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CYCLIC GROUPS OF AUTOMORPHISMS OF COMPACT NON-ORIENTABLE KLEIN SURFACES WITHOUT BOUNDARY

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We obtain the minimum genus of the compact non-orientable Klein surfaces of genus $p \geq 3$ without boundary which have a given cyclic group of automorphisms.

1. Introduction. Let X be a compact Klein surface [**1**]. Singerman [**8**] showed that the order of a group of automorphisms of a surface X without boundary of algebraic genus $g \geq 2$ is bounded above by $84(g-1)$, and May [7] proved that if X has nonempty boundary, this bound is $12(g-1)$.

These bounds may be considered as particular cases of the general problem of finding the minimum genus of surfaces for which a given finite group G is a group of automorphisms. The study of cyclic groups is a necessary preliminary to this, since it leads to limitations on the orders of elements within a general group. In this paper we consider the above problem for the case of cyclic groups of automorphisms of compact non-orientable Klein surfaces without boundary. The corresponding problem for compact orientable Klein surfaces without boundary was solved by Harvey [**5**].

2. Compact non-orientable Klein surfaces without boundary. By a non-Euclidean crystallographic (NEC) group, we shall mean a discrete subgroup Γ of the group of isometries G of the non-Euclidean plane, with compact quotient space, including those which reverse orientation, reflections and glide reflections. We say that Γ is a proper NEC group if it is not a Fuchsian group. We shall denote by Γ^+ the Fuchsian group $\Gamma \cap G^+$, where G^+ is the subgroup of G whose elements are the orientation-preserving isometries.

NEC groups are classified according to their signature. The signature of an NEC group Γ is either of the form

$$(*) \qquad \left(g;\ +;\ [m_1,\ldots,m_\tau];\quad \left\{\left(n_{i1},\ldots,n_{is_i}\right)_{i=1,\ldots,k}\right\}\right)$$

or

$$(**) \qquad \left(g;\ -;\ [m_1,\ldots,m_\tau];\quad \left\{\left(n_{i1},\ldots,n_{is_i}\right)_{i=1,\ldots,k}\right\}\right);$$

the numbers m_i are the periods and the brackets $(n_{i1},\ldots,n_{is_i})$, the period cycles.

A group Γ with signature $(*)$ has the presentation given by generators

$$x_i,\quad i=1,\ldots,\tau,\qquad c_{ij},\quad i=1,\ldots,k, j=0,\ldots,s_i,$$
$$e_i,\quad i=1,\ldots,k,\qquad a_j, b_j, j=1,\ldots,g,$$

and relations

$$x_i^{m_i}=1,\quad i=1,\ldots,\tau,\qquad c_{is_i}=e_i^{-1}c_{i0}e_i,\quad i=1,\ldots,k,$$
$$c_{ij-1}^2=c_{ij}^2=(c_{ij-1}\cdot c_{ij})^{n_{ij}}=1,\qquad i=1,\ldots,k, j=1,\ldots,s_i,$$
$$x_1\cdots x_\tau e_1\cdots e_k\, a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}=1.$$

A group Γ with signature $(**)$ has the presentation given by generators

$$x_i,\quad i=1,\ldots,\tau,\qquad c_{ij},\quad i=1,\ldots,k, j=0,\ldots,s_i,$$
$$e_i,\quad i=1,\ldots,k,\qquad d_j,\quad j=1,\ldots,g,$$

and relations

$$c_{is_i}=e_i^{-1}c_{i0}e_i,\quad i=1,\ldots,k,\qquad x_i^{m_i}=1,\quad i=1,\ldots,\tau,$$
$$c_{ij-1}^2=c_{ij}^2=(c_{ij-1}\cdot c_{ij})^{n_{ij}}=1,\qquad i=1,\ldots,k, j=1,\ldots,s_i,$$
$$x_1\cdots x_\tau e_1\cdots e_k\, d_1^2\cdots d_g^2=1.$$

From now on, we will denote by x_i, e_i, c_{ij}, a_i, b_i, d_i the above generators associated to the NEC groups.

(2.1) Definition. We shall say that an NEC group Γ_g is the group of an orientable surface if Γ_g has the signature $(g;\ +;\ [-];\ \{-\})$ where $[-]$ indicates that the signature has no periods and $\{-\}$ indicates that the signature has no period cycles.

(2.2) Definition. An NEC group Γ_p is the group of a non-orientable surface if Γ_p has the following signature $(p;\ -;\ [-];\ \{-\})$.

For a given Γ_p we have that the orbit space D/Γ_p (where $D=C^+$) is a non-orientable surface of genus p. The canonical projection $\pi\colon D\to D/\Gamma_p$ induces an analytic and anti-analytic structure on D/Γ_p, which establishes a structure of compact non-orientable Klein surface without boundary of genus p in D/Γ_p.

From now on, Klein surfaces appearing in this paper are supposed to be compact without boundary.

Singerman has shown in [**8**] the following

(2.3) PROPOSITION. *If G is a group of automorphisms of a non-orientable Klein surface of genus $p \geq 3$, then G is finite.*

(2.4) THEOREM. *A necessary and sufficient condition for a finite group G to be a group of automorphisms of a non-orientable Klein surface of genus $p \geq 3$ is that there exist a proper NEC group Γ and a homomorphism θ: $\Gamma \to G$ such that the kernel of θ is a surface group and $\theta(\Gamma^+) = G$.*

As a consequence of this theorem, we have that if G is a finite group of automorphisms of a non-orientable Klein surface of genus $p \geq 3$ then $G \simeq \Gamma/\Gamma_p$, where Γ is a proper NEC group and Γ_p is the group of a non-orientable surface; thus

$$\text{order}(G) = |\Gamma_p|/|\Gamma| = 2\pi(p-2)/|\Gamma|,$$

where $|\ |$ denotes the non-Euclidean area of a fundamental region of the group.

(2.5) THEOREM. *If G is a finite group, G is a group of automorphisms of a non-orientable Klein surface of genus $p \geq 3$.*

Proof. Let us suppose that G has n generators $g_1, g_2, \ldots, g_n$. There exists a proper NEC group Γ_{2n+1} that is the group of a non-orientable surface, and therefore it has the following generators and relations:

$$\{a_1, a_2, \ldots, a_{2n+1} \mid a_1^2 \cdot a_2^2 \cdots a_{2n+1}^2 = 1\}.$$

We establish a homomorphism θ: $\Gamma_{2n+1} \to G$, by defining

$$\theta(a_1) = g_1, \quad \theta(a_3) = g_2 \cdots \theta(a_{2n-1}) = g_n, \quad \theta(a_{2n+1}) = 1,$$
$$\theta(a_2) = g_1^{-1}, \quad \theta(a_4) = g_2^{-1}, \quad \theta(a_{2n}) = g_n^{-1}.$$

θ is an epimorphism. $\ker\theta$ is a normal subgroup of Γ_{2n+1} with finite index, and therefore, $\ker\theta$ is an NEC group.

As Γ_{2n+1} has neither periods nor period-cycles, and $\ker\theta$ is a normal subgroup of Γ_{2n+1}, by [**2**] and [**3**], $\ker\theta$ has neither periods nor period-cycles, and thus it is a surface group.

Moreover, as $a_1 \cdot a_{2n+1}, a_3 \cdot a_{2n+1}, \ldots, a_{2n-1} \cdot a_{2n+1}$ belong to Γ_{2n+1}^+ and $\theta(a_1 \cdot a_{2n+1}) = g_1$, $\theta(a_3 \cdot a_{2n+1}) = g_2, \ldots, \theta(a_{2n-1} \cdot a_{2n+1}) = g_n$, then $\theta(\Gamma_{2n+1}^+) = G$.

By (2.4) G is a group of automorphisms of a non-orientable Klein surface of genus $p \geq 3$.

3. Non-orientable surface-kernel homomorphisms.

(3.1) DEFINITION. A homomorphism θ of a proper NEC group Γ into a finite group is a non-orientable surface-kernel homomorphism if $\ker \theta$ is the group of a surface and $\theta(\Gamma^+) = G$.

From [2], [3] and (2.4) we get

(3.2) PROPOSITION. *A homomorphism θ of a proper NEC group Γ of signature $(g; \pm; [m_1,\ldots,m_\tau]; \{(n_{11},\ldots,n_{1s_1}) \cdots (n_{k1},\ldots,n_{ks_k})\})$ into a finite group G is a non-orientable surface-kernel homomorphism if and only if $\theta(c_{ij})$ has order 2, $\theta(x_i)$ has order m_i, $\theta(c_{ij-1} \cdot c_{ij})$ has order n_{ij} and $\theta(\Gamma^+) = G$.*

(3.3) COROLLARY. *Let G be a finite group with odd order. Then there is no proper NEC group Γ with period cycles for which there exists a non-orientable surface-kernel homomorphism θ: $\Gamma \to G$.*

(3.4) COROLLARY. *There does not exist any proper NEC group Γ with period cycles having some non-empty period cycle for which there is a non-orientable surface-kernel homomorphism θ: $\Gamma \to Z_n$ with n even.*

Proof. If there were a non-orientable surface-kernel homomorphism θ: $\Gamma \to Z_n$, we would have that for every $c_{ij} \in \Gamma$, $\theta(c_{ij})$ would have order 2 in Z_n; if Γ has some non-empty period cycle, there would be two reflections c_{ij}, $c_{ij+1} \in \Gamma$ such that $(c_{ij} \cdot c_{ij+1})^{n_{ij}} = 1$ and, by (3.2), the order of $\theta(c_{ij} \cdot c_{ij+1})$ would be n_{ij}, but this is impossible because

$$\theta(c_{ij} \cdot c_{ij+1}) = \theta(c_{ij}) + \theta(c_{ij+1}) = \bar{n}/2 + \bar{n}/2 = \bar{n},$$

where $\bar{p}$ denotes the equivalence class of the element p of Z_n.

(3.5) THEOREM. *Let Γ be a proper NEC group with signature*

$$\left(g;\ +;\ [m_1,\ldots,m_\tau];\ \left\{(-)(-),\overset{k}{\ldots},(-)\right\}\right)$$

and let n be even. Then there exists a non-orientable surface-kernel homomorphism θ: $\Gamma \to Z_n$ if and only if:

(i) $m_i \backslash n\ \forall\, i \in I,\ I = \{1,\ldots,\tau\}$;

(ii) *if* $g = 0$, $k = 1$, *then* $\text{l.c.m.}(m_1 \cdots m_\tau) = n$.

Proof. If there is a non-orientable surface-kernel homomorphism θ: $\Gamma \to Z_n$, then, by (3.2), $\theta(\Gamma^+) = Z_n$.

By Theorem 2 of [**9**] and Theorem 4 of [**5**], (i) and (ii) hold.

If we suppose that the elements of the signature Γ fulfill (i) and (ii), we define the homomorphism θ: $\Gamma \to Z_n$ in the following way:

if $g \neq 0$:

$$\theta(a_1) = \bar{1}, \qquad \theta(a_i) = \bar{n}, \quad i = 2,\ldots,g, \qquad \theta(x_i) = \frac{\bar{n}}{m_i},$$

$$\theta(b_1) = \bar{1}, \qquad \theta(b_i) = \bar{n}, \qquad \theta(c_i) = \frac{\bar{n}}{2},$$

$$\theta(e_1) = \overline{-\sum_{i=1}^{\tau} \frac{n}{m_i}}, \quad \theta(e_i) = \bar{n}, \qquad i = 2,\ldots,k;$$

if $g = 0$, $k = 1$:

$$\theta(x_i) = \frac{\bar{n}}{m_i}, \quad \theta(c_j) = \frac{\bar{n}}{2}, \quad \theta(e_1) = \overline{-\sum_{i=1}^{\tau} \frac{n}{m_i}};$$

if $g = 0$, $k > 1$:

$$\theta(x_i) = \frac{\bar{n}}{m_i}, \qquad \theta(c_i) = \frac{\bar{n}}{2}, \quad \theta(e_1) = \bar{1},$$

$$\theta(c_2) = \overline{-1 - \sum_{i=1}^{\tau} \frac{n}{m_i}}, \quad \theta(e_i) = \bar{n}, \qquad i = 3,\ldots,k;$$

in every case there is a $\gamma \in \Gamma^+$ such that $\theta(\gamma) = \bar{1}$:

if $g \neq 0$, $\gamma = a_1$;

if $g = 0$, $k = 1$, by (ii) l.c.m.$(m_1 \cdots m_\tau) = n$, for there exist integers $\alpha_1,\ldots,\alpha_\tau$ such that $\alpha_1 n/m_1 + \cdots + \alpha\tau n/m_\tau = 1$, therefore $\gamma = x_1^{\alpha_1} \cdots x_\tau^{\alpha_\tau}$;

if $g = 0$, $k > 1$, $\gamma = e_1$.

Therefore $\theta(\Gamma^+) = Z_n$ and θ is a non-orientable surface-kernel homomorphism.

(3.6) THEOREM. *Let Γ be a proper NEC group of signature*

$$\left(g;\ -;\ [m_1,\ldots,m_\tau];\ \left\{(-)(-)\overset{k}{,\ldots,}(-)\right\}\right)$$

and let θ be a non-orientable surface-kernel homomorphism θ: $\Gamma \to Z_n$ with n even. Then

(i) $m_i \backslash n\ \forall\, i \in I$, $I = \{1,2,\ldots,\tau\}$;

(ii) *if* $g = 1$, $k = 0$, *then* l.c.m.$(m_1 \cdots m_\tau) = n$.

Proof. The Conditions (i) and (ii) hold by Theorem 2 of **[9]** and Theorem 4 of **[5]**.

(3.7) THEOREM. *Let Γ be a proper NEC group of signature* $(g; -; [m_1 \cdots m_\tau])$ *and let n be odd. Then there exists a non-orientable surface-kernel homomorphism* $\theta\colon \Gamma \to Z_n$ *if and only if*

(i) $m_i \backslash n \ \forall\, i \in I, I = \{1, \ldots, \tau\}$;

(ii) *if* $g = 1$, *then* $\text{l.c.m.}(m_1 \cdots m_\tau) = n$.

Proof. The necessity is similar to (3.6). Let us see the sufficiency. If we suppose that the elements of Γ fulfill (i) and (ii) we define the homomorphism $\theta\colon \Gamma \to Z_n$ in the following way: assume $\overline{\Sigma_{i \in I} n/m_i} = \bar{p}$.

If $g = 1$ and p odd:

$$\theta(x_i) = \frac{\bar{n}}{m_i}, \qquad \theta(a_1) = \overline{\frac{1}{2}(n - p)}.$$

If $g = 1$ and p even:

$$\theta(x_i) = \frac{\bar{n}}{m_i}, \qquad \theta(a_1) = \overline{-\frac{1}{2}p}.$$

If $g > 1$ and p odd:

$$\theta(x_i) = \frac{\bar{n}}{m_i}, \qquad \theta(a_2) = \overline{\frac{n - 2p - 1}{2}},$$
$$\theta(a_1) = \overline{\frac{p + 1}{2}}, \quad \theta(a_i) = \bar{n}, \quad i > 2.$$

If $g > 1$ and p even:

$$\theta(x_i) = \frac{\bar{n}}{m_i}, \qquad \theta(a_2) = \overline{\frac{n + 1}{2}},$$
$$\theta(a_1) = \overline{\frac{-p - n - 1}{2}}, \quad \theta(a_i) = \bar{n}, \quad i > 3.$$

In every case there is $\gamma \in \Gamma^+$ such that $\theta(\gamma) = \bar{1}$: if $g = 1$, by (ii) $\text{l.c.m.}(m_1 \cdots m_\tau) = n$, for there exist integers $\alpha_1, \ldots, \alpha_\tau$ such that $\alpha_1 n/m_1 + \cdots + \alpha_\tau n/m_\tau = 1$, therefore $\gamma = x_1^{\alpha_1} \cdots x_\tau^{\alpha_\tau}$;

if $g > 1$ and p odd, $\gamma = a_1^4 \cdot a_2^2$;

if $g > 1$ and p even, $\gamma = a_2^2$.

Therefore $\theta(\Gamma^+) = Z_n$ and θ is a non-orientable surface-kernel homomorphism.

4. Minimum genus. In this section we shall compute the minimum genus of a non-orientable Klein surface which has a cyclic group of automorphisms. We know by (2.4) that if G is a group of automorphisms of a non-orientable Klein surface of genus $p \geq 3$, then $G \simeq \Gamma/\Gamma_p$, where Γ is a proper NEC group, and Γ_p is a group of a non-orientable surface. Thus if order$(G) = n$, we have

$$n = 2\pi(p-2)/|\Gamma|$$

and $p = 2 + (n/2\pi)|\Gamma|$, so we can reduce the problem to the search of a proper NEC group for which there exists a non-orientable surface-kernel homomorphism $\theta\colon \Gamma \to Z_n$ which minimizes p.

(4.1) THEOREM. *If $n = 1$, q prime, then the minimum genus p of a non-orientable Klein surface with a group of automorphisms isomorphic to Z_n is*:

$$\begin{aligned} &\textit{if } q = 2, \quad p = 3, \\ &\textit{if } q \neq 2, \quad p = q. \end{aligned}$$

Proof. If $q = 2$, we consider an NEC group of signature

$$(0;\ +;\ [2,2,2];\ \{(-)\}).$$

This group fulfills the conditions of Theorem (3.5), so

$$(p-2)/2 = 1/2, \quad \text{i.e.} \quad p = 3.$$

If $q \neq 2$, we have that an NEC group of signature $(1;\ -;\ [q,q])$ fulfills the conditions of Theorem (3.7), therefore it is the group of a surface and

$$(p-2)/q = 1 - 2/q, \quad \text{i.e.} \quad p = q.$$

Now let us see that q is the minimum genus.

If we take any other NEC group Γ with the conditions of Theorem (3.7), Γ would have the signature $(g;\ -;\ [\overset{\tau}{q, \ldots, q}])$ and

$$\frac{p-2}{q} = g - 2 + \tau\left(1 - \frac{1}{q}\right) = (\tau + g - 2) - \frac{\tau}{q},$$
$$p = 2 + (\tau + g - 2)q - \tau,$$

since $\tau > 1$ if $g = 1$, and $g \geq 1$, then the following expression is always $\geq q$.

(4.2) THEOREM. *If* $n = 2^{\beta} q_1^{r_1} \cdots q_\alpha^{r_\alpha}$, *where* $2 < q_1 < \cdots < q_\alpha$ *and* $q_1 \cdots q_\alpha$ *are prime, then the minimum genus* p *of a non-orientable Klein surface with group of automorphisms isomorphic to* Z_n *is*

$$\begin{aligned} p &= n/2 && \text{if } \beta = 1, \\ p &= n/2 + 1 && \text{if } \beta > 1. \end{aligned}$$

Proof. If $\beta = 1$, we consider an NEC group Γ of signature

$$(0;\ +;\ [2, n/2];\ \{(-)\}).$$

This group fulfills the conditions of Theorem (3.5), so

$$\frac{p-2}{n} = \frac{1}{2} - \frac{2}{n}, \quad \text{i.e.} \quad p = \frac{n}{2}.$$

Now let us see that $n/2$ is the minimum genus. If we take any other group Γ in the conditions of (3.5), Γ would have the signature $(g;\ +;\ [m_1 \cdots m_\tau];\ \{(-), \overset{k}{\ldots}, (-)\})$, where $m_i \backslash n$, so

$$p = 2 + n(2g - 2 + k) + n \sum_{i \in I} \left(1 - \frac{1}{m_i}\right).$$

If $2g - 2 + k > 0$, then the genus would be greater than the one we had calculated before; if $2g - 2 + k \leq 0$ as $g \geq 0$ and $k \geq 1$, we have that only the following cases can hold: $g = 0$, $k = 1$; $g = 0$, $k = 2$. If $g = 0$, $k = 2$, as $|\Gamma| > 0$ then $\tau \geq 1$.

$$p = 2 + n \sum_{i \in I} \left(1 - \frac{1}{m_i}\right) > \frac{n}{2}.$$

If $g = 0$, $k = 1$,

$$p = 2 - n + n \sum_{i \in I} \left(1 - \frac{1}{m_i}\right),$$

as $p \geq 3$, $\tau \geq 2$ necessarily. But $\sum_{i=1}^{\tau}(1 - 1/m_i) < 2$, since if it is greater or equal, the genus would be greater than the one calculated before. Thus τ can only be 2 or 3. In both cases, keeping in mind that l.c.m. $(m_1 \cdots m_\tau) = n$, one can check easily that the minimum genus one gets is $\geq n/2$.

If we take an NEC group Γ with signature

$$\left(g;\ -;\ [m_1, \ldots, m_\tau];\ \left\{(-), \overset{k}{\ldots}, (-)\right\}\right)$$

then

$$p = 2 + n(g - 2 + k) + n \sum_{i \in I} \left(1 - \frac{1}{m_i}\right).$$

If $g - 2 + k > 0$, then the genus would be greater than the one we had calculated before. If $g - 2 + k \leq 0$, then, necessarily:

$$\begin{aligned} g &= 1, \quad k = 1, \\ g &= 1, \quad k = 0, \\ g &= 2, \quad k = 0. \end{aligned}$$

In the three cases, using Theorem (3.6), we have $p \geq n/2$.

If $\beta \neq 1$, we consider an NEC group Γ of signature

$$(0;\ +;\ [n, 2];\ \{(-)\}).$$

This group fulfills the conditions of Theorem (3.5), so

$$\frac{p - 2}{n} = \frac{1}{2} - \frac{1}{n}, \quad \text{i.e.} \quad p = \frac{n}{2} + 1.$$

If we take any other group Γ, by (3.5) and (3.6) and operating in the same way as before, we get that $n/2 + 1$ is the minimum genus.

(4.3) THEOREM. *Let* $n = q_1^{r_1} \cdots q_\alpha^{r_\alpha}$, *with* $q_1 < q_2 < \cdots < q_\alpha$ *being prime numbers and* $q_1 \neq 2$. *Then the minimum genus p of a non-orientable Klein surface with group of automorphisms isomorphic to* Z_n *is*

$$\begin{aligned} p &= 2 - q_1 + n - n/q_1 \quad && \text{if } r_1 = 1, \\ p &= 1 + n - n/q_1 && \text{if } r_1 > 1. \end{aligned}$$

Proof. Similar to the proof of the above theorem, bearing in mind (3.7).

The following corollary has also been obtained by W. Hall in [**4**]. The corresponding result for orientable Klein surfaces without boundary is due to A. Wiman [**10**].

(4.4) COROLLARY. *The maximum order for an automorphism of a non-orientable Klein surface of genus $p \geq 3$ is*

$$\begin{array}{ll} 2p & \text{if } p \text{ is odd}, \\ 2(p-1) & \text{if } p \text{ is even}, \end{array}$$

and it is always reached.

Proof. Given a non-orientable Klein surface of genus $p \geq 3$, we have by Theorems (4.1), (4.2) and (4.3) that the genus p satisfies $p \geq n/2$, i.e. $2p \geq n$. If $p = n/2$, then $n = 2$ and $n \neq 4$, so that bound is only reached when p is odd: in fact, given an NEC group Γ of signature $(0; +; [2, p]; \{(-)\})$, by (3.5) there is a non-orientable Klein surface of genus p, with a group of automorphisms isomorphic to Z_{2p}.

If p is even, the maximum order for an automorphism is $2(p-1)$, since given an NEC group Γ of signature $(0; +; [2(p-1), 2]; \{(-)\})$, by (3.5) there is a non-orientable Klein surface of genus p, with a group of automorphisms isomorphic to $Z_{2(p-1)}$.

If p is the topological genus of a compact non-orientable Klein surface without boundary, the algebraic genus is $g = p - 1$.

If we express the above corollary in terms of algebraic genus, these bounds are the same as the ones obtained by C. L. May in [**6**] for the order of an automorphism of an orientable bordered Klein surface.

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