any two graphs G and H, let $H \subset G$ mean that H is an induced subgraph of G.

THEOREM 1. *There exists a t-embedding ($t < 1$) of a graph G in R^n if and only if there is a $G_0 \subset G$ with g_0 ($\leq n+1$) vertices such that*

(i) *for any $F \subset G_0$,* $\operatorname{sign} \mathrm{CM}(F; t) = \operatorname{sign} \mathrm{CM}(F; 0)$;

(ii) *for any $G_0 \subsetneqq H \subset G$,* $\mathrm{CM}(H; t) = 0$.

In this case, $\dim(G, t) = g_0 - 1$.

Let $s^+(G)$ be the minimum positive root of the polynomial $\mathrm{CM}(G; s)$, if it exists, and ∞ otherwise. For example, $s^+(K_2) = 1$, and $s^+(\overline{K}_n) = \infty$. Let $s^-(G)$ be the maximum negative root of $\mathrm{CM}(G; s)$, if it exists, and $-\infty$ otherwise.

LEMMA 1. *For* $H \subset G$, $s^-(H) \leq s^-(G) < s^+(G) \leq s^+(H)$.

Proof. We shall only show that $s^+(G) \leq s^+(H)$. Let s_0 be the minimum value of $s^+(F)$ for $F \subset G$. It is sufficient to show that $s_0 \geq s^+(G)$. If $s_0 = \infty$ then clearly $s_0 = s^+(G) = \infty$. Suppose $s_0 < \infty$ and $\mathrm{CM}(F_0; s_0) = 0$ for some $F_0 \subset G$. In this case, $s_0 \leq 1$, because $s^+(K_2) = 1$. Since $\operatorname{sign}\mathrm{CM}(F; s) = \operatorname{sign}\mathrm{CM}(F; 0)$ for $F \subset G$ and for $0 \leq s < s_0$, it follows from Theorem 1 that for every $0 \leq s < s_0$, there is an s-embedding $f_s\colon G \to R^n$ of G where $n + 1 \geq g := |V(G)|$, the cardinality of the vertex set $V(G)$ of G. Since $\mathrm{Vol}(f_s(V(F_0)))^2$ is the product of $\mathrm{CM}(F_0; s)$ by a constant, and $\mathrm{CM}(F_0; s_0) = 0$, we have

$$\mathrm{Vol}(f_s(V(F_0))) \to 0 \quad \text{as } s \to s_0.$$

Hence we have

$$\mathrm{Vol}(f_s(V(G))) \to 0 \quad \text{as } s \to s_0.$$

Then by the continuity, $\mathrm{CM}(G; s_0) = 0$, and hence $s_0 \geq s^+(G)$.
Note that if G contains at least one edge, then $s^+(G) \leq 1$.

THEOREM 2. *For every* $s^-(G) < s < \min(s^+(G), 1)$, *there is an s-embedding of G, and* $\dim(G, s) = g - 1$. *If* $-\infty < s^*(G) < 1$ *then there is an* $s^*(G)$*-embedding of G, where* $s^*(G) = s^-(G)$ *or* $s^+(G)$.

Proof. We shall only prove the existence of an $s^+(G)$-embedding of G, provided that $s^+(G) < 1$. Let H be a maximal induced subgraph of G such that $\mathrm{CM}(H; s^+(G)) \neq 0$. Then

(i) if $F \subset H$ then $s^+(G) < s^+(H) \leq s^+(F)$, and hence

$$\operatorname{sign}\mathrm{CM}(F; s^+(G)) = \operatorname{sign}\mathrm{CM}(F; 0);$$

(ii) if $H \subsetneqq F \subset G$, then $\mathrm{CM}(F; s^+(G)) = 0$ by the maximality of H. Hence there is an $s^+(G)$-embedding of G, by Theorem 1.

4. Calculation of $\mathrm{CM}(G; s)$. Let $\mathbf{I}_r$ and $\mathbf{J}_r$ denote, respectively, the identity $r \times r$ matrix and $r \times r$ matrix each entry of which is 1. (In the following, the subscripts are often omitted.) Put $\mathbf{K}_r = \mathbf{J}_r - \mathbf{I}_r$ and

$$\mathbf{B}(G) = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \mathbf{A}(G) & \\ 0 & & \end{bmatrix},$$

where $\mathbf{A}(G)$ is the adjacency matrix of G, and put $g = |V(G)|$. Then, by the definition of $\mathrm{CM}(G;s)$,

$$\mathrm{CM}(G;s) = |\mathbf{K}_{g+1} - s\mathbf{B}(G)| = |s\mathbf{K}|\,|(1/s)\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)|\,.$$

Since $\mathbf{K}^{-1} = (1/g)\mathbf{J} - \mathbf{I}$,

$$\begin{aligned}\mathbf{K}^{-1}\mathbf{B}(G) &= (1/g)\mathbf{J}\mathbf{B}(G) - \mathbf{B}(G)\\ &= \begin{bmatrix} 0 & d_1/g & \cdots & d_g/g \\ \vdots & & & \vdots \\ 0 & d_1/g & \cdots & d_g/g \end{bmatrix} - \mathbf{B}(G),\end{aligned}$$

where d_i is the sum of entries in the ith column of $\mathbf{A}(G)$. In the matrix $x\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)$, by subtracting the top row from other rows, we have

$$|x\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)| = \begin{vmatrix} x & -d_1/g & \cdots & -d_g/g \\ -x & & & \\ \vdots & & x\mathbf{I} + \mathbf{A}(G) & \\ -x & & & \end{vmatrix}.$$

On the right-hand side, adding to the top row the product of the ith row by $1/g$, $i = 2,\ldots,g+1$, we have

$$\begin{aligned}|x\mathbf{I} - \mathbf{K}^{-1}\mathbf{B}(G)| &= \begin{vmatrix} 0 & x/g & \cdots & x/g \\ -x & & & \\ \vdots & & x\mathbf{I} + \mathbf{A}(G) & \\ -x & & & \end{vmatrix}\\
&= -x^2/g \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & x\mathbf{I} + \mathbf{A}(G) & \\ 1 & & & \end{vmatrix}\\
&= -x^2/g \left\{ \begin{vmatrix} x & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & x\mathbf{I} + \mathbf{A}(G) & \\ 1 & & & \end{vmatrix} - x\,|x\mathbf{I} + \mathbf{A}(G)| \right\}\\
&= -x^2/g\{|x\mathbf{I}_{g+1} + \mathbf{A}(G + K_1)| - x\,|x\mathbf{I} + \mathbf{A}(G)|\}\end{aligned}$$

(where $G + K_1$ is the join of G and K_1, defined by $\overline{G + K_1} = \overline{G} \cup \overline{K}_1$)

$$\begin{aligned}&= -x^2/g\{(-1)^{g+1}\,|(-x)\mathbf{I} - \mathbf{A}(G + K_1)| \\ &\qquad\qquad + (-1)^{g+1}x\,|(-x)\mathbf{I} - \mathbf{A}(G)|\}\\
&= (-1)^g x^2/g\{\phi(G + K_1); -x) + x\phi(G; -x)\}.\end{aligned}$$

Using Cvetković's theorem ([3], p. 57):

$$\begin{aligned}\phi(G_1+G_2;x)&\\ &=(-1)^{g_2}\phi(G_1;x)\phi(\overline{G}_2;-x-1)\\ &\quad+(-1)^{g_1}\phi(G_2;x)\phi(\overline{G}_1;-x-1)\\ &\quad-(-1)^{g_1+g_2}\phi(\overline{G}_1;-x-1)\phi(\overline{G}_2;-x-1),\qquad g_i=|V(G_i)|.\end{aligned}$$

After a brief calculation, we have

$$|x\mathbf{I}-\mathbf{K}^{-1}\mathbf{B}(G)|=(-1)^g x^2/g\{\phi(G;-x)-(-1)^g\phi(\overline{G};x-1)\}.$$

Since $|(1/x)\mathbf{K}_{g+1}|=(-1)^g g(1/x)^{g+1}$, we have the following:

THEOREM 3.

$$\mathrm{CM}(G;1/x)=(1/x)^{g-1}\{\phi(G;-x)-(-1)^g\phi(\overline{G};x-1)\}.$$

5. Bounds on the parameter s. Put

$$\Phi(G;x)=\phi(G;-x)-(-1)^g\phi(\overline{G};x-1),$$

where g is the number of vertices of G. Then Theorem 3 says

$$\mathrm{CM}(G;s)=s^{g-1}\Phi(G;1/s).$$

Note that $s_0\neq 0$ is a root of $\mathrm{CM}(G;s)$ if and only if $1/s_0$ is a root of $\Phi(G;x)$. Thus we have the following theorem:

THEOREM 4. *The polynomial $\Phi(G;x)$ has a positive root if and only if $s^+(G)<\infty$. In this case, $1/s^+(G)$ is the maximum root of $\Phi(G;x)$. The polynomial $\Phi(G;x)$ has a negative root if and only if $s^-(G)>-\infty$. In this case, $1/s^-(G)$ is the minimum root of $\Phi(G;x)$.*

Now let $V(G)=\{v_1,\dots,v_g\}$, $g\geq 2$, and put

$$G_{i_1\cdots i_k}=G-v_{i_1}-\cdots-v_{i_k},\qquad k\leq g-1.$$

LEMMA 2.

$$\frac{d^k}{dx^k}\Phi(G;x)=(-1)^k k!\sum_{\{i_1\cdots i_k\}}\Phi(G_{i_1\cdots i_k};x),$$

where the summation extends over all k-subsets of $\{1,\dots,g\}$.

Proof. Since

$$\frac{d}{dx}\phi(G; x) = \sum_{i=1}^{i=g} \phi(G_i; x)$$

(see [6], p. 331), we have

$$\begin{aligned}\frac{d}{dx}\Phi(G; x) &= \frac{d}{dx}\{\phi(G; -x) - (-1)^g\phi(\bar{G}; x-1)\}\\ &= -\sum\{\phi(G_i; -x) - (-1)^g\phi(\bar{G}_i; x-1)\}\\ &= -\sum\Phi(G_i; x).\end{aligned}$$

Differentiating repeatedly, we have the lemma.

LEMMA 3. *If* $-\infty < s^*(G) < \infty$, *and the multiplicity of the root* $x^* = 1/s^*(G)$ *of* $\Phi(G; x)$ *equals* $k+1$, *then*

$$\Phi\left(G_{i_1\cdots i_j}; x^*\right) = 0 \quad \text{for } j \le k,\ \{i_1,\ldots,i_j\} \subset \{1,\ldots,g\},$$

where $s^*(G) = s^-(G)$ *or* $s^+(G)$.

Proof. Since $\Phi^{(j)}(G;\ x^*) = 0$ for $j \le k$ it follows from the above lemma that

$$(\#) \qquad (-1)^j j! \sum \Phi\left(G_{i_1\cdots i_j}; x^*\right) = 0.$$

Since there is an s-embedding of G for every $s^-(G) < s < s^+(G)$, it follows by the continuity that

$$\operatorname{sign} \mathrm{CM}\left(G_{i_1\cdots i_j}; s^*(G)\right) = (-1)^{g-j} \quad \text{or} \quad 0.$$

Since $\mathrm{CM}(G_{i_1\cdots i_j};\ s^*(G)) = s^*(G)^{g-j-1}\Phi(G_{i_1\cdots i_j};\ x^*)$, it follows that the non-zero term of the left-hand side of (#) must have the same sign, which is impossible. Hence $\Phi(G_{i_1\cdots i_j};\ x^*) = 0$.

THEOREM 5. *If there is a t-embedding of G then*

$$s^-(G) \le t \le s^+(G).$$

Proof. It is clear that the theorem holds true for graphs with fewer vertices than three. Assume that there exists a graph for which the theorem does not hold, and let H be one of such graphs which is minimal in the number of vertices. Then there is a t-embedding of H such that

$t < s^-(H)$ or $s^+(H) < t$. Suppose that $s^+(H) < t$. (The case $t < s^-(H)$ is similar, and is omitted.) Let $V(H) = \{v_1, \ldots, v_h\}$, and put $H_i = H - v_i$, $i = 1, \ldots, h$; $x^+ = 1/s^+(H)$. Then x^+ is the maximum root of $\Phi(H; x)$ and $1/t < x^+$. By the minimality of H, $t \le s^+(H_i)$, $i = 1, \ldots, h$.

Now we show that x^+ is a simple root of $\Phi(H; x)$. If x^+ is a multiple root, then $\Phi(H_i; x^+) = 0$ by Lemma 3, which implies that $s^+(H_i) = s^+(H) < t$, a contradiction. Thus x^+ must be a simple root of $\Phi(H; x)$.

Since $\Phi(H; x)$ changes sign when x passes through x^+, a simple root, $\mathrm{CM}(H; s)$ also changes sign when s passes through $s^+(H)$. Since $\operatorname{sign} \mathrm{CM}(H; t) = \operatorname{sign} \mathrm{CM}(H; 0)$ or $\mathrm{CM}(H; t) = 0$ (because there is a t-embedding of H), and $s^+(H) < t$, there must be a root s_1 of $\mathrm{CM}(H; s)$ such that $s^+(H) < s_1 \le t$. Thus $\Phi(H; x)$ has a root $x_1 = 1/s_1$ such that $1/t \le x_1 < x^+$. Then, by Rolle's theorem, there is a ξ, $x_1 < \xi < x^+$, such that $\Phi'(H; \xi) = 0$. But since $1/\xi < 1/x_1 \le t \le s^+(H_i)$, there is a $(1/\xi)$-embedding of H_i, and $\Phi(H_i; \xi)$ is non-zero and has the same sign for every i. This contradicts the fact that $0 = \Phi'(H; \xi) = -\sum \Phi(H_i; \xi)$.

6. The dimension of a critical embedding. Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_g\}$, and put

$$G_{i_1 \cdots i_j} = G - v_{i_1} - \cdots - v_{i_j}, \qquad j < g.$$

THEOREM 6. *If* $-\infty < s^*(G) < 1$, *and the multiplicity of the root* $x^* = 1/s^*(G)$ *of* $\Phi(G; x)$ *equals* k, *then*

$$\dim(G, s^*(G)) = g - k - 1,$$

where $s^*(G) = s^-(G)$ *or* $s^+(G)$.

Proof. Since

$$0 \ne \Phi^{(k)}(G; x^*) = (-1)^k k! \sum_{\{i_1 \cdots i_k\}} \Phi\left(G_{i_1 \cdots i_k}; x^*\right),$$

there is a $\{j_1, \ldots, j_k\}$ such that $\Phi(G_{j_1 \cdots j_k}; x^*) \ne 0$. By Lemma 1, it follows easily that if $F \subset G_{j_1 \cdots j_k}$ then

$$\operatorname{sign} \mathrm{CM}(F; s^*(G)) = \operatorname{sign} \mathrm{CM}(F; 0).$$

Using Lemma 3, it follows that if $G_{j_1 \cdots j_k} \subsetneqq H \subset G$ then $\mathrm{CM}(H; s^*(G)) = 0$. Hence $\dim(G; s^*(G)) = g - k - 1$, by Theorem 1.

7. On regular graphs. If G is a regular ρ-valent graph with g vertices, then by Sachs' theorem ([**6**], p. 56),

$$\phi(\bar{G}; x) = (-1)^g \frac{x+\rho+1-g}{x+\rho+1} \phi(G; -x-1).$$

Hence we have

$$\Phi(G; x) = \frac{g}{x+\rho} \phi(G; -x).$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g$ be the eigenvalues of $\mathbf{A}(G)$. Then $\lambda_1 = \rho$ is a simple root of $\phi(G; x)$, and $\lambda_g < 0$. Therefore

$$s^+(G) = -1/\lambda_g$$

and

$$s^-(G) = \begin{cases} -1/\lambda_2 & \text{if } \lambda_2 > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Example. Let G be the Petersen graph. The characteristic polynomial of G is $(x-3)(x-1)^5(x+2)^4$. Hence $s^+(G) = 1/2$, $s^-(G) = -1$, and

$$\dim(G, s) = \begin{cases} 4, & s = -1, \\ 5, & s = 1/2, \\ 9, & -1 < s < 1/2. \end{cases}$$

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References

[1] B. Alspach and M. Rosenfeld, *On embedding triangle-free graphs in unit spheres*, Discrete Math., **19** (1977), 103–111.

[2] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Chelsea, New York (1970).

[3] D. Cvetkovič, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York (1980).

[4] P. Erdős, F. Harary and W. T. Tutte, *On the dimension of a graph*, Mathematika, **12** (1965), 118–122.

[5] F. S. Roberts, *On the boxicity and cubicity of a graph*, in Recent Progress in Combinatorics (ed. by W. T. Tutte) Academic Press, New York (1969), pp. 301–310.

[6] A. J. Schwenk and R. J. Wilson, *On the Eigenvalue of a Graph*, in Selected Topics in Graph Theory (ed. by L. W. Beineke and R. J. Wilson), Academic Press, New York (1978).

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