PACIFIC JOURNAL OF MATHEMATICS
Vol. 111, No. 1, 1984

CLOSED INCOMPRESSIBLE SURFACES IN COMPLEMENTS OF STAR LINKS

ULRICH OERTEL

A star link is constructed by plumbing bands with twists "according to a star graph". In this paper the closed incompressible surfaces in most star link complements are determined. It is shown that Dehn surgery on star knots yields Haken manifolds in most cases, but also some non-Haken manifolds.

1. Statement of results. To each $p/q \in \mathbf{Q} \cup \{1/0\}$ with $(p, q) = 1$ is associated a *rational tangle* constructed by drawing slope p/q lines on a square "pillowcase" starting at the four corners (Figure 1.1(a)). The *star link* $K = K(p_1/q_1, \ldots, p_k/q_k)$ is obtained by joining the rational tangles corresponding to the fractions p_i/q_i in a circle, reading clockwise, as shown in Figure 1.1(b). (F. Bonahon and L. Siebenmann have made a study of arborescent links, a larger class of links which includes star links **[B-S]**.)

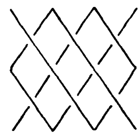
To rule out trivial or previously studied cases, we always assume $q_i \geq 2$ for each i. If some $q_i = 0$, K splits into links which are sums of 2-bridge links. If some $q_i = 1$, say q_1, then the first tangle can be incorporated in an adjacent tangle:

$$K\left(\frac{p_1}{1}, \frac{p_2}{q_2}, \ldots, \frac{p_k}{q_k}\right) = K\left(\frac{p_1 q_2 + p_2}{q_2}, \frac{p_3}{q_3}, \ldots, \frac{p_k}{q_k}\right).$$

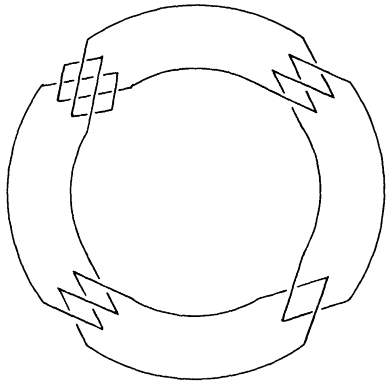
If $q_i = 1$ for all i, K is a torus link. In all these special cases the only closed incompressible surfaces in $S^3 - K$ are peripheral tori; see **[H-T]** for the case of 2-bridge links.

The hypothesis $q_i \geq 2$ implies $S^3 - K$ is irreducible, as will be shown.

A star link complement $S^3 - K$ contains some obvious 4-punctured spheres whose boundary curves are meridians of K. Namely, in the plane of projection of K in Figure 1.1(b), take a simple closed curve C which intersects K transversely at four points on the arcs of K joining tangles, and then cap off C with the two discs it bounds above and below the projection plane to obtain a 4-punctured sphere $S_C \subset S^3 - K$. As will be shown, S_C is incompressible in $S^3 - K$ if and only if there are at least two tangles on each side of C. Up to isotopy there are evidently exactly $k(k - 3)/2$ such curves C.



(a) slope 2/3 rational tangle



(b) $K(\frac{2}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3})$

FIGURE 1.1

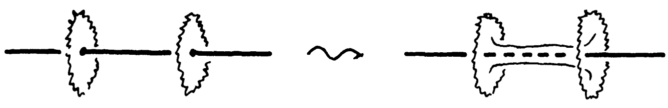


FIGURE 1.2

Starting with a finite collection of these obvious 4-punctured spheres S_C which are disjoint, one can construct a closed surface in $S^3 - K$ by a sequence of *peripheral tubing* operations (Figure 1.2).

Here are the main results, assuming always that $q_i \geq 2$ for each i:

THEOREM 1. *If $\Sigma_{i=1}^{k} p_i/q_i \neq 0$, in particular if K is a knot, then every closed incompressible surface in $S^3 - K$ (except the peripheral tori) is isotopic to a surface obtained from a finite collection of the disjoint incompressible spheres S_C by a sequence of peripheral tubing operations. When $\Sigma_{i=1}^{k} p_i/q_i = 0$ there is, in addition, just one other isotopy class of closed incompressible surfaces in $S^3 - K$; surfaces in this class have Euler characteristic $l(2 - k + \Sigma_{i=1}^{k} 1/q_i)$ where $l = \text{l.c.m.}(q_1, \ldots, q_k)$.*

THEOREM 2. *If $q_i \geq 3$ for each i, then a surface obtained from disjoint incompressible S_C's by a sequence of tubing operations is incompressible if and only if each tube passes through at least one rational tangle.*

Examples of how tubing an incompressible S_C can yield a compressible closed surface if some q_i's are 2 are shown in Figure 1.3.

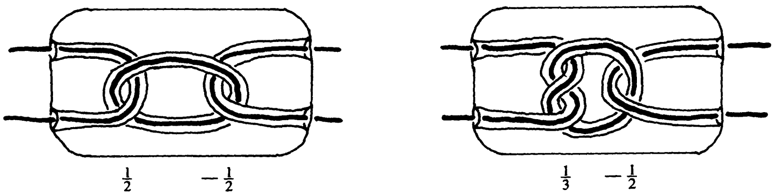


FIGURE 1.3

$K(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ $K(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ $K(\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4})$ $K(\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6})$

FIGURE 1.4

COROLLARY 3. *Suppose $q_i \geq 3$ for each i. If $k \geq 5$ and $q_i = 1 \pmod 2$ for at least one i or $k \geq 4$ and K is a knot, then $S^3 - K$ contains closed incompressible surfaces of every genus ≥ 2.*

COROLLARY 4. *Suppose the star link K is a knot.*

(a) *If $k \leq 3$ the only closed incompressible surface in $S^3 - K$ is the peripheral torus. Hence, all but finitely many Dehn surgeries on K yield irreducible non-Haken manifolds.*

(b) *If $k \geq 4$, $q_i \geq 3$, $S^3 - K$ contains closed incompressible surfaces which remain incompressible in every closed 3-manifold obtained by a non-trivial Dehn surgery on K.*

COROLLARY 5. *$S^3 - K$ has a complete hyperbolic structure if K is not a torus link and K is not equivalent to $K(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$, $K(\frac{2}{3}, \frac{-1}{3}, \frac{-1}{3})$, $K(\frac{1}{2}, \frac{-1}{4}, \frac{-1}{4})$, $K(\frac{1}{2}, \frac{-1}{3}, \frac{-1}{6})$ or the mirror images of these links.*

Bonahon and Siebenmann **[B-S]** have determined which star links are torus links. The other exceptions in Corollary 5 are links whose complements contain incompressible tori which are evident in the projections shown in Figure 1.4.

To prove Theorem 1 a fibered orbifold structure is given to (S^3, K). We show in §2 that a closed "incompressible" 2-suborbifold is either vertical or horizontal, i.e. either transverse to fibers or a union of fibers. This fact is not unexpected since the concept of a fibered orbifold is a generalization of the concept of a Seifert-fibered 3-manifold in which incompressible surfaces are known to be isotopic to horizontal or vertical ones. The closed incompressible surfaces in $S^3 - K$ are shown to be either horizontal 2-orbifolds or obtained from vertical 2-orbifolds (S_C's) by tubing.

The proof of Theorem 2 is less routine; it uses a result on branched surfaces **[F-O]** together with some technical lemmas.

2. Fibered orbifolds, proof of Theorem 1. The reader is referred to **[T]** or **[D]** for definitions of "orbifold" and "orbifold fibration". The theory of 3-orbifolds is analogous to the theory of 3-manifolds. We use the concepts of Euler characteristic, incompressible 2-orbifold, irreducible 3-orbifold, and orientable orbifold, all of which are generalizations from manifold theory. If O is an orbifold we use X_O to denote the underlying space of O.

A Seifert-fibered manifold is a particular example of a 3-orbifold with an orbifold fibration having generic fiber S^1. The main goal of this section is to prove a generalization of the fact that incompressible surfaces in Seifert-fibered manifolds are either vertical or horizontal, i.e. isotopic either to a surface which is a union of fibers or to a surface transverse to fibers. The proof of the generalization is modelled on the proof in **[H]** for the Seifert-fibered case.

We use the term "4-punctured sphere" to denote the 2-orbifold with underlying space the sphere and a rotation-$\mathbf{Z}_2$ singular locus of four isolated points. Similarly a "2-punctured disc" is a 2-orbifold with underlying space a disc and a rotation-$\mathbf{Z}_2$ singular locus of 2 isolated points.

In the following lemma note that a ∂-compressing 2-orbifold is the same as a ∂-compressing disc, i.e. a disc with no singularities.

LEMMA 2.1. *Let F be an incompressible 2-orbifold in the irreducible, orientable 3-orbifold O^3 such that ∂F is contained in components of ∂O which are either tori or 4-punctured spheres. If F is not ∂-incompressible, then F is a boundary-parallel annulus or it is a boundary-parallel 2-punctured disc.*

Proof. Notice first that if F has boundary in a 4-punctured sphere then the components of ∂F separate one pair of punctures from the other pair. (Otherwise, since F is incompressible, F is a punctured disc or a disc, so that $\chi(F) > 0$. But, by definition, for F to be incompressible it is necessary that $\chi(F) \leq 0$.)

Let D be a ∂-compressing disc for F, $D \cap \partial M = \alpha \subset \partial D$. To prove the lemma we consider two cases: If α lies in a 4-punctured sphere of ∂O, then the components of ∂F which contain $\partial \alpha$ are as shown in Figure 2.2(a) or (b). In Figure 2.2(a) β bounds a compressing disc for F, whence, using

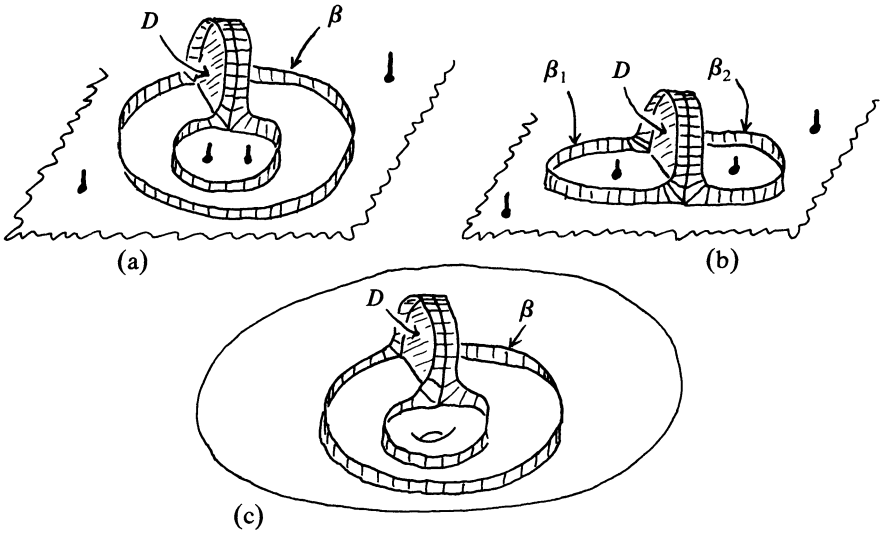


FIGURE 2.2

irreducibility, F is a ∂-parallel annulus. In Figure 2.2(b) β_1 and β_2 bound punctured discs which compress F, whence F is a ∂-parallel 2-punctured disc. Next, we consider the case where α lies on a torus boundary component of O. The components of ∂F containing $\partial\alpha$ must be as shown in Figure 2.2(c). The curve β bounds a disc in O, hence, using irreducibility, F is a ∂-parallel annulus. □

PROPOSITION 2.3. *Let p: $O^3 \to W^2$ be an orbifold fibration of an irreducible, orientable orbifold O^3 (with generic fiber S^1). Then every orientable incompressible, ∂-incompressible 2-suborbifold F^2 of O^3 can be isotoped so it is either a union of fibers (vertical) or transverse to fibers (horizontal).*

Proof. Let C_i $(i = 1,\dots,n)$ be a set of fibers of the fibration which project to points c_i of W including all cyclic-rotation singular points and possibly some isolated non-singular points. Let B_i $(i = 1,\dots,m)$ be a set of fibers which project to points b_i of W including all dihedral, non-$\mathbf{Z}_2$ singular points and possibly some isolated reflection-$\mathbf{Z}_2$ singular points. Let $N(C_i)$ $(N(B_i))$ be a small tubular neighborhood of $C_i(B_i)$ which is a union of fibers. Let $p(N(C_i)) = N(c_i)$ and $p(N(B_i)) = N(b_i)$. We may assume that F intersects each $N(C_i)$ and each $N(B_i)$ in a collection of discs transverse to fibers possibly with one cyclic-rotation singular point each (of the same order).

∂O consists of tori and 4-punctured spheres; the 4-punctured spheres are fibered like $\partial(G \times I)$, where G is a 2-punctured disc fibered as shown in Figure 2.4. Clearly F can be isotoped so each component of ∂F is either a fiber or transverse to fibers.

Let $O_0 = O - [(\bigcup_{i=1}^{n} N(C_i)) \cup (\bigcup_{i=1}^{m} N(B_i))]$. Using the incompressibility of F and the irreducibility of O, we isotope F so that $F \cap O_0$ is incompressible in O_0. We may assume $F \cap O_0$ is also ∂-incompressible otherwise, by Lemma 2.1, $F \cap O_0$ is an annulus or 2-punctured disc ∂-parallel in O_0 and $\chi(F) > 0$, contrary to the definition of "incompressible", or F is a ∂-parallel annulus or a ∂-parallel 2-punctured disc in O.

We now decompose O_0 further by cutting W along enough arcs α_u joining $N(b_i)$'s and $N(c_i)$'s so that $W - [(\bigcup_i N(c_i)) \cup (\bigcup_i N(b_i))]$ cut along the α_u's contains only components with underlying space a disc and a singular locus which is either empty or consists of a single reflection-$\mathbf{Z}_2$ arc on ∂X_W. The union of the fibers over α_u is an annulus A_u. O_0 cut open on the annuli A_u has components O_{1j} of two types: Those fibered as $D^2 \times S^1$ and those fibered as $G \times I$ where G is a 2-punctured disc fibered as shown in Figure 2.4. Since F is ∂-incompressible in O_0 and F is transverse to fibers in $\partial N(B_i)$, $\partial N(C_i)$, it follows that F can be isotoped so that $F \cap A_u$ consists of arcs transverse to fibers, closed curves coinciding with fibers, and closed curves bounding discs in A_u. We isotope $F \cap O_0$ (rel ∂O_0) to ensure that $F \cap O_{1j}$ is incompressible. This removes curves from $F \cap A_u$ which bound discs in A_u. If some component of $F \cap O_{1j}$ is ∂-incompressible, it can be shown to be a disc transverse to the fibers of O_{1j}. Thus $F \cap O_{1j}$ consists entirely of discs transverse to fibers. If, on the other hand, all components of $F \cap O_{1j}$ are ∂-compressible, then, by Lemma 2.1, $F \cap O_{1j}$ consists of ∂-parallel annuli and 2-punctured discs. If $\partial(F \cap O_{1j})$ were horizontal, then $F \cap O_0$ would be ∂-compressible; therefore $F \cap O_{1j}$ is a collection of annuli and 2-punctured discs which can be isotoped (rel ∂O_{1j}) to be vertical. Clearly, if $F \cap O_{1j}$ is vertical (horizontal) for one j then it must be vertical (horizontal) for all j. Hence F is vertical or horizontal. □

We now describe a fibered orbifold O_K with underlying space S^3 and $\mathbf{Z}_2$-singular set $K = K(p_1/q_1, \ldots, p_k/q_k)$. We do this by giving another description of a slope p/q rational tangle. Let $(T, p/q)$ be a solid

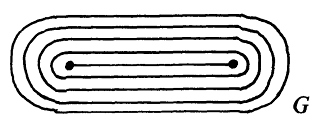


FIGURE 2.4

Seifert-fibered torus with one exceptional fiber having invariants $(q, -p)$. The "invariant" p is only determined mod q, but if we choose a closed curve x on ∂T intersecting each fiber transversely at a single point, then p, q are uniquely determined: if D is a meridian disc for T, the curve ∂D lifts to a line of slope p/q in the universal cover of ∂T relative to a coordinate system with a lift of the curve x as the x-axis and a lift of a fiber as the y-axis. We obtain $(L, p/q)$ as the orbit space of a 180° rotation of T about a diameter of the exceptional fiber of $(T, p/q)$, where the rotation preserves the set of fibers (see Figure 2.5). If the axis of rotation intersects T in arcs κ_0 and κ_1, then κ_0 and κ_1 become the arcs of the tangle. $(L, p/q)$ is a fibered 3-orbifold with rotation-$\mathbf{Z}_2$ singular set $\kappa_0 \cup \kappa_1$. The generic fibers are circles. The exceptional fibers are arcs with reflection-$\mathbf{Z}_2$ singular ends on $\kappa_0 \cup \kappa_1$. We shall view $(L, p/q)$ as a fibered orbifold or as a ball L with embedded arcs κ_0 and κ_1 and a "coordinate system" for ∂L. The coordinate system consists of two closed curves on ∂L: a closed fiber called the *y-axis* or *axis* and the projection of the curve x called the *x-axis* (see Figure 2.5).

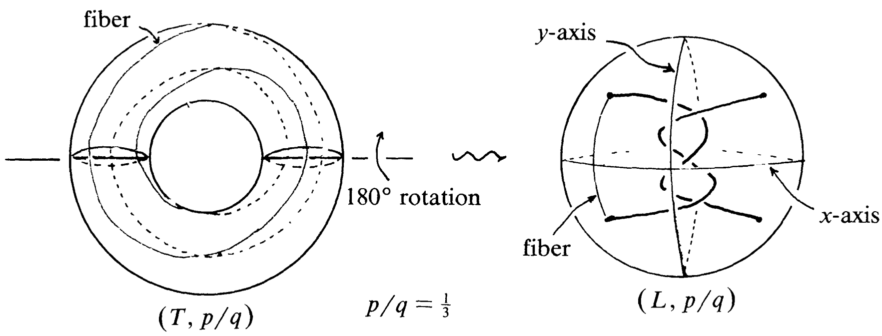


FIGURE 2.5

To construct the 3-orbifold O_K, first deform each tangle $(L_i, p_i/q_i)$ so that it is lens-shaped with its axis at the edge, then identify the left face of L_i with the right face of L_{i+1} (subscripts mod k), as shown in Figure 2.6, so that fibers, punctures, and axes are identified and so that half of one x-axis is identified with half of the next x-axis. The 3-orbifold O_K is fibered over the 2-orbifold shown in Figure 2.7.

Figure 2.5 shows that the double cover of $(L, p/q)$ branched over $\kappa_0 \cup \kappa_1$ is $(T, p/q)$. By identifying the $(T_i, p_i/q_i)$ along annuli corresponding to the faces of the lenses $(L_i, p_i/q_i)$, one shows that the double cover of S^3 branched over K is a Seifert-fibered manifold with base S^2 and exceptional fibers having invariants $(q_i, -p_i)$, $i = 1, 2, \dots, k$.

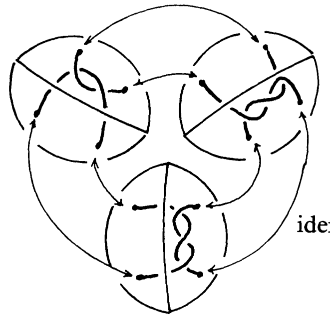


FIGURE 2.6

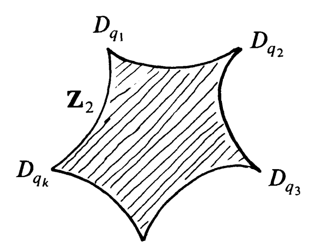


FIGURE 2.7

To apply Proposition 2.3 to (S^3, K), we reinterpret the meaning of "incompressible 2-orbifold" when the 2-orbifold is viewed as a punctured surface in (S^3, K). A 2-orbifold in O_K is incompressible if and only if the corresponding punctured surface in (S^3, K) is incompressible and "peripheral incompressible". By "incompressible" we mean incompressible in $S^3 - K$. A punctured surface S in (S^3, K) is *peripheral incompressible* if for every disc D with $D \cap S = \partial D$ meeting K transversely at a single point P, there is a disc $D' \subseteq S$ with $\partial D = \partial D'$ meeting K transversely at a single point P'. (S is really the pair (S, punctures).)

If we assume O_K is irreducible as an orbifold (as will be shown in Proposition 2.10), then Proposition 2.3 implies the following corollary.

COROLLARY 2.8. *An incompressible, peripheral incompressible punctured surface in (S^3, K) is either vertical or horizontal in the orbifold fibration of O_K.*

LEMMA 2.9. *The obstruction to the existence of a horizontal surface in O_K is $\Sigma_{i=1}^k p_i/q_i$. If $\Sigma_{i=1}^k p_i/q_i = 0$, there is exactly one connected horizontal surface S in O_K up to isotopy. The Euler characteristic of S is $\chi(S) = l(2 - k + \Sigma_{i=1}^k 1/q_i)$ where $l = \text{l.c.m.}(q_1, \ldots, q_k)$. S intersects the tangle $(L_i, p_i/q_i)$ in l/q_i separating discs, where a separating disc is a disc transverse to the fibers of $(L_i, p_i/q_i)$ and separating the arcs of the tangle.*

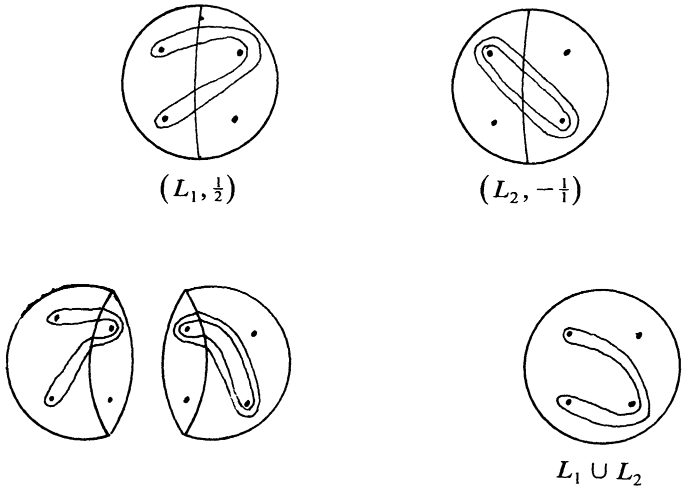


FIGURE 2.10

Proof of Lemma 2.9. If F is a closed surface transverse to fibers in O_K, then $F \cap (L_i, p_i/q_i)$ is a collection of separating discs. $F \cap \partial L_i$ is a collection of closed curves of slope p_i/q_i. (A closed curve of slope p_i/q_i lifts to a slope p_i/q_i line in $\mathbf{R}^2$, the orbifold universal cover of the 4-punctured sphere ∂L_i, when the y-axis and x-axis are chosen to be lifts of the y-axis on ∂L_i.) We fit the curves on ∂L_1 to the curves on ∂L_2 as shown in Figure 2.10. The number of separating discs required in L_i is l.c.m.$(q_1, q_2)/q_i$, $i = 1, 2$. We now have a surface in $L_1 \cup L_2$ whose boundary consists of slope $(p_1/q_1 + p_2/q_2)$ curves on $\partial(L_1 \cup L_2)$. We continue joining local sections of the orbifold fibration until we obtain a surface in $L_1 \cup L_2 \cup \cdots \cup L_k$ whose boundary is a collection of slope $\Sigma_{i=1}^k p_i/q_i$ closed curves on $\partial(L_1 \cup L_2 \cup \cdots \cup L_k)$. The final identification to obtain O_K with a closed surface transverse to fibers is possible if and only if $\Sigma_{i=1}^k p_i/q_i = 0$. □

PROPOSITION 2.11. *If $q_i \geq 2$, O_K is irreducible. This implies $S^3 - K$ is irreducible and K is prime.*

Proof. If O_K were reducible, i.e. if there existed a closed 2-suborbifold F of positive Euler characteristic in O_K not bounding a 3-orbifold of positive Euler characteristic, then a procedure like that in the proof of Proposition 2.3 allows one to change F so it is vertical or horizontal and still has positive Euler characteristic. Throughout the procedure one uses surgery instead of isotopies made possible by the irreducibility of the

3-orbifold. F cannot be vertical since vertical orbifolds are all 4-punctured spheres, hence not elliptic. Thus F is horizontal and $\chi(F) = l(2 - k + \Sigma_{i=1}^{k} 1/q_i)$. If $k \geq 4$, since $q_i \geq 2$, $\chi(F) \leq 0$, a contradiction. When $k = 3$ (and $\Sigma_{i=1}^{k} p_i/q_i = 0$), not all the q_i equal 2. By checking possibilities, we verify $\chi(F) \leq 0$. In the case $k = 2$ (where K is a 2-bridge link), $p_1/q_1 + p_2/q_2 = 0$ implies K is a trivial link. □

If the rational tangles $(L_i, p_i/q_i)$, $i = 1, 2, \ldots, k$, are glued as in Figure 2.6 except that the right face of ∂L_k is not glued to the left face of ∂L_1, the result is a *Seifert tangle* $(B;\ p_1/q_1, \ldots, p_k/q_k)$. The Seifert tangle consists of a ball with two embedded arcs and possibly some embedded closed curves. The union of the closed curves and arcs is denoted κ. The Seifert tangle can also be viewed as a fibered orbifold with $\mathbf{Z}_2$-rotation singular locus κ fibered over the 2-orbifold W^2 shown in Figure 2.12.

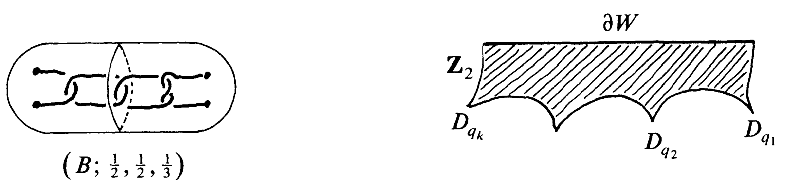


FIGURE 2.12

PROPOSITION 2.13. *If $k \geq 2$, $q_i \geq 2$, the orbifold ∂B is incompressible in the orbifold $(B;\ p_1/q_1, \ldots, p_k/q_k)$. This means the punctured surface ∂B is incompressible (in $B - \kappa$) and ∂B is peripheral incompressible in (B, κ).*

Proof. Using surgery as before, a compressing orbifold can be replaced by one which is vertical or horizontal to the fibering of the Seifert tangle. Vertical surfaces are 2-punctured discs or 4-punctured spheres, not of positive Euler characteristic, therefore the compressing orbifold F must be horizontal. There is a unique horizontal surface whose Euler characteristic is

$$\chi(F) = l\left(1 - k + \sum_{i=1}^{k} \frac{1}{q_i}\right) \leq 0,$$

where $l = \text{l.c.m.}(q_1, \ldots, q_k)$. This contradicts the fact that a compressing orbifold has positive Euler characteristic. □

COROLLARY 2.14. *A vertical 4-punctured sphere in O_K is incompressible as an orbifold if and only if it bounds a Seifert tangle on each side which is not a rational tangle.*

Proof. If a vertical 4-punctured sphere bounds a rational tangle on one side, it is compressible; a separating disc for the rational tangle is a compressing disc. Proposition 2.13 shows that all other vertical orbifolds are incompressible. □

Proof of Theorem 1. Suppose S is a closed incompressible surface in $S^3 - K$. If S is peripheral compressible, we perform surgery on S using a peripheral compressing disc (see Figure 2.15).

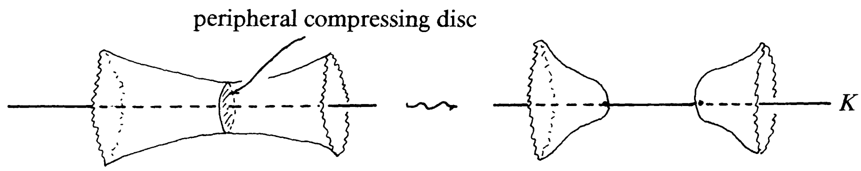


FIGURE 2.15

Clearly surgery on a peripheral compressing disc is the inverse of the tubing operation of Figure 1.2. Repeated peripheral surgery results in an incompressible, peripheral incompressible punctured surface $\hat{S}$ in (S^3, K) which can also be viewed as an incompressible orbifold in (S^3, K). If $\hat{S}$ is horizontal, then $S = \hat{S}$ is horizontal; if $\hat{S}$ is vertical it is a collection of vertical 4-punctured spheres (the S_C's of §1) and S can be recovered from $\hat{S}$ be a sequence of tubing operations.

Lemma 2.9 proves the properties of horizontal surfaces except incompressibility. An easy proof of the incompressibility of horizontal surfaces uses the fact that the double cover of S^3 branched over K is Seifert-fibered over S^2. We use the previous notation: $(T_i, p_i/q_i)$ is the double cover of $(L_i, p_i/q_i)$ branched over the arcs κ_0 and κ_1. Since a horizontal surface S in O_K intersects $(L_i, p_i/q_i)$ in horizontal separating discs, S lifts to a surface $\tilde{S}$ in the Seifert-fibered manifold $\tilde{O}_K$ which intersects $(T_i, p_i/q_i)$ in horizontal discs. To show S is incompressible in $S^3 - K$, let D be a disc such that $D \cap S = \partial D$, $D \cap K = \varnothing$. Then D lifts to a disc $\tilde{D}$ in $\tilde{O}_K$ with $\partial \tilde{D} \subseteq \tilde{S}$. Since $\tilde{S}$ is horizontal in $\tilde{O}_K$ and since horizontal surfaces in Seifert-fibered manifolds are incompressible, $\tilde{S}$ is incompressible and $\partial \tilde{D}$ bounds a disc $\tilde{D}'$ in $\tilde{S}$. The projection of $\tilde{D}'$ to S shows ∂D is null-homotopic in S, so S is incompressible. □

3. Proof of Theorem 2. The proof of Theorem 2 depends on recent work on branched surfaces [**F-O**]. For the purposes of this paper we need only consider branched surfaces locally modelled on the space $\mathcal{U} \hookrightarrow \mathbf{R}^3$ shown in Figure 3.1.

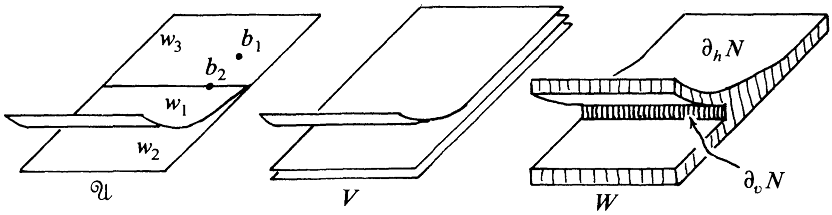


FIGURE 3.1

A subspace $\mathfrak{B}$ of an orientable, irreducible M^3 is a *branched surface* embedded in M if a neighborhood of $b \in \mathfrak{B}$ in $(M, \mathfrak{B})$ is modelled on a neighborhood of b_1 or $b_2 \in \mathfrak{U}$ in $(\mathbf{R}^3, \mathfrak{U})$.

A surface S is carried by $\mathfrak{B}$ if it is related to $\mathfrak{B}$ as V is related to $\mathfrak{U}$ in Figure 3.1. S is determined by integer weights on components of $\mathfrak{B}$ − (branch set), which satisfy obvious conditions like the condition $w_1 + w_2 = w_3$ satisfied by the weights on $\mathfrak{U}$. We will use a fibered neighborhood $N = N(\mathfrak{B})$ which is to $\mathfrak{B}$ as W is to $\mathfrak{U}$ in Figure 3.1. The boundary of N is divided into $\partial_h N$, the portion transverse to fibers, and $\partial_v N$, the portion contained in a union of fibers. A version of the main theorem of **[F-O]** follows:

THEOREM 3.2. *Suppose $\mathfrak{B}$ is a branched surface in M disjoint from ∂M such that no branch circle bounds a disc in M. Further, suppose*:

(1) *There are no monogons in $M \setminus \mathfrak{B}$.* I.e. *there is no disc D with $D \cap N = \partial D$, $\partial D = \alpha \cup \beta$ where α is a fiber of $\partial_v N$ and $\beta \subseteq \partial_h N$* (Figure 3.3).

(2) *$\partial_h N$ is incompressible in $M - \mathring{N}$.*

If S is carried by $\mathfrak{B}$ with positive weights then S is incompressible.

To apply Theorem 3.2 to the proof of Theorem 2 we need two technical lemmas analyzing the compressing discs in "rational tangle exteriors" and "Seifert tangle exteriors". If κ denotes the arcs and closed curves embedded in a ball B to construct a Seifert tangle, the *Seifert tangle exterior* is $B \setminus \mathring{N}(\kappa)$.

FIGURE 3.3

Recall that the tangling of a rational tangle $(L, p/q)$ is measured against a coordinate system on ∂L consisting of an x-axis and a y-axis or axis. If the 4 points of $\kappa \cap \partial L$ are regarded as punctures, then ∂L is a 4-punctured sphere. Thus we can represent ∂L using a square "pillowcase" model where each edge of the pillowcase is either disjoint from the x-axis or from the y-axis. A slope p/q arc on ∂L is constructed by drawing a slope p/q line on the pillowcase starting at a corner or puncture and wrapping around edges (Figure 1.1a). Similarly a slope p/q closed curve is constructed by drawing a slope p/q line on the pillowcase which does not intersect any puncture. It is then easy to see from the definition of "rational tangle" in §2 that each arc of the tangle can be isotoped (rel endpoints) to a slope p/q arc on ∂L. Also, the boundary of a separating disc for $(L, p/q)$ is a slope p/q closed curve. A slope p/q arc γ has geometric intersection numbers $i(\gamma, y\text{-axis}) = q$ and $i(\gamma, x\text{-axis}) = |p|$. For a slope p/q closed curve δ, $i(\delta, y\text{-axis}) = 2q$ and $i(\delta, x\text{-axis}) = 2|p|$.

Before dealing with the compressing discs in a rational tangle exterior of its boundary, we consider compressing discs in a 3-manifold M of its boundary. If D is a compressing disc for ∂M in M and H is a disc embedded in M so that $H \cap D = \alpha$, $H \cap \partial M = \beta$, $\alpha \cup \beta = \partial H$, and $\alpha \cap \beta = S^\circ$, then *half-disc surgery* (Figure 3.4) divides D into two discs D_1 and D_2, at least one of which is a compressing disc. Even after $(D_1, \partial D_1)$ and $(D_2, \partial D_2)$ have been isotoped disjointly in $(M, \partial M)$, D can be recovered (up to isotopy of D) as the *disc-sum of* D_1 *and* D_2 *along* ρ for a suitable arc ρ in ∂M joining ∂D_1 to ∂D_2 (Figure 3.5). The disc-sum of D_1 and D_2 along ρ is denoted $D_1 \#_\rho D_2$.

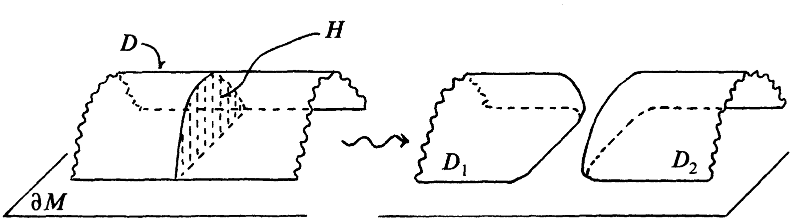


FIGURE 3.4

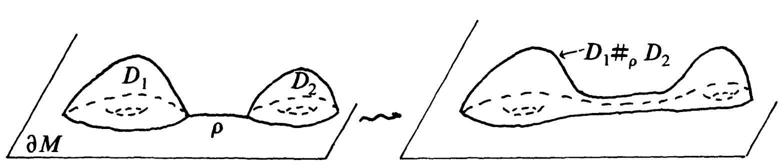
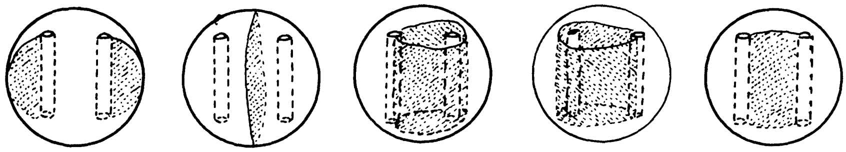


FIGURE 3.5

Figure 3.6 (a) through (e) shows some obvious compressing discs for $\partial(L - \mathring{N}(\kappa))$ in the rational tangle exterior $L - \mathring{N}(\kappa)$. In the figure the coordinates on ∂L have been changed; relative to the new coordinates the axis is a slope s/q closed curve where s is some integer with $(s, q) = 1$.



(a) (b) (c) (d) (e)

FIGURE 3.6

LEMMA 3.7. *Among all compressing discs D of $\partial(L - \mathring{N}(\kappa))$ in $L - \mathring{N}(\kappa)$, the minimum value of the geometric intersection number $i(\partial D$, axis$)$ is taken when $(D, \partial D)$ is isotopic in $(L - \mathring{N}(\kappa), \partial(L - \mathring{N}(\kappa)))$ to one of the compressing discs in Figure* 3.6 (a) *or* (e) *up to a change of coordinates on ∂L fixing the slope of the boundary of a separating disc, i.e. up to a twist of $L - \mathring{N}(\kappa)$ fixing a separating disc.*

Proof. Denote the curve system consisting of the four closed curves $\partial(\partial N(\kappa))$ at the ends of the tubes $\partial N(\kappa)$ by λ. Let D be a compressing disc for $\partial(L - \mathring{N}(\kappa))$ with ∂D transverse to the axis and λ. Let $I(\partial D$, axis) denote the actual number of intersections of ∂D with the axis. We isotope D to minimize $I(\partial D$, axis) so that $I(\partial D$, axis) $= i(\partial D$, axis).

Without increasing $I(\partial D$, axis), we can replace D by a compressing disc having the following two properties:

(a) There is no half-disc $H \subseteq \partial(L - \mathring{N}(\kappa))$ with $\partial H = \mu \cup \nu$ where ν is an arc in λ and μ is an arc in ∂D.

(b) If there is a half-disc H in $L - \mathring{N}(\kappa)$ with $\partial H = \mu \cup \nu$, ν an arc in $\partial N(\kappa)$ and μ an arc in D, then there is a half-disc $H' \subseteq D$ with $\partial H' = \mu \cup \nu'$ where $\nu' \subseteq \partial N(\kappa)$.

For if D fails to satisfy (a) we can isotope $\mu \subseteq \partial D$ to ν and beyond, reducing $I(\partial D, \lambda)$. If D satisfies (a) and fails to satisfy (b), then the half-disc H in (b) can be used to perform half-disc surgery on D, and D can be replaced by one of the two resulting compressing discs. This operation also reduces $I(\partial D, \lambda)$.

Let D be a compressing disc satisfying (a) and (b). We will show D is one of the discs in Figure 3.6 up to a twist fixing a separating disc. Let E be a separating disc transverse to D for the tangle. We isotope ∂E to minimize intersections with ∂D, then we isotope $E(\text{rel}\,\partial E)$ to eliminate closed curves of $E \cap D$. Suppose an arc $D \cap E$ edgemost in E cuts a half-disc H from E. Half-disc surgery using H splits D into two discs D_1 and D_2. If D_1 or D_2 failed to satisfy (a) then D would not satisfy (b), a contradiction. If D_1 or D_2 failed to satisfy (b), then D would not satisfy (b). Thus $D = D_1 \#_\rho D_2$, where ρ is disjoint from $\partial N(\kappa)$. Now let $\mathfrak{D} = D_1 \cup D_2$. We repeat the above process on $\mathfrak{D}$: if H is a half-disc cut from E by an arc of $\mathfrak{D} \cap E$ edgemost in E, we perform half-disc surgery on a disc in $\mathfrak{D}$ and replace the disc by the two resulting discs. Eventually, we obtain $\mathfrak{D}$ with $\mathfrak{D} \cap E = \varnothing$ and $\mathfrak{D}$ a union of compressing discs, each satisfying (a) and (b). D can be recovered from $\mathfrak{D}$ by performing disc-sums between pairs of discs in $\mathfrak{D}$. Further, if D_1 and D_2 are discs in $\mathfrak{D}$ we need only consider disc-sums $D_1 \#_\rho D_2$ such that $\rho \subseteq \partial L - \mathring{N}(\kappa)$ and $D_1 \#_\rho D_2$ satisfies (a) and (b). Since each of the discs of $\mathfrak{D}$ is a compressing disc disjoint from E, it must be one of the discs shown in Figure 3.6 (a) or (b).

To construct all the possibilities for D, the reader should first consider $D_1 \#_\rho D_2$, where D_1 and D_2 are of the types shown in (a) or (b). The only possibilities for $D_1 \#_\rho D_2$, up to a twist fixing E, are those shown in Figure 3.6 (a)–(e). Next, the reader can verify that $D_1 \#_\rho D_2$ yields no new discs if D_1 and D_2 are each isotopic to one of the discs in (a)–(e).

We have shown that for compressing discs D, $i(\partial D, \text{axis})$ takes its minimum value when D is one of the discs in Figure 3.6. But for each disc D of the type shown in (b), $i(\partial D, \text{axis})$ is twice as large as for a disc of the type shown in (a). For each disc of the types shown in (c) or (d), $i(\partial D, \text{axis})$ is twice as large as for a disc of the type shown in (e). Hence $i(\partial D, \text{axis})$ takes its minimum value when D is one of the discs in (a) or (e). □

LEMMA 3.8. *Let $(L, p/q)$ be a rational tangle with $q \geq 2$. Then any compressing disc D for $\partial(L - \mathring{N}(\kappa))$ in $L - \mathring{N}(\kappa)$ satisfies $i(\partial D, axis) \geq 2$.*

Proof. By Lemma 3.7 we need only check $i(\partial D, \text{axis}) \geq 2$ when D is one of the discs in Figure 3.6 (a) or (e). A disc D like that in (a) gives an isotopy of an arc of the tangle to an arc in ∂L whose intersection number with the axis is $i(\partial D, \text{axis})$. Therefore $i(\partial D, \text{axis}) = q \geq 2$. To check $i(\partial D, \text{axis}) \geq 2$ when D is a disc like that in (e), recall that the axis is a slope s/q closed curve on ∂L (in the coordinates of Figure 3.6) and note that closed curves of every slope except $0/1$ intersect the two slope $0/1$ arcs of $\partial D \cap (\partial L - \mathring{N}(\kappa))$ at least twice. □

LEMMA 3.9. *Let $(B; p_1/q_1, \ldots, p_k/q_k)$ be a Seifert tangle with $q_i \geq 2$, $i = 1, 2, \ldots, k$, $q_1 \geq 3$, $q_k \geq 3$ and $k \geq 2$. Let κ denote the arcs and closed* *curves embedded in the ball B to construct the Seifert tangle. Then* $\partial(B - \mathring{N}(\kappa))$ *is incompressible in* $B - \mathring{N}(\kappa)$.

Proof. $(B;\ p_1/q_1,\ldots,p_k/q_k)$ is constructed using the k tangles $(L_i,\ p_1/q_i)$, $i = 1,2,\ldots,k$, as described in §2. Let

$$C_i = (\partial L_i \cap \partial L_{i+1}) \setminus \mathring{N}(\kappa), \qquad i = 1,2,\ldots,k-1.$$

Suppose D is a compressing disc for $\partial(B - \mathring{N}(\kappa))$ which is transverse to $\bigcup_{i=1}^{k-1} C_i$. Consider $D \cap (\bigcup_{i=1}^{k-1} C_i)$ (Figure 3.10). An innermost closed curve bounding a disc D' in D is either isotopic to the axis in some C_i or it bounds a disc in C_i. If $\partial D'$ is isotopic in C_i to the axis, then D' is a separating disc for L_i or L_{i+1} and, since $\partial D'$ does not intersect the axis, $p_i/q_i = 1/0$ or $p_{i+1}/q_{i+1} = 1/0$. If $\partial D'$ bounds a disc in C_i, we eliminate the circle of intersection by an isotopy.

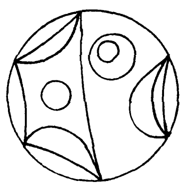


FIGURE 3.10

Whenever an arc of $D \cap (\bigcup_i C_i)$ is isotopic to an arc of the axis in some C_i, i.e. whenever the arc cuts a half-disc from C_i, use half-disc surgery to eliminate the arc of intersection. One of the resulting discs is a compressing disc whose boundary intersects the axis in fewer points than the boundary of the original disc. When all possible half-disc surgeries of this type have been done, the remaining arcs of $D \cap (\bigcup C_i)$ must be non-trivial in C_i.

Let H be a disc cut from D by an arc of $D \cap (\bigcup_i C_i)$ edgemost in D. H must be a compressing disc for $L_1 - \mathring{N}(\kappa)$ or $L_k - \mathring{N}(\kappa)$; assume it is a compressing disc for $L_1 - \mathring{N}(\kappa)$. ∂H intersects ∂C_1 twice, therefore $i(\partial H, \text{axis}) = 0, 1$, or 2. By Lemma 3.8, $i(\partial H, \text{axis}) = 2$, therefore ∂H does not intersect the other components of ∂C_1 (which are ends of tubes in $\partial(L_1 - \mathring{N}(\kappa))$). ∂H passes through at most one tube of the boundary of the first tangle. Using half-disc surgery with half-discs H disjoint from $\partial L_1 - \mathring{N}(\kappa)$, we can replace H by one of the compressing discs in Figure 3.6. The disc in (e) is ruled out because its boundary passes through both tubes. The discs in (b), (c), and (d) are ruled out because $i(\partial H, \text{axis})$ does not take the minimum value (among all compressing discs H). Thus H is a disc in (a) and $q_1 = 2$ contrary to assumption.

We conclude that $D \cap (\bigcup C_i) = \varnothing$, whence ∂D is a slope $1/0$ closed curve in ∂L_1 or ∂L_k and D is a separating disc. But this would imply $q_1 = 0$ or $q_k = 0$, so there are no compressing discs. $\square$

Proof of Theorem 2. If a closed surface in $S^3 - K$, obtained by taking a collection of vertical incompressible 4-punctured spheres, is constructed using a tube which does not pass through a rational tangle, the surface is compressible. It follows that every incompressible tubed surface is carried by the branched surface $\mathfrak{B}$ shown in Figure 3.11 for $K = K(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2})$.

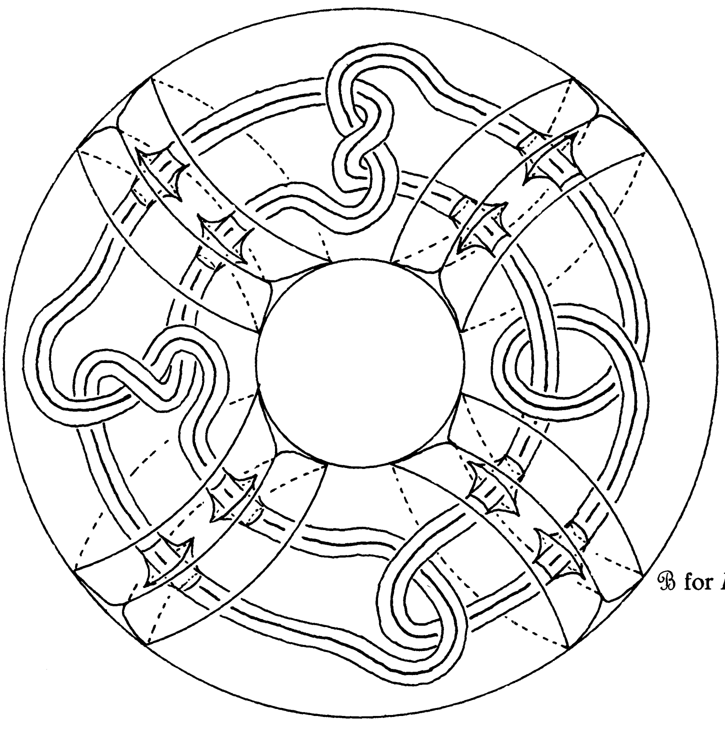


FIGURE 3.11

We must prove that when $q_i \geq 3$, $i = 1, 2, \ldots, k$, a surface F carried by $\mathfrak{B}$, obtained by tubing vertical incompressible 4-punctured spheres, is incompressible. Let $\mathfrak{B}_1$ be a branched subsurface of $\mathfrak{B}$ carrying F with positive weights and let $N = N(\mathfrak{B}_1)$ be a fibered regular neighborhood of $\mathfrak{B}_1$. Let M be the manifold $S^3 - \mathring{N}(K)$. To prove that F is incompressible it is enough, by Theorem 3.2, to show that there are no monogons in $M - \mathring{N}$, that $\partial_h N$ has no compressing discs, and that no branch circle bounds a disc in M. Every branch circle is either a meridian circle, which is certainly not null-homotopic in M, or it is isotopic to the axis. To prove that the axis does not bound a disc in M, consider the orbifold $\hat{O}_K$ obtained from O_K by removing a small regular neighborhood of the axis which is a union of fibers. If D were a disc in M with $\partial D =$ axis, then $D \subset O_K$ yields another disc D with $\partial D \subseteq \partial \hat{O}_K$ non-trivial. This disc can be replaced (as in the proof of Proposition 2.10) by one which is either vertical or horizontal. As usual we check that no vertical or horizontal surface is a disc.

Next we show there are no monogons in $M - \mathring{N}$ and no compressing discs for $\partial_h N$ in $M - \mathring{N}$.

We insert an annulus E_i in every peripheral tube of $\partial_h N$ as shown in Figure 3.12. After possibly modifying $\mathcal{B}_1$ as shown in Figure 3.13, every component M_j of $M - \mathring{N}$ cut open on $\bigcup_i E_i$ is topologically either

(1) $B - \mathring{N}(\kappa)$, where B with embedded curves κ is a Seifert tangle $(B; p_{r+1}/q_{r+1}, \dots, p_{r+s}/q_{r+s})$, $2 \le s \le k - 2$; or

(2) $L - \mathring{N}(\kappa)$, where L with embedded arcs κ is a rational tangle $(L, p_r/q_r)$.

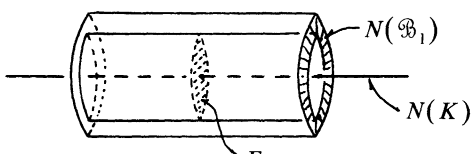


FIGURE 3.12

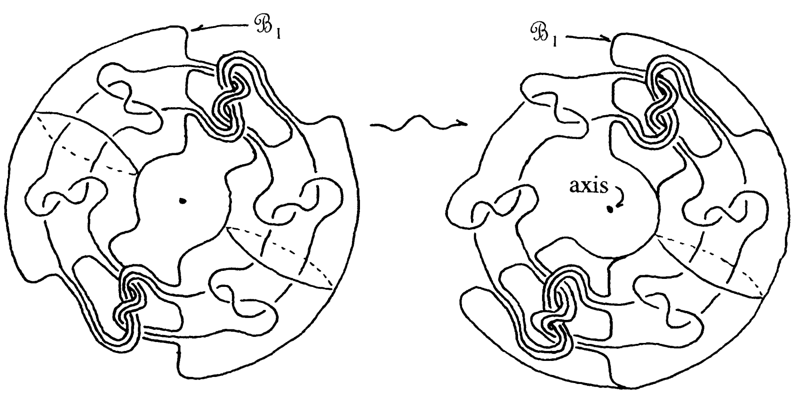


FIGURE 3.13

If M_j is a rational tangle exterior, ∂M_j must contain an annulus of $\partial_v N$ isotopic in ∂M_j to a regular neighborhood in ∂M_j of the axis. ∂M_j also includes parts of $\partial_h N$ and possibly annuli E_i, annuli in $\partial N(\kappa)$ and other components of $\partial_v N$.

If D is a compressing disc for $\partial_h N$, or a monogon, and D intersects some E_i, then half-disc surgery on D results in a compressing disc or monogon disjoint from $\bigcup E_i$. Therefore we may assume $D \subseteq M_j$ for some j. If D is a compressing disc for $\partial_h N$ contained in a rational tangle exterior

M_j, then, since ∂D does not intersect $\partial_v N$, we may assume it does not intersect the axis of ∂M_j and, by Lemma 3.8, ∂D bounds a disc D' in ∂M_j. In fact, $D' \subseteq \partial_h N$, otherwise some branch circle of $\mathfrak{B}_1$ would bound a disc in M or a component of some ∂E_i would bound a disc. D could not have been a compressing disc. Similarly if D is a disc in a Seifert tangle exterior M_j with $D \cap \partial_h N = \partial D$, we use Lemma 3.9 to show there is a $D' \subseteq \partial_h N$ with $\partial D' = \partial D$.

If D is a monogon in some rational tangle exterior M_j, then ∂D intersects the axis of ∂M_j at most once and by Lemma 3.8 there is a disc D' in ∂M_j with $\partial D' = \partial D$. But this implies ∂D intersects $\partial_v N$ in an even number of fibers, a contradiction. Similarly if D is a monogon in a Seifert tangle exterior M_j, we get a contradiction from Lemma 3.9. □

4. Proofs of corollaries.

Proof of Corollary 3. If $k \geq 5$, $q_1 = 1 \pmod 2$ and $q_i \geq 3$ for all i, then an arc of the first tangle $(L_1, p_1/q_1)$ goes from the left side of the tangle to the right side. Let U be a 4-punctured sphere isotopic to $\partial(L_2 \cup L_3)$ and let V be a 4-punctured sphere isotopic to $\partial(L_4 \cup L_5 \cup \cdots \cup L_k)$. We construct a tubed closed incompressible surface using n copies, $U_1, U_2, \ldots, U_n$, of U and one copy of V by tubing as shown schematically in Figure 4.1. All the remaining punctures of the U_i's and V can be paired using tubes passing through at least one rational tangle. The tubes connect pairs (U_1, V), (U_2, V), $(U_3, U_1), \ldots, (U_n, U_{n-2})$, hence the surface is connected. The genus of the surfaces is $n + 2$.

If $k \geq 4$ and K is a knot, we choose any incompressible 4-punctured sphere U and let $U_1, \ldots, U_n$ be n copies of U. Tube as shown in Figure 4.2. The tubes connect pairs (U_i, U_i) $(i = 1, 2, \ldots, n)$, (U_{n-1}, U_n), (U_{n-2}, U_n), (U_{n-3}, U_{n-1}), $(U_{n-4}, U_{n-2}), \ldots, (U_1, U_3)$. again it is possible to connect the remaining punctures with non-trivial tubes to give a surface of genus $n + 1$. □

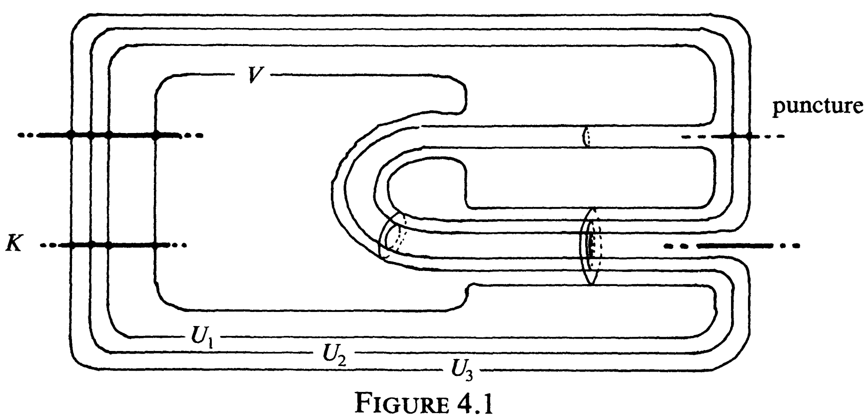


FIGURE 4.1

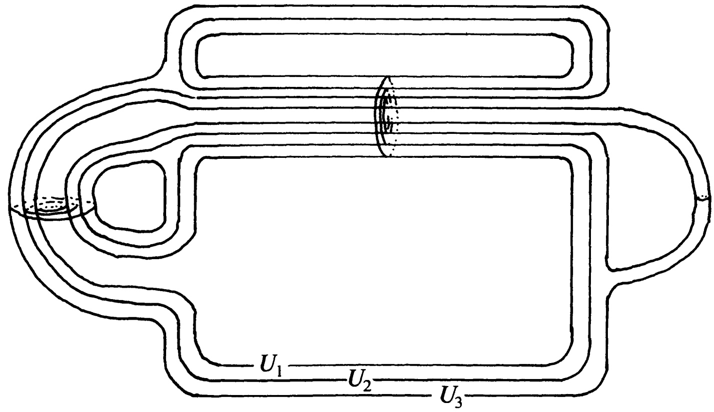


FIGURE 4.2

To prove Corollary 4 we need the following result.

THEOREM. (*Menasco*) *Let K be a knot and suppose there is a closed incompressible surface S embedded in $S^3 - K$ having the property that there exist disjoint peripheral compressing discs D_1 and D_2, with D_i meeting K transversely at a single point such that ∂D_1 is not isotopic to ∂D_2 in S. Then S remains incompressible in any manifold obtained from K using non-trivial Dehn surgery.*

A proof can be found in [**Me**].

Proof of Corollary 4. Part (b) is an immediate application of Menasco's theorem using a tubed incompressible surface.

For part (a) we note that in the complement of a star knot of three tangles there are no incompressible 4-punctured spheres, hence no incompressible tubed surfaces. It can be shown that when $\Sigma_{i=1}^{k} p_i/q_i = 0$, K is a link of at least two components, hence when K is a knot there are no horizontal surfaces. Every closed incompressible surface is a peripheral torus.

In order to conclude that all but finitely many Dehn surgeries on K yield non-Haken manifolds we use a result of A. Hatcher [**H1**] which states that in a knot exterior $S^3 - \mathring{N}(K)$ only finitely many isotopy classes of closed curves in the boundary torus are realized as the boundaries of incompressible, ∂-incompressible surfaces. If there is an incompressible surface S (an S^2 not bounding a ball) in the Dehn surgery manifold $M_{r/s}(K)$ of a knot K, one can show that there is an incompressible,

∂-incompressible surface in $S^3 - \mathring{N}(K)$ whose boundary components have slopes r/s or there is a non-peripheral closed incompressible surface (an S^2 not bounding a ball) in $S^3 - \mathring{N}(K)$. The details of this argument can be found in [T]. Thus for all slopes r/s not realized as boundaries of incompressible, ∂-incompressible surfaces, $M_{r/s}(K)$ is irreducible (non-Haken) if $S^3 - \mathring{N}(K)$ is irreducible (contains no closed incompressible surfaces). □

Proof of Corollary 5. We apply Thurston's theorem on the existence of hyperbolic structures. To do this we must show that $S^3 - N^\circ(K)$ is atoroidal and anannular. An incompressible torus, T^2 in $S^3 - \mathring{N}(K)$, when K is a star knot, must be horizontal since tubed surfaces have genus ≥ 2. Thus $\Sigma_{i=1}^k p_i/q_i = 0$ and $\chi(T^2) = 0 = l(2 - k + \Sigma_{i=1}^k 1/q_i)$ where $l = \text{l.c.m.}(q_1, q_2, \ldots, q_k)$. The only solutions occur when: (1) $k = 3$ and the q_i take values 3, 3, and 3 or 2, 4, and 4 or 2, 3, and 6; or (2) $k = 4$ and the q_i take values 2, 2, 2, and 2. In fact, every link corresponding to a solution of the two equations above is equivalent to one of the links in the statement because the integral part of the slope of a rational tangle represents a vertical twist of the two right (or left) ends of the tangle. In a star link the vertical twist can be transferred fromone tangle to the next so

$$K\left(\frac{p_1}{q_1}, \ldots, \frac{p_r}{q_r}, \frac{p_{r+1}}{q_{r+1}}, \ldots, \frac{p_k}{q_k}\right)$$

is equivalent to

$$K\left(\frac{p_1}{p_1}, \ldots, \frac{p_r - q_r}{q_r}, \frac{p_{r+1} + q_{r+1}}{q_{r+1}}, \ldots, \frac{p_k}{q_k}\right).$$

To complete the proof we must show that $S^3 - \mathring{N}(K)$ is also anannular. But an atoroidal manifold whose boundary is a collection of tori is also anannular unless it is Seifert-fibered (see [H]). Further, the only link exteriors which are Seifert-fibered are the exteriors of torus links. Therefore $S^3 - \mathring{N}(K)$ is anannular when K is not a torus link. □

Acknowledgements. This paper contains the most significant results from my dissertation, submitted to the University of California, Los Angeles in 1980. Many of the improvements in this version, including better proofs and better corollaries, were suggested by A. Hatcher. W. Dunbar carefully read the manuscript and suggested further improvements and corrections.

References

[B-S] Francis Bonahon and Larry Siebenmann, *Geometric splittings of classical knots and the algebraic knots of Conway*, to appear L.M.S. lecture notes.

[D] William Dunbar, *Fibered orbifolds and crystallographic groups*, dissertation, Princeton University, 1981.

[F-O] William Floyd and Ulrich Oertel, *Incompressible surfaces via branched surfaces*, to appear in Topology.

[H] Allen Hatcher, *Torus decomposition*, xeroxed notes.

[H1] ______, *On the boundary curves of incompressible surfaces*, Pacific J. Math., **99** (1982), 373–377.

[H-T] Allen Hatcher and William Thurston, *Incompressible surfaces in 2-bridge knot complements*, preprint.

[Me] William Menasco, *Incompressible surfaces in the complement of alternating knots and links*, dissertation, University of California, Berkeley, 1981.

[O] Ulrich Oertel, *Incompressible surfaces in complements of star links*, thesis, University of California, Los Angeles, 1980.

[T] William Thurston, *The topology and geometry of 3-manifolds*, xeroxed notes.

Received January 27, 1982 and in revised form May 21, 1982.

MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824