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BESSEL FUNCTIONS ON P_n

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In this paper Bessel functions are defined in the homogeneous symmetric space $GL(n, \mathbf{R})/O(n)$. Two definitions are given. One is an integral over the space itself, the other is a Euclidean integral. A relation between the integrals is proved. The use of this relation is shown explicitly in a low dimensional case. Some additional properties of these Bessel functions are then noted.

I. Introduction. The Riemannian symmetric space P_n of $n \times n$ symmetric matrices over the real numbers has appeared in many areas of mathematics. It has been studied in connection with multivariate statistics [**J**], [**I**], with analytic number theory [**S**], [**M**], and with the study of other higher rank symmetric spaces, such as the Siegel upper half space of $n \times n$ complex matrices with imaginary part in P_n. The above references have been cited merely as examples of the uses of P_n.

One way to approach questions about P_n is to use harmonic analysis and special functions. P_n has many coordinate systems. In the geodesic polar coordinate system a general theory has been worked out by Harish-Chandra [**HC**] and Helgason [**Hel**]. Using some computations of Bhanu-Murti [**BM**], Audrey Terras has been able to make this theory explicit for P_n [**T**]. However, it seems that the so-called partial Iwasawa coordinate system is needed in the study of some questions that arise in number theory. This is the coordinate system that is used in this note. In fact, the functions and formulas proved here have already found application in the Fourier series expansion of Eisenstein series for $GL(3, \mathbf{Z})$ due to recent work of Kaori Imai and Audrey Terras [**I-T**].

The purpose of this note will be to define some Bessel functions for P_n and prove some of their properties. To begin, however, we establish some notation and recall some basic facts about P_n. We will only mention those details of structure that directly concern us here.

II. Basic facts and notation. Let P_n be the space of all $n \times n$ symmetric matrices over the real numbers. Let $Y = (y_{ij})$ be in P_n and let A be in $GL(n, \mathbf{R})$. Then $GL(n, \mathbf{R})$ acts on P_n by sending Y to $Y[A] = {}'AYA$ where ${}'A$ denotes the transpose of A. The orthogonal matrices $O(n)$ fix the identity I in P_n. The action is sufficiently nice that P_n can be identified as the symmetric space $K \backslash G$ where $G = GL(n, \mathbf{R})$ and $K = O(n)$. The invariant volume on P_n is given by $dv_n = |Y|^{-(n+1)/2}\,dY$ where $|Y|$ is the determinant of Y and $dY = \prod_{1\le i\le j\le n} dy_{ij}$ and dy_{ij} is ordinary Euclidean measure.

Denote by ∂_Y the matrix $(((1+\delta_{ij})/2)\partial/\partial y_{ij})$ where δ_{ij} is the usual Kronecker delta. Then an algebraically independent basis for the commutative polynomial ring of invariant differential operators on P_n is given by $\mathrm{Tr}((Y\partial_Y)^i)$, $i = 1,\ldots,n$.

Let Y_i be the upper left $i\times i$ corner of Y and let $s = (s_1,\ldots,s_n)$ be in $\mathbf{C}^n$. Then define the power function $p_s(Y) = \prod_{i=1}^{n} |Y_i|^{s_i}$. These are joint eigenfunctions for the ring of all invariant differential operators. Furthermore, the action of an invariant differential operator on these functions determines the operator. The power functions were introduced by Selberg in the 1950s. Details for the above facts can be found in Terras [**T**] or Maass [**M**].

Carl Herz [**Her**] in the early 1950s considered many special functions on P_n. However, at that time the symmetric space structure was not well understood. For example, he uses the measure $|Y|^{-(n+1)/2}\,dY$ but does not seem to recognize it as an invariant volume, and he did not know about the power functions. Consequently, his generalizations were only of the special case $s = (0,\ldots,0,s_n)$ in $\mathbf{C}^n$. However, due to the recent work of Terras and Imai already cited [**I-T**] it is clear that more than this special case needs to be considered. We will do that here.

Let Y be in P_n. Then Y has the decomposition $Y = {}^tTT$ where $T = (t_{ij})$ is upper triangular, that is $t_{ij} = 0$ for i greater than j. The invariant volume can be computed and the result is

$$dv_n = 2^n \prod_{i=1}^{n} (t_{ii})^{-i}\,dT \quad \text{where } dT = \prod_{1\le i\le j\le n} dt_{ij}$$

and the dt_{ij} are ordinary Euclidean measures.

Again for Y in P_n we have the partial Iwasawa decomposition: $Y = \left(\begin{smallmatrix} F & 0\\ 0 & G\end{smallmatrix}\right)\left[\begin{smallmatrix} I & H\\ 0 & I\end{smallmatrix}\right]$ with F in P_p, G in P_q, and H in $\mathbf{R}^{p\times q}$ with $p+q=n$. The invariant volume here is $dv_n = |F|^{q-(n+1)/2}\,|G|^{-(n+1)/2}\,dF\,dG\,dH$ with the following usual products of Euclidean measures:

$$dF = \prod_{1\le i\le j\le p} df_{ij},\quad dG = \prod_{1\le i\le j\le q} dg_{ij},\quad \text{and}\quad dH = \prod_{1\le i\le p,1\le j\le q} dh_{ij}.$$

With the same notation we can also decompose Y by $Y = \left(\begin{smallmatrix} F & 0\\ 0 & G\end{smallmatrix}\right)\left[\begin{smallmatrix} I & 0\\ {}^tH & I\end{smallmatrix}\right]$ (F, G, and H are all the same size and in the same places as before.) The invariant volume for these coordinates is

$$dv_n = |F|^{-(n+1)/2}\,|H|^{p-(n+1)/2}\,dF\,dG\,dH.$$

These computations are done, for example, in Terras [**T**].

III. The K-Bessel and k-Bessel functions on P_n. The classical one dimensional K-Bessel function, sometimes called MacDonald's function, can be defined by

$$K_v(z) = (1/2)\int_0^\infty w^v e^{-(1/2)z(w+1/w)}\,dw/w \quad \text{for } v \text{ and } z \text{ in } \mathbf{C}$$

with $\operatorname{Re} z > 0$. Audrey Terras has extended this definition for s in $\mathbf{C}^n$, A and B real symmetric matrices to

$$K_n(s \mid A, B) = \int_{P_n} p_s(Y)\operatorname{etr}(-AY - BY^{-1})\,dv_n$$

$$\text{where } \operatorname{etr}(A) = \exp(\operatorname{trace}(A))$$

is a notation taken from Herz. In the one dimensional case the K-Bessel function is related to the Euclidean integral

$$k_w(z) = \int_{-\infty}^{\infty} (1 + w^2)^{-t} e^{itz}\,dt$$

by a 'complete the square' argument. This case arises in number theory. In that application, the Fourier series expansion of an Eisenstein series is computed by using Poisson summation. Poisson summation relates the sum of a function at integer points to the sum of the Fourier transform of the function at those same points. In this way the k-Bessel function, which is a Fourier transform, arises. Hence we expect to relate the K-Bessel function to a Euclidean integral in order to carry out the expansions in higher dimensional cases. After we find the K-Bessel function as a Euclidean integral a correct generalization of the k-Bessel function as a Fourier transform can be made.

The complete the square argument in the one dimensional case introduces a gamma function. We will need the generalization:

$$\Gamma_n(s) = \int_{P_n} p_s(Y)\operatorname{etr}(-Y)\,dv_n \quad \text{for } s \text{ in } \mathbf{C}^n.$$

This, it turns out, factors into one dimensional gamma functions:

$$\Gamma_n(s) = \pi^{n(n-1)/4} \prod_{i=1}^{n} \Gamma(s_i + \cdots + s_n - (i-1)/2).$$

We can now show

THEOREM 1.

$$\Gamma_n(s)\int_{X \text{ in } \mathbf{R}^{m\times n}} p_s\big((A+'XX)^{-1}\big)\operatorname{etr}(2i'RX)\,dx = \pi^{mn/2}K_n(s^* \mid A,'RR)$$

with A in P_n, R in $\mathbf{R}^{m\times n}$, and

$$s=(s_1,\dots,s_n)\quad \text{in } \mathbf{C}^n, \qquad s^* = s+(0,\dots,0,-m/2).$$

Proof. Our proof will put aside questions of convergence, which will be considered in more detail later. We start with the left hand side and transform it into the right. To begin, write Γ_n as an integral and interchange the order of integration to obtain

$$\int_X\int_Y p_s(Y)p_s\big((A+'XX)^{-1}\big)\operatorname{etr}(-Y)\,dv_n\operatorname{etr}(2i'RX)\,dX.$$

Since A is in P_n, $A+'XX$ is also in P_n. Write $A+'XX = T'T$ with T upper triangular. Make the change of variables $Y \to Y[T]$ and get

$$\int_X\int_Y p_s(Y[T])p_s\big((T'T)^{-1}\big)\operatorname{etr}(-Y[T])\,dv_n\operatorname{etr}(2i'RX)\,dX.$$

Now note that

$$\begin{aligned} p_s(Y[T])p_s\big((T'T)^{-1}\big) &= p_s(Y)p_s('TT)p_s('T^{-1}T^{-1}) \\ &= p_s(Y)p_s('T^{-1}T^{-1}[T]) = p_s(Y), \end{aligned}$$

and that $\operatorname{Tr}(Y[T]) = \operatorname{Tr}('TYT) = \operatorname{Tr}(T'TY)$. Our left hand side has now become

$$\int_Y p_s(Y)\operatorname{etr}(-AY)\int_X \operatorname{etr}(-'XXY+2i'RX)\,dX\,dv_n.$$

The next part is the part where we complete the square in the X integral. We do this by writing $Y=V^2$, V in P_n, and making the change of variables $X \to XV^{-1}$, $dX \to |V^{-1}|^m\,dX = |Y|^{-m/2}\,dX$. Thus

$$\begin{aligned} &\int_X \operatorname{etr}(-'XXY+2i'RX)\,dX \\ &\qquad = \int_X \operatorname{etr}\big(-'(XV^{-1})(XV^{-1})V^2 + 2iRXV^{-1}\big)\,|Y|^{-m/2}\,dX. \end{aligned}$$

It takes five steps to verify that $\mathrm{Tr}({}'(XV^{-1})(XV^{-1})V^2) = \mathrm{Tr}({}'XX)$, which was why we made this substitution in the first place. To finish completing the square, let $C = iRV^{-1}$ and note that

$$\mathrm{Tr}(I[X - C]) + \mathrm{Tr}(Y^{-1}[{}'R]) = \mathrm{Tr}({}'XX - 2i{}'RXV^{-1})$$

by a short computation which we omit. Thus our X integral is

$$|Y|^{-m/2}\int_X \mathrm{etr}(-I[X - C])\mathrm{etr}(-Y^{-1}[{}'R])\, dX.$$

All together our left hand side is

$$\int_Y p_s(Y)\,|Y|^{-m/2}\mathrm{etr}(-AY - {}'RRY^{-1})\, dv_n \int_X \mathrm{etr}(-I[X - C])\, dX.$$

We need to make a change of variables $X \to X + C$, $dX \to dX$ to evaluate the X integral. This involves integration in several complex variables, which is omitted. However, it reduces the X integral to

$$\int_X \mathrm{etr}(-{}'XX)\, dX = \pi^{mn/2}.$$

Now we have brought the left hand side to the definition of the right hand side, and so we are done.

Next we turn to the definition of the k-Bessel function.

$$k_{m,n-m}(s \mid Y, N) = \int_{X \text{ in } \mathbf{R}^{m\times n-m}} p_s\left(Y\begin{bmatrix} I_m & 0 \\ {}'X & I_{n-m}\end{bmatrix}\right)\mathrm{etr}(2i{}'NX)\, dX$$

where Y is in P_n, s is in $\mathbf{C}^n$, N is in $\mathbf{R}^{m\times n-m}$, and dX is the usual Euclidean measure. This definition is a Fourier transform of the power function and hence ordinary Fourier inversion gives us

$$p_s\left(Y\begin{bmatrix} I & 0 \\ {}'X & I\end{bmatrix}\right) = (2/\pi)^{m(n-m)}\int_N k_{m,n-m}(s \mid Y, N)\mathrm{etr}(-2i{}'NX)\, dN.$$

One can also see that the k-Bessel functions transform under the action of the abelian subgroup $\{(\begin{smallmatrix} I & 0 \\ X & I\end{smallmatrix})\}$ of $GL(n, \mathbf{R})$ by a character of that subgroup since the following formula is a consequence of the definition.

$$k_{m,n-m}\left(s \mid Y\begin{bmatrix} I & 0 \\ {}'Q & I\end{bmatrix}, N\right) = \mathrm{etr}(-2i{}'NQ)k_{m,n-m}(s \mid Y, N).$$

This establishes a connection between these functions and group representations.

We will next relate this function, for restricted values of s in $\mathbf{C}^n$, to the K-Bessel function. We will show

THEOREM 2.

$$\Gamma_m(\tilde{s})k_{m,n-m}\left(s_{\text{ext}} \,\middle|\, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, N\right) = |B|^{-m/2}\pi^{m(n-m)/2}K_m(\hat{s} \mid B^{-1}['N], A)$$

where $s = (s_1,\ldots,s_m)$ *is in* $\mathbf{C}^m$, $s_{\text{ext}} = (s_1,\ldots,s_m,0,\ldots,0)$ *is in* $\mathbf{C}^n$, $\tilde{s} = (s_{m-1},\ldots,s_1, -\sum_{i=1}^m s_i)$, *and* $\hat{s} = s + (0,\ldots,0,(n-m)/2)$.

Proof. Only an outline of the proof will be given. The first step is to rewrite the previous theorem as follows:

$$\Gamma_m(s)\int_{X \text{ in } \mathbf{R}^{m\times n-m}} p_s((A + 'XX)^{-1})\operatorname{etr}(2iR'X)\,dX$$
$$= K_m(s^* \mid A, R'R)\pi^{m(n-m)/2}$$

with A in P_m, R in $\mathbf{R}^{m\times n-m}$ and $s^* = (0,\ldots,0, -(n-m)/2) + s$ in $\mathbf{C}^m$. Now for $\tilde{s} = (s_{m-1},\ldots,s_1, -\sum_{i=1}^m s_i)$ and

$$U = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$$

a computation will show that $p_{\tilde{s}}(Y^{-1}) = p_s(Y[U])$. So we put $\tilde{s}$ for s in the statement of the previous theorem to obtain

$$\Gamma_m(\tilde{s})\int_{X\in\mathbf{R}^{m\times n-m}} p_s(A[U] + ('XX)[U])\operatorname{etr}(2i'RX)\,dX$$
$$= K_m((\tilde{s})^* \mid A, R'R)\pi^{m(n-m)/2}.$$

Next note that

$$p_{s_{\text{ext}}}\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{bmatrix} I & 0 \\ 'X & I \end{bmatrix}\right) = p_s(A + B['X]).$$

First make the change of variable $X \to UXV$ where $V^2 = B$ and $dX \to |\det U|^{n-m}|\det V|^m\,dX = |B|^{m/2}\,dX$ and then make the change $'R \to V^{-1}{}'RU$, $A \to A[U]$. When done these substitutions will produce the statement of the theorem.

As consequences of the above theorem relating the K-Bessel function to the k-Bessel function we can note that since the power function p_s is an eigenfunction for the invariant differential operators on P_n, so is $k_{m,n-m}$, and in this way one can obtain differential equations for the K-Bessel function. Convergence information is also available by this formula. We will next discuss this.

Audrey Terras has shown that the K-Bessel function is given by a convergent integral whenever the arguments are in P_n. [T] For certain singular arguments the integral also makes sense. For example, one can show that $K_n(s \mid 0, B) = \Gamma_n(\tilde{s})p_s(B)$, where $\tilde{s} = (s_{n-1}, \ldots, s_1, -\sum_{i=1}^n s_i)$. In fact, this author has extended this result to show that

$$K_n\left(s \,\middle|\, \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\right)$$
$$= \pi^{m(n-m)/2} \mid A \mid^{-(n-m)/2} p_{\hat{s}}(B)\Gamma_{n-m}(\tilde{\hat{s}})K_m(s^* \mid C, A)$$

where $s = (s_1, \ldots, s_n)$ is in $\mathbf{C}^n$, $\hat{s} = (s_{m+1}, \ldots, s_n)$ is in $\mathbf{C}^{n-m}$, $\tilde{\hat{s}} = (s_{n-1}, \ldots, s_{m+1}, -\sum_{i=m+1}^n s_i)$ is in $\mathbf{C}^{n-m}$, and $s^* = (s_1, \ldots, s_{m-1}, s_m + \cdots + s_n + (n-m)/2)$ is in $\mathbf{C}_m$, and C and A are in P_m, B is in P_{n-m}, and X is in $\mathbf{R}^{m \times n-m}$. This result is proved using the partial Iwasawa decomposition in the integral defining the K-Bessel function. It is the Jacobians involved that produce some of the mess. The same techniques also will show

$$K_n\left(s \,\middle|\, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{bmatrix} I & 0 \\ 'X & I \end{bmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}\right)$$
$$= \pi^{m(n-m)/2} \mid B \mid^{-m/2} p_{s^*}(A^{-1})\Gamma_m(s^*)K_{n-m}(\hat{s} \mid B, D)$$

with s in $\mathbf{C}^n$, $s^* = (s_1, \ldots, s_{m-1}, s_m + \cdots + s_{n-1} + s_n)$ in $\mathbf{C}^m$, $\hat{s} = (s_{m+1}, \ldots, s_{n-1}, s_n - m/2)$ in $\mathbf{C}^{n-m}$, A in P_m, B and D in P_{n-m}, and X in $\mathbf{R}^{m \times n-m}$.

IV. A low dimensional illustration. As an example to show what can be done by using Theorem 2 explicitly in a low dimensional case we take B in P_2, $'N = (n_1, n_2)$ in $\mathbf{R}^2$, and consider $K_2(s_1, s_2 \mid N'N, B)$. We express this as an integral over Y in P_2 and write

$$Y = 'TT \quad \text{with} \quad T = \begin{pmatrix} t_1 & t_{12} \\ 0 & t_2 \end{pmatrix}.$$

Now

$$p_s(Y) = (t_1)^{2(s_1+s_2)}(t_2)^{2s_2}$$

and

$$\operatorname{etr}(-N'NY - BY^{-1})$$
$$\leq \exp\left(t_1^2 n_1^2 - 2t_1 t_{12} n_1 n_2 - n_2^2 t_2^2 - b\left(t_1^{-2} + t_{12}^2 t_1^{-2} t_2^{-2} + t_2^{-2}\right)\right)$$

where b is the smallest eigenvalue of B. In checking for absolsute convergence we are led to examine

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty (t_1)^{2s_1+2s_2-1}(t_2)^{2s_2-2} \exp(\cdots)\, dt_{12}\, dt_1\, dt_2.$$

The substitution

$$t_{12} \to t_{12}\left(n_2^2 + bt_1^{-2}t_2^{-2}\right)^{-1/2}, \qquad dt_{12} \to \left(n_2^2 + bt_1^{-2}t_2^{-2}\right)^{-1/2} dt_{12}$$

enables us to complete the square in the t_{12} integral, which evaluates to $(\pi)^{1/2}$. We are left with checking that

$$\int_0^\infty \int_0^\infty (t_1)^{2s_1+2s_2-1}(t_2)^{2s_2-2}\left(n_2^2 + bt_1^{-2}t_2^{-2}\right)^{-1/2} \exp(w)\, dt_1\, dt_2$$

converges, where

$$w = -t_1^2n_1^2 - n_2^2t_2^2 - bt_1^{-2} - bt_2^{-2} + n_1^2n_2^2t_1^2/\left(n_2^2 + bt_1^{-2}t_2^{-2}\right).$$

There is no trouble near the zero end of t_1 or t_2, so we are left with looking at

$$\int_1^\infty \int_1^\infty (t_1)^{2s_1+2s_2-1}(t_2)^{2s_2-2}\left(n_2^2 + bt_1^{-2}t_2^{-2}\right)^{-1/2} \exp(v)\, dt_1\, dt_2$$

where

$$v = -t_2^2n_2^2 - n_1^2bt_2^{-2}/\left(n_2^2 + bt_1^{-2}t_2^{-2}\right).$$

Consider what happens when $n_2 \neq 0$. Then the second term in v is less than or equal to zero so it can be thrown away. Note also

$$\left(n_2^2 + bt_1^{-2}t_2^{-2}\right)^{-1/2} \leq |n_2|^{-1}.$$

Thus we have convergence when $\mathrm{Re}(2s_2 + 2s_1 - 1) < -1$ and all s_2.

When $n_2 = 0$ we are looking at a different thing. The integrand simplifies quite a bit and we are left with

$$\int_1^\infty \int_1^\infty (t_1)^{2s_1+2s_2}(t_2)^{2s_2-1} \exp\left(-n_1^2t_1^2\right) dt_1\, dt_2.$$

If $n_2 = 0$ and $n_1 \neq 0$, then this says that we have convergence whenever $\mathrm{Re}(2s_2 - 1) < -1$. If both $n_1 = 0$ and $n_2 = 0$, then this says that we have convergence whenever $\mathrm{Re}(2s_2 - 1) < -1$ and $\mathrm{Re}(2s_1 + 2s_2) < -1$. It is interesting to note how the region of convergence varies with n_2. Thus $K_2(s_1, s_2 \mid N'N, B)$ converges whenever $N = 0$ and $\mathrm{Re}(s_2) < 0$ and $\mathrm{Re}(s_1 + s_2) < -1/2$, or $n_2 = 0$ and $n_1 \neq 0$ and $\mathrm{Re}(s_2) < 0$, or $n_2 \neq 0$ and $\mathrm{Re}(s_1 + s_2) < 0$. Note that as the symmetric matrix $N'N$ moves away from 0 toward P_n some of the restrictive conditions for convergence are removed.

This can now be used to make sense of the integral defining $k_{2,1}$ past its region of absolute convergence. Some of the zeros of $k_{2,1}$ will also be found. Let $s=(s_1,s_2)$, $\tilde{s}=(s_1,-s_1-s_2)$, $\hat{s}=(s_1,s_2+1/2)$ be in $\mathbf{C}^2$, $s_{\text{ext}}=(s_1,s_2,0)$ be in $\mathbf{C}^3$, $N={}'(n_1,n_2)$ be in $\mathbf{R}^2$, A be in P_2, and $B=b>0$ be a real number. We have

$$\Gamma_2(s_1,-s_2-s_1)=\Gamma(-s_2)\Gamma(-s_2-s_1-1/2).$$

Thus Theorem 2 says

$$\pi^{1/2}\Gamma(-s_2)\Gamma(-s_2-s_1-1/2)k_{2,1}\left(s_1,s_2,0\,\middle|\left(\begin{matrix}A & 0\\ 0 & b\end{matrix}\right),{}'(n_1,n_2)\right)$$
$$=b^{-1}\pi K_2\left(s_1,s_2+1/2\,|\,{}'(n_1,n_2)b^{-1}(n_1,n_2),A\right).$$

Suppose $n_2\neq 0$. Then our previous discussion gives the convergence of the right hand side when $\mathrm{Re}(s_2+1/2)<-\mathrm{Re}(s_1)$, or equivalently $\mathrm{Re}(s_1)<-1/2-\mathrm{Re}(s_2)$. This enables us to display some zeros of $k_{2,1}$ as follows. Let $s_2=0,1,2,\ldots$ and let $s_1=-s_2-d$ where $d>1/2$. Then $\mathrm{Re}(s_1)=-s_2-d<-1/2-\mathrm{Re}(s_2)$ and the condition for convergence on the right is satisfied. Next note that $-s_2-s_1-1/2=-s_2-(-s_2-d)-1/2=d-1/2$ and so $\Gamma(-s_2-s_1-1/2)$ converges, while $\Gamma(-s_2)$ has a pole. Thus

$$k_{2,1}\left(-j-d,j,0\,\middle|\left(\begin{matrix}A & 0\\ 0 & b\end{matrix}\right),{}'(n_1,n_2)\right)=0$$

for $n_2\neq 0, j=0,1,2,\ldots$ and $d>1/2$.

In the case $n_2=0$, $n_1\neq 0$, the condition for convergence on the right becomes $\mathrm{Re}(s_2)+1/2<0$, or equivalently $\mathrm{Re}(s_2)<-1/2$, in which case $\Gamma(-s_2)$ always converges. However, $\Gamma(-s_1-s_2-1/2)$ has poles at $s_1=j-1/2-s_2$ for $j=0,1,2,\ldots$ The zeros of $k_{2,1}$ are then at $s_1=j-1/2-s_2$ for $j=0,1,2,\ldots$.

V. Properties of the Bessel functions. In this section, a few properties of the Bessel functions will be discussed. These will, hopefully, serve to demonstrate the elegance and utility of the Bessel functions.

In the one dimensional case a Mellin transform is an integration against a power function. Here we compute an integral of a K-Bessel function against a power function. Let s and t be $\mathbf{C}^n$. Then

$$\int_{A \text{ in } P_n} p_s(A)K_n(t\,|\,I,A)\,dv_n=\Gamma_n(s)\Gamma_n(s+t).$$

To see this, note that the left hand side is

$$\iint_{P_n \times P_n} p_s(A) p_t(Y) \operatorname{etr}(-Y - AY^{-1})\, dv_n\, dv_n.$$

Let $Y = {}'TT$, T upper triangular, and make the change $A \to A[T]$. A short computation will then produce the result. This generalizes a result in the one dimensional case.

One can connect the arguments Y and N of $k_{n-1,1}(s \mid Y, N)$. If Y has the partial Iwasawa decomposition $Y = \left(\begin{smallmatrix} G & 0 \\ 0 & h \end{smallmatrix}\right)\left[\begin{smallmatrix} I & 0 \\ 'Q & I \end{smallmatrix}\right]$ where G is in P_{n-1}, Q is in $\mathbf{R}^{n-1}$ a column vector and h is a positive real number, and $G = {}'TT$ where T is upper triangular, then a computation will show that

$$k_{n-1,1}(s \mid Y, N) = \operatorname{etr}(-2i'NQ) p_{\hat{s}}\left(\begin{pmatrix} G & 0 \\ 0 & h \end{pmatrix}\right) k_{n-1,1}(s \mid I, TNh^{-1/2})$$

where

$$\hat{s} = (s_1, \ldots, s_{n-2}, s_{n-1} + n/2, s_n - (n-1)/2).$$

The final result we will mention is one that builds $k_{n-1,1}$ out of an integral of $k_{n-2,1}$. This is, in fact, motivated by a formula of Kaori Imai and a suggestion of Audrey Terras. Imai's formula used the K-Bessel function, which obscured the correct generalization **[I-T]**.

Here we consider the case $n > 3$.

$$\begin{aligned} k_{n-1,1}(s \mid I, N) &= \int_{X \in \mathbf{R}^{n-1}} p_s\left(I\begin{bmatrix} I & 0 \\ 'X & 1 \end{bmatrix}\right) \operatorname{etr}(2i'NX)\, dX \\ &= \int_X (1 + x_1^2)^{s_1}(1 + x_1^2 + x_2^2)^{s_2} \cdots (1 + x_1^2 + \cdots + x_{n-1}^2)^{s_{n-1}} \\ &\quad \times \operatorname{etr}(2i'NX)\, dX. \end{aligned}$$

The above step is not immediate, but follows as follows. The upper left $i \times i$ corner of $I\left[\begin{smallmatrix} I & 0 \\ 'X & 1 \end{smallmatrix}\right]$ is $I_i + {}'(x_1, \ldots, x_i)(x_1, \ldots, x_i)$. We are interested in the determinant of this. The matrix ${}'(x_1, \ldots, x_i)(x_1, \ldots, x_i)$ is $i \times i$ with rank 1. Its eigenvector ${}'(x_1, \ldots, x_i)$ is associated with its unique non-zero eigenvalue $x_1^2 + \cdots + x_i^2$. Thus a diagonalization will produce

$$\begin{pmatrix} \|X\|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Any diagonalization leaves I fixed, and the determinant invariant. Our desired determinant is

$$\left| I + \begin{pmatrix} \|X\|^2 & 0 \\ 0 & 0 \end{pmatrix} \right| = 1 + x_1^2 + \cdots + x_i^2.$$

Thus we have the above step. Next, make the substitution $(x_2,\ldots,x_{n-1}) \to (1+x_1^2)^{1/2}(x_2,\ldots,x_{n-1})$ and $dX \to (1+x_1^2)^{(n-2)/2}\,dX$. A direct computation now shows that

$$k_{n-1,1}(s \mid I, N) = \int_{u=-\infty}^{\infty} (1+u^2)^t e^{2in_1u} k_{n-2,1}\left(s^* \mid I, (1+u^2)^{1/2}\,'N_2\right) du$$

where $s=(s_1,\ldots,s_n)$ is in $\mathbf{C}^n$, $N={}'(n_1,\ldots,n_{n-1})$ is in $\mathbf{R}^{n-1}$,

$$t = s_1 + \cdots + s_{n-1} + (n-2)/2, \qquad 'N_2 = (n_2,\ldots,n_{n-1})$$

and $s^* = (s_2,\ldots,s_{n-2},0)$.

These last two facts also show that induction proofs on P_n should work well.

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