PACIFIC JOURNAL OF MATHEMATICS
Vol 111, No 1, 1984

A MARCINKIEWICZ CRITERION FOR L^p-MULTIPLIERS

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Suppose m is a bounded measurable function on the n-dimensional Euclidean space $\mathbf{R}^n$. Define a linear operator T_m by $(T_m f)\hat{} = mf\hat{}$, where $f \in L^2 \cap L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, and $f\hat{}$ denotes the Fourier transform of f:

$$f\hat{}(\xi) := \int f(x) e^{-ix\xi}\, dx \qquad \left(x\xi := \sum_{j=1}^{n} x_j \xi_j \right).$$

(We omit the domain of integration if it is the whole $\mathbf{R}^n$.) If T_m is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$, then m is called an L^p-(Fourier) multiplier, denoted $m \in M_p(\mathbf{R}^n)$. The norm of m coincides with the operator norm of T_m.

THEOREM 1. ***Let m and m' be locally absolutely continuous on $(0, \infty)$ and***

$$B := \|m\|_\infty + \sup_{j \in Z} \int_{2^j}^{2^{j+1}} r\,|m''(r)|\,dr < \infty.$$

Then $m(|\xi|) \in M_p(\mathbf{R}^n)$ for all p with $1 \le 2n/(n+3) < p < 2n/(n-3) \le \infty$; in particular, $\|m\|_{M_p(\mathbf{R}^n)} \le cB$ with c independent of m.

1. To prove Theorem 1 we need a result stated in Theorem 2 about the following Littlewood-Paley function:

$$(1.1) \qquad g_\lambda(f)(x) = \left(\int_0^\infty |S_t^{\lambda+1}(f;x) - S_t^\lambda(f;x)|^2 u(t) \frac{dt}{t} \right)^{1/2},$$

where

$$S_t^\lambda(f;x) = \int \left(1 - \frac{|\xi|^2}{t^2} \right)_+^\lambda f\hat{}(\xi) e^{i\xi x}\, d\xi \qquad (r_+ = \max(0, r))$$

denotes the Bochner-Riesz means of f of order λ, u is a nonnegative measurable function on $(0, \infty)$ satisfying

$$(1.2) \qquad t \le R(t) = \int_0^t u(s)\, ds \le ct, \qquad t > 0,$$

and f belongs to S, the space of all infinitely differentiable rapidly decreasing functions on $\mathbf{R}^n$.

THEOREM 2. *Let λ and p be such that*

$$1 < p < 2(n+1)/(n+3), \qquad \lambda > n(1/p - 1/2) - 1/2,$$

are valid. Then

$$\|g_\lambda(f)\|_p \leq c\|f\|_p$$

holds uniformly for $f \in S$.

By c or C we always denote a constant that may be different on various occasions.

The above g_λ-function is a modification of the g_δ^*-function of Bonami and Clerc [**1**; p. 242], used by them for deriving sufficient criteria of Marcinkiewicz type for zonal multipliers of expansions into spherical harmonics, and can be regarded as a variant of Stein's g_δ-function [**7**; p. 130], which in our context reads as follows:

$$g_\lambda^*(f)(x) = \left(\int_0^\infty |S_t^{\lambda+1}(f;x) - S_t^\lambda(f;x)|^2 \frac{dt}{t}\right)^{1/2}.$$

Its L^p-behaviour has been investigated by Igari and Kuratsubo in [**6**] where they have shown via analytic interpolation between the points $(\lambda_0, 1/p_0)$ and $(\lambda_1, 1/p_1)$, $\lambda_0 = (n-1)/2 + \varepsilon$, $1/p_0 = 1 - \varepsilon$ or $1/p_0 = \varepsilon$, and $\lambda_1 = -\frac{1}{2} + \varepsilon$, $1/p_1 = \frac{1}{2}$ $(\varepsilon \to 0+)$ that

$$\text{(1.3)} \qquad c_1\|f\|_p \leq \|g_\lambda^*(f)\|_p \leq c_2\|f\|_p, \qquad 1 < p < \infty, \quad \lambda > n|1/p - 1/2| - 1/2,$$

where each $c_j > 0$ is independent of $f \in S$. Had we applied the interpolation argument of [**6**] to the g_λ-function defined in (1.1) as Bonami and Clerc [**1**; pp. 240, 242] did for their g_δ^*-function, we could only take $(\lambda_1, 1/p_1)$, $\lambda_1 = \varepsilon$, $1/p_1 = 1/2$ $(\varepsilon \to 0+)$ as a second interpolation point. We should have then obtained

$$\|g_\lambda(f)\|_p \leq c\|f\|_p, \qquad 1 < p < \infty, \quad \lambda > (n-1)|1/p - 1/2|$$

uniformly for $f \in S$, hence the same result as that of Bonami and Clerc for their g_δ^*-function, which is not a good estimate in view of (1.3). In Theorem 2 we give an improvement of the above estimate in the sense of (1.3). The method of proof used here is a modification of techniques of Fefferman [**2**; pp. 28–33] in combination with the Tomas and Stein restriction theorem [**9**] for the Fourier transform. This theorem is applied at a crucial point of the proof and implies the restriction

$p \leq 2(n+1)/(n+3)$, which is subsequently sharpened to $p < 2(n+1)/(n+3)$ after the use of the Marcinkiewicz interpolation theorem. Proceeding analogously to Bonami and Clerc [**1**; pp. 246–7] we derive Theorem 1 from Theorem 2.

The plan of the paper is the following. In §2 we prove Theorem 2. In §3 we derive Theorem 1 and make several remarks; in particular we show that Theorem 1 is best possible regarded as a Marcinkiewicz type criterion.

2. Let us recall the following decomposition Lemma, which is an essential tool for the proof of Theorem 2 (see [**2**; p. 15]).

LEMMA. *Let $f \in L^p(\mathbf{R}^n)$ and $\alpha > 0$ be given. Then there exist two functions h and b and a collection $\{I_j\}_{j \in \mathbf{N}}$ of pairwise disjoint cubes with the following properties*:

(2.1) $f = h + b, \quad \|h\|_p + \|b\|_p \leq A\|f\|_p.$

(2.2) $|h(x)| \leq A\alpha$ *for almost every* $x \in \mathbf{R}^n$.

(2.3) $b(x) = 0$ *for every* $x \notin \Omega := \bigcup_{j \in \mathbf{N}} I_j$.

(2.4) $\int_{I_j} |b(x)|^p \, dx \leq A\alpha^p |I_j|, \quad \int_{I_j} b(x)\, dx = 0$ *for every* I_j,

where $|I_j|$ *denotes Lebesgue measure of* I_j.

(2.5) $|\Omega| = \sum_{j \in \mathbf{N}} |I_j| \leq A\alpha^{-p}\|f\|_p^p.$

(2.6) *Each cube has diameter equal to* 2^k *for some* $k \in Z$.

Let I_j^* *be a cube with the same center as* I_j *but with*

(2.7) *sides twice as large. Then no point* $x \in \mathbf{R}^n$ *belongs to more than* N *of the cubes* I_j^*.

Proof of Theorem 2. Let $f \in S$ be given. In view of the Marcinkiewicz interpolation theorem [**8**; p. 21], it suffices to show that

$$|\{x: g_\lambda(f)(x) > \alpha > 0\}| \leq c\alpha^{-p}\|f\|_p^p$$

holds uniformly in α and $f \in S$. By (2.1) we have

$$(2.8) \quad |\{x: g_\lambda(f)(x) > \alpha\}| \leq |\{x: g_\lambda(h)(x) > \alpha/2\} + |\{x: g_\lambda(b)(x) > \alpha/2\}.$$

Thus we may estimate each term on the right side separately. Let us begin with the first one, to which we apply the standard argument (see [**8**; p. 20])

$$(2.9)\qquad |\{x: g_\lambda(f)(x) > \alpha\}| \leq \alpha^{-2}\|g_\lambda(h)\|_2^2.$$

Now by theorems of Fubini and Plancherel we obtain

$$\begin{aligned}\|g_\lambda(h)\|_2^2 &= \int_0^\infty \|S_t^{\lambda+1}(h;\cdot) - S_t^\lambda(h;\cdot)\|_2^2 u(t)\,\frac{dt}{t}\\ &= \int |\hat{h}(\xi)|^2 \left\{\int_0^\infty \left(\frac{|\xi|}{t}\right)^4 \left(1 - \frac{|\xi|^2}{t^2}\right)_+^{2\lambda} u(t)\,\frac{dt}{t}\right\} d\xi.\end{aligned}$$

By (1.2) we may replace t by $R(t)$ and estimate the inner integral by

$$\begin{aligned}\{\cdots\} &\leq C\int_0^\infty \left(\frac{|\xi|}{R(t)}\right)^4 \left(1 - \left(\frac{|\xi|}{R(t)}\right)^2\right)_+^{2\lambda} \frac{dR(t)}{R(t)}\\ &= C\int_0^\infty s^4(1-s^2)_+^{2\lambda}\,\frac{ds}{s} = C_\lambda.\end{aligned}$$

Hence, again by the Plancherel theorem, (2.2), and (2.1)

$$\|g_\lambda(h)\|_2^2 \leq C\int |h(x)|^2\,dx \leq C\alpha^{2-p}\int |h(x)|^p\,dx \leq C\alpha^{2-p}\|f\|_p^p,$$

and thus, by (2.9),

$$|\{x: g_\lambda(h)(x) > \alpha\}| \leq C\alpha^{-p}\|f\|_p^p.$$

To estimate the second term on the right side of (2.8) let us define the operators T_t, $t > 0$, by the equation

$$(T_t f)\hat{\ }(\xi) := \left(S_t^{\lambda+1}(f;\cdot) - S_t^\lambda(f,\cdot)\right)\hat{\ }(\xi) = m_\lambda(|\xi|/t)f\hat{\ }(\xi).$$

Note that $m_\lambda(|\xi|) = -|\xi|^2(1-|\xi|^2)_+^\lambda$ is a C^∞-function for $|\xi| \neq 1$, vanishing outside the unit ball. Then following Fefferman [**2**] decompose m by means of a C^∞-function $\theta(s)$ defined on $\mathbf{R}$ such that $0 \leq \theta(s) \leq 1$, $\theta(s) = 0$ for $|s| \geq \frac{1}{2}$, $\theta(s) = 1$ for $|s| \leq \frac{1}{4}$ holds. Choose an arbitrary, small, positive number δ. With the notation

$$\theta_k(|\xi|) := \theta\left(2^{k(1-\delta)}(|\xi|-1)\right),\quad \Phi_k(|\xi|) := 1 - \theta_k(|\xi|),\qquad k \in \mathbf{N},$$

the decomposition of m reads

$$\begin{aligned}m_\lambda(|\xi|/t) &= m_\lambda(|\xi|/t)\theta_k(|\xi|/t) + m_\lambda(|\xi|/t)\Phi_k(|\xi|/t)\\ &= m_\lambda(|\xi|/t)(s_{k,t})\hat{\ }(\xi) + (r_{k,t})\hat{\ }(\xi),\qquad t > 0.\end{aligned}$$

Obviously $s_{k,t}$ and $r_{k,t}$ belong to S. In order to state the basic decomposition of $T_t b$, which modifies Fefferman's approach slightly, define the operators $K_{k,t}$ by

$$(2.10)\quad (K_{k,t}f)\hat{}(\xi) := \begin{cases} m_\lambda(|\xi|/t)f\hat{}(\xi), & \text{if } ||\xi|/t - 1| \leq 2^{-k(1-\delta)-1}, \\ 0, & \text{otherwise,} \end{cases}$$

set $J_k := \{j \in \mathbf{N}\colon \text{diameter}(I_j) = 2^k\}$ and, denoting by χ_E the characteristic function of the set E, put $\beta_j := b\chi_{I_j}$, $b_k := \sum_{j\in J_k}\beta_j$. Let $[r]$ be the largest integer not greater than r, set $\lg r := \log_2 r$ and $k_t := (k - [\lg(1/t)])_+$. Then

$$T_t b = \sum_{k\in Z} T_t b_k = \sum_{k\leq[\lg(1/t)]} \{K_{0,t}(s_{0,t} * b_k) + r_{0,t} * b_k\}$$
$$+ \sum_{k>[\lg(1/t)]} \left\{K_{k_t,t}\left(s_{k_t,t} * b_k\right) + r_{k_t,t} * b_k\right\}.$$

Hence, by Minkowski's inequality,

$$g_\lambda(b)(x) = \left(\int_0^\infty \left|\sum_{k\leq[\lg(1/t)]} (r_{0,t} * b_k)(x)\right|^2 u(t)\,\frac{dt}{t}\right)^{1/2}$$
$$+ \left(\int_0^\infty \left|\sum_{k>[\lg(1/t)]} \left(r_{k_t,t} * b_k\right)(x)\right|^2 u(t)\,\frac{dt}{t}\right)^{1/2}$$
$$+ \left(\int_0^\infty \left|\sum_{k\leq[\lg(1/t)]} \left(K_{0,t} \sum_{j\in J_k} (s_{0,t} * \beta_j)\chi_{\mathbf{R}^n\setminus I_j^*}\right)(x)\right|^2 u(t)\,\frac{dt}{t}\right)^{1/2}$$
$$+ \left(\int_0^\infty \left|\sum_{k>[\lg(1/t)]} \left(K_{k_t,t} \sum_{j\in J_k} \left(s_{k_t,t} * \beta_j\right)\chi_{\mathbf{R}^n\setminus I_j^*}\right)(x)\right|^2 u(t)\,\frac{dt}{t}\right)^{1/2}$$
$$+ \left(\int_0^\infty \left|\sum_{k\leq[\lg(1/t)]} \left(K_{0,t} \sum_{j\in J_k} (s_{0,t} * \beta_j)\chi_{I_j^*}\right)(x)\right|^2 u(t)\,\frac{dt}{t}\right)^{1/2}$$
$$+ \left(\int_0^\infty \left|\sum_{k>[\lg(1/t)]} \left(K_{k_t,t} \sum_{j\in J_k} \left(s_{k_t,t} * \beta_j\right)\chi_{I_j^*}\right)(x)\right|^2 u(t)\,\frac{dt}{t}\right)^{1/2}$$
$$= \sum_{i=1}^{6} g_{\lambda,i}(x).$$

$g_{\lambda,6}$ and $g_{\lambda,5}$ are the essential contributions. $g_{\lambda,6}$ will be estimated with the aid of the Tomas and Stein restriction theorem, which implies the restrictions $1 \le p \le 2(n+1)/(n+3)$ and $\lambda > n(1/p - 1/2) - 1/2$; the fractional integration theorem used to estimate $g_{\lambda,5}$ requires the condition $p > 1$. The remaining $g_{\lambda,i}$-functions will be estimated by L^1-arguments.

2.1. *Estimate of* $g_{\lambda,6}$. Choose an arbitrary sequence $\{w_k\}$ with $w_k > 0$ and $\sum_{k=1}^{\infty} w_k^{-2} = C_w < \infty$ and apply Hölder's inequality to obtain

$$(g_{\lambda,6}(x))^2 \le C_w \int_0^\infty \sum_{k>[\lg(1/t)]} w_{k_t}^2 \left| \left(K_{k_t,t} \sum_{j\in J_k} (s_{k_t,t} * \beta_j)\chi_{I_j^*} \right)(x) \right|^2 u(t)\, \frac{dt}{t}.$$

From (2.10) it follows that the L^2-operator norm of $K_{k_t,t}$ is bounded by $C2^{-k_t(1-\delta)\lambda}$. Hence, after interchanging the order of integration,

$$\begin{aligned}
\|g_{\lambda,6}\|_2^2 &\le C \int_0^\infty \sum_{k>[\lg(1/t)]} w_{k_t}^2 2^{-2k_t(1-\delta)\lambda} \\
&\quad \times \int \left| \sum_{j\in J_k} \left((s_{k_t,t} * \beta_j)\chi_{I_j^*} \right)(x) \right|^2 dx\, u(t)\, \frac{dt}{t} \\
&\le CN \int_0^\infty \sum_{k>[\lg(1/t)]} w_{k_t}^2 2^{-2k_t(1-\delta)\lambda} \\
&\quad \times \int \sum_{j\in J_k} |\left((s_{k_t,t} * \beta_j)\chi_{I_j^*} \right)(x)|^2\, dx\, u(t)\, \frac{dt}{t} \\
&\le C \sum_{k\in Z} \sum_{j\in J_k} \int_{c2^{-k}}^\infty w_{k_t}^2 2^{-2k_t(1-\delta)\lambda} \\
&\quad \times \left[\int \left| \theta_{k_t}\left(\frac{|\xi|}{t} \right) \right|^2 |\hat{\beta_j}(\xi)|^2\, d\xi \right] u(t)\, \frac{dt}{t}.
\end{aligned}$$

For the second inequality we use Hölder's inequality and (2.7), for the third, Plancherel's theorem and an interchange of summation and integration. Introduce polar coordinates in the inner integral and apply the restriction theorem **[9]** valid for $p \le 2(n+1)/(n+3)$ to derive

$$\begin{aligned}
[\cdots] &= \int_0^\infty \left| \theta_{k_t}\left(\frac{r}{t} \right) \right|^2 r^{n-1} \left[\int_{|\xi'|=1} |\hat{\beta_j}(r\xi')|^2\, d\xi' \right] dr \\
&\le C\|\beta_j\|_p^2 \int_0^\infty \left| \theta_{k_t}\left(\frac{r}{t} \right) \right|^2 r^{2n/p-n-1}\, dr \\
&\le C\alpha^2 2^{2nk/p - k_t(1-\delta)} t^{2n/p-n},
\end{aligned}$$

where (2.4) and (2.6) are used for the last inequality. Observe that (1.2) yields, for $\gamma < 0$,

$$\int_{2^{-k}}^{\infty} t^{\gamma} u(t)\,\frac{dt}{t} \leq C \int_{cR(2^{-k})}^{\infty} (R(t))^{\gamma}\,\frac{dR(t)}{R(t)} \leq C 2^{-k\gamma}.$$

Choose $w_k = 2^{\delta k}$ and note that in view of the condition $\lambda > n(1/p - 1/2) - 1/2$, the number δ can be determined so small that $\gamma = 2n/p - n - 2(1-\delta)\lambda - 1 + 3\delta < 0$ holds. Then, by (2.6) and (2.5), we arrive at

$$\begin{aligned} \|g_{\lambda,6}\|_2^2 &\leq C\alpha^2 \sum_{k\in Z} \sum_{j\in J_k} 2^{2nk/p - 2k(1-\delta)\lambda - k + 3\delta k} \\ &\quad \times \int_{c2^{-k}}^{\infty} t^{2n/p - n - 2(1-\delta)\lambda - 1 + 3\delta} u(t)\,\frac{dt}{t} \\ &\leq C\alpha^2 \sum_{k\in Z} \sum_{j\in J_k} 2^{nk} = C\alpha^2 \sum_{j\in \mathbf{N}} |I_j| \leq C\alpha^{2-p}\|f\|_p^p. \end{aligned}$$

Thus analogously to (2.9),

$$|\{x \in \mathbf{R}^n \colon g_{\lambda,6}(x) > \alpha\}| \leq C\alpha^{-p}\|f\|_p^p.$$

2.2. *Estimate of* $g_{\lambda,5}$. First recall that the L^2-operator norm of $K_{0,t}$ is bounded by a constant, then interchange the order of integration and apply Hölder's inequality together with (2.7). Then

$$\begin{aligned} \|g_{\lambda,5}\|_2^2 &\leq C\int_0^{\infty} \Big| \sum_{k\leq[\lg(1/t)]} \sum_{j\in J_k} \big((s_{0,t} * \beta_j)\chi_{I_j^*}\big)(x) \Big|^2 dx\, u(t)\,\frac{dt}{t} \\ &\leq CN \int_0^{\infty} \int \sum_{k\leq[\lg(1/t)]} \sum_{j\in J_k} |\big((s_{0,t} * \beta_j)\chi_{I_j^*}\big)(x)|^2\, dx\, u(t)\,\frac{dt}{t}. \end{aligned}$$

Again by the theorems of Fubini and Plancherel, after interchanging the summation and integration, we obtain

$$\|g_{\lambda,5}\|_2^2 \leq C \sum_{k\in Z} \sum_{j\in J_k} \int\int_0^{2^{-k+1}} \left|\theta_0\left(\frac{|\xi|}{t}\right)\right|^2 u(t)\,\frac{dt}{t}\,|\hat{\beta}_j(\xi)|^2\, d\xi.$$

By the definition of θ_0,

$$\int_0^{2^{-k+1}} \left|\theta_0\left(\frac{|\xi|}{t}\right)\right|^2 u(t)\,\frac{dt}{t} \leq \begin{cases} 0, & 2^k|\xi| \geq 3 \\ C & \text{otherwise} \end{cases} \leq C\big(2^k|\xi|\big)^{-2a}$$

with $a := n(1/p - 1/2), p \leq 2$; thus, it follows that

$$\|g_{\lambda,5}\|_2^2 \leq C \sum_{k\in Z} \sum_{j\in J_k} 2^{-2ak} \int |\xi|^{-2a} |\hat{\beta}_j(\xi)|^2\, d\xi.$$

The integral represents the Fourier transform of the Riesz potential of order a; thus, applying Plancherel's theorem, the theorem on fractional integration [**8**, pp. 117], (2.4), (2.6) and (2.5), there holds for $p > 1$,

$$\begin{aligned}\|g_{\lambda,5}\|_2^2 &\leq C \sum_{k\in Z}\sum_{j\in J_k} 2^{-2ak}\|\beta_j\|_p^2 \\ &\leq C\alpha^2 \sum_{k\in Z}\sum_{j\in J_k} |I_j||I_j|^{-2/p}|I_j|^{2/p} \leq C\alpha^{2-p}\|f\|_p^p,\end{aligned}$$

and finally,

$$|\{x\colon g_{\lambda,5}(x) > \alpha\}| \leq C\alpha^{-p}\|f\|_p^p.$$

2.3. *Estimates of* $g_{\lambda,1}$ *to* $g_{\lambda,4}$. The following inequalities are the starting point.

$$\begin{aligned}(2.11)\quad |\{x\colon g_{\lambda,i}(x) > \alpha\}| &\leq |\Omega^*| + |\{x \in \mathbf{R}^n\setminus\Omega^*\colon g_{\lambda,i}(x) > \alpha\}| \\ &\leq A2^n\alpha^{-p}\|f\|_p^p + \alpha^{-2}\int |(g_{\lambda,i}\chi_{\mathbf{R}^n\setminus\Omega^*})(x)|^2\,dx, \qquad i = 1,2.\end{aligned}$$

$$(2.12)\qquad |\{x\colon g_{\lambda,i}(x) > \alpha\}| \leq \alpha^{-2}\int (g_{\lambda,i}(x))^2\,dx, \qquad i = 3,4.$$

Here we set $\Omega^* = \bigcup_{j\in\mathbf{N}} I_j^*$ in (2.11), use (2.5) and the argument (2.9). First note that for $i = 1,2$,

$$\begin{aligned}(2.13)\quad &(g_{\lambda,i}\chi_{\mathbf{R}^n\setminus\Omega^*})(x) \\ &\leq \left(\int_0^\infty \left| \sum_{k\geq(-1)^i[\lg(1/t)]} \sum_{j\in J_k} \left(\left(r_{k_i,t} * \beta_j\right)\chi_{\mathbf{R}^n\setminus I_j^*}\right)(x)\right|^2 u(t)\,\frac{dt}{t}\right)^{1/2}.\end{aligned}$$

Thus, provided we can show on the one hand for $i = 1,2$,

$$(2.14)\qquad \left| \sum_{k\geq(-1)^i[\lg(1/t)]} \sum_{j\in J_k} \left(\left(r_{k_i,t} * \beta_j\right)\chi_{\mathbf{R}^n\setminus I_j^*}\right)(x)\right| = \Sigma_i \leq C\alpha,$$

with C independent of x, t and α, and, on the other hand,

$$\begin{aligned}(2.15)\quad &\int\int_0^\infty \left| \sum_{k\geq(-1)^i[\lg(1/t)]} \sum_{j\in J_k} \left(\left(r_{k_i,t} * \beta\right)\chi_{\mathbf{R}^n\setminus I_j^*}\right)(x)\right| \frac{u(t)}{t}\,dx \\ &\qquad = Q_i \leq C\alpha^{1-p}\|f\|_p^p,\end{aligned}$$

we have established, via argument (2.9),

$$|\{x: g_{\lambda,i}(x) > \alpha\}| \leq C\alpha^{-p}\|f\|_p^p, \qquad i = 1,2.$$

For the verification of (2.14) and (2.15) observe that, since $r_{k_t,t} \in S$,

$$|r_{k_t,t}(x)| \leq C_m t^n \left(1 + 2^{-k_t(1-\delta)} t|x|\right)^{-m} = P_{k_t,t}(x), \tag{2.16}$$

$$|\operatorname{grad}(r_{0,t}(x))| \leq CtP_{0,t}(x), \tag{2.17}$$

where we may choose m so large that $m\delta \geq n + 1$ holds. Further note that

$$c_1|x - y_j| \leq |x - y| \leq c_2|x - y_j| \tag{2.18}$$

is true for all $x \in I_j^*$ and $y \in I_j$, with y_j denoting the center of I_j. Starting with (2.14) apply (2.16), (2.4) and (2.18). Hence,

$$\begin{aligned}
\Sigma_1 &\leq \sum_{k \leq [\lg(1/t)]} \sum_{j \in J_k} \chi_{\mathbf{R}^n \setminus I_j^*}(x) \int |r_{0,t}(x-y)||\beta_j(y)|\, dy \\
&\leq \sum_{k \leq [\lg(1/t)]} \sum_{j \in J_k} \chi_{\mathbf{R}^n \setminus I_j^*}(x) \sup_{y \in I_j} |r_{0,t}(x-y)| \int |\beta_j(y)|\, dy \\
&\leq C\alpha \sum_{k \leq [\lg(1/t)]} \sum_{j \in J_k} P_{0,t}(x - y_j)|I_j| \\
&\leq C\alpha \sum_{k \leq [\lg(1/t)]} \sum_{j \in J_k} \int_{I_j} P_{0,t}(x-y)\, dy \\
&\leq C\alpha \int P_{0,t}(x-y)\, dy \leq C\alpha,
\end{aligned}$$

where we also used the fact that the I_j's are pairwise disjoint. Since

$$\int_{|x| > c2^k} P_{k_t,t}(x)\, dx \leq C2^{-k_t},$$

with C independent of k and t, we obtain analogously

$$\begin{aligned}
\Sigma_2 &\leq C\alpha \sum_{k \geq [\lg(1/t)]} \int_{\bigcup_{j \in J_k} I_j} P_{k_t,t}(x-y)\, dy \\
&\leq C\alpha \sum_{k \geq [\lg(1/t)]} \int_{|x-y| \geq c2^k} P_{k_t,t}(x-y)\, dy \\
&\leq C\alpha \sum_{k \geq [\lg(1/t)]} 2^{-k_t} \leq C\alpha.
\end{aligned}$$

Now consider (2.15), apply (2.4), then interchange the order of integration to derive

$$(2.19)\quad Q_1 = \int\int_0^\infty \left| \sum_{k\le[\lg(1/t)]} \sum_{j\in J_k} \chi_{\mathbf{R}^n\setminus I_j^*}(x) \times \int (r_{0,t}(x-y) - r_{0,t}(x-y_j))\beta_j(y)\,dy \right| u(t)\,\frac{dt}{t}\,dx$$

$$\le \sum_{k\in Z}\sum_{j\in J_k}\int |\beta_j(y)| \times \left[\int_{\mathbf{R}^n\setminus I_j^*}\int_0^\infty |r_{0,t}(x-y) - r_{0,t}(x-y_j)|\,u(t)\,\frac{dt}{t}\,dx\right] dy.$$

The mean value theorem together with (2.17) yields for $0<q<1$,

$$|r_{0,t}(x-y) - r_{0,t}(x-y_j)| \le C|y-y_j|\,tP_{0,t}\big(x-y_j+q(y_j-y)\big) \le C2^k t^{n+1}\big(1+t|x-y_j|\big)^{-m},$$

since $|x-y_j+q(y_j-y)|\ge c|x-y_j|$ holds for all $x\notin I_j^*$, $y\in I_j$ and $P_{0,t}$ is nonincreasing. Replacing t by $R(t)$ we estimate the expression in brackets on the right side of (2.19) as follows:

$$[\cdots] \le C2^k\int_{\mathbf{R}^n\setminus I_j^*}\int_0^\infty (R(t))^n\big(1+R(t)|x-y_j|\big)^{-m}\,dR(t)\,dx$$

$$\le C2^k\int_{|x-y_j|\ge c2^k}|x-y_j|^{-n-1}\,dx \le C.$$

Thus, by (2.4) and (2.5),

$$Q_1 \le C\sum_{k\in Z}\sum_{j\in J_k}\int|\beta_j(y)|\,dy \le C\alpha\sum_{j\in\mathbf{N}}|I_j| \le C\alpha^{1-p}\|f\|_p^p.$$

Consider again (2.15); an interchange of the integration and summation orders gives

$$Q_2 = \int\int_0^\infty \left| \sum_{k\ge[\lg(1/t)]}\sum_{j\in J_k}\chi_{\mathbf{R}^n\setminus I_j^*}(x)\int r_{k_t,t}(x-y_j)\beta_j(y)\,dy\right| u(t)\,\frac{dt}{t}\,dx$$

$$\le \sum_{k\in Z}\sum_{j\in J_k}\int|\beta_j(y)|\left[\int_{\mathbf{R}^n\setminus I_j^*}\int_{c2^{-k}}^\infty |r_{k_t,t}(x-y)|\,u(t)\,\frac{dt}{t}\,dx\right] dy.$$

Replace t by $R(t)$ and use (2.16) to obtain

$$[\cdots] \le C\int_{\mathbf{R}^n\setminus I_j^*}\int_{c2^{-k}}^{\infty}(R(t))^{n-1}$$
$$\times\left\{1+(R(t))^{\delta}2^{-k(1-\delta)}|x-y|\right\}^{-m}dR(t)\,dx$$
$$\le C2^{mk-nk}\int_{|x-y|\ge c2^k}|x-y|^{-m}\,dx \le C.$$

Hence, by (2.4) and (2.5),

$$Q_2 \le C\sum_{k\in Z}\sum_{j\in J_k}\int|\beta_j(y)|\,dy \le C\alpha^{1-p}\|f\|_p^p.$$

Next, (2.12) can be treated in the same manner as (2.11) if we first use the same arguments as at the beginning of 2.1 (with $w_k = 2^{k(1-\delta)\lambda}$) and 2.2. Finally collecting all the $g_{\lambda,i}$-estimates, the proof of Theorem 2 is completed by the observation

$$|\{x\colon g_\lambda(b)(x)>\alpha\}| \le \sum_{i=1}^{6}|\{x\colon g_{\lambda,i}(x)>\alpha/6\}|.$$

3. Proof of Theorem 1. The general idea of the proof is to show

$$g_1^*(h;x)\le CBg_1(f;x),\qquad h\hat{\ }(\xi) := m(|\xi|)f\hat{\ }(\xi). \tag{3.1}$$

Then in view of (1.3) and Theorem 2 the following norm inequalities prove Theorem 1 (F^{-1} denotes the inverse Fourier transformation):

$$\|F^{-1}\{m(|\xi|)f\hat{\ }(\xi)\}\|_p \le C\|g_1^*(h)\|_p \le cB\|g_1(f)\|_p \le cB\|f\|_p.$$

To this end, set

$$k(r) := \frac{-r^{n+1}}{R^2}\int_{|\xi'|=1} f(r\xi')\exp(i\,r\xi'\cdot x)\,d\xi',$$

introduce polar coordinates and integrate by parts to obtain

$$S_R^2(h;x)-S_R^1(h;x) = \int_0^R\left(1-\frac{r^2}{R^2}\right)m(r)k(r)\,dr$$
$$= -\int_0^R\frac{\partial}{r\partial r}\left\{\left(1-\frac{r^2}{R^2}\right)m(r)\right\}\left\{r\int_0^r k(s)\,ds\right\}dr$$
$$= m(R)\int_0^R\left(1-\frac{r^2}{R^2}\right)k(r)\,dr$$
$$+\frac{1}{R}\int_0^R\frac{r^4}{2R}\frac{\partial}{\partial r}\left\{\frac{\partial}{r\partial r}\left\{\left(1-\frac{r^2}{R^2}\right)m(r)\right\}\right\}\left(S_r^2(f;x)-S_r^1(f;x)\right)dr,$$

where we used (cf. [4])

$$|rm'(r)| \leq C, \quad \int_0^r (r^2 - s^2)k(s)\,ds = O(r^3), \quad r \to 0+ .$$

Since

$$\left|\frac{1}{2}\frac{1}{R}r^4\frac{\partial}{\partial r}\left\{\frac{\partial}{r\partial r}\left\{\left(1 - \frac{r^2}{R^2}\right)m(r)\right\}\right\}\right| \leq C\{|rm'(r)| + |r^2m''(r)|\} =: v(r),$$

we obtain by Minkowski's and Hölder's inequalities:

$$\begin{aligned}
g_1^*(h)(x) &= \left(\int_0^\infty |S_R^2(h;x) - S_R^1(h;x)|^2 \frac{dR}{R}\right)^{1/2} \\
&\leq \sup_{r>0}|m(r)|\,g_1^*(f)(x) \\
&\quad + \left(\int_0^\infty \left|\frac{1}{R}\int_0^R |S_r^2(f;x) - S_r^1(f;x)|\,v(r)\,dr\right|^2 \frac{dR}{R}\right)^{1/2} \\
&\leq Bg_1^*(f)(x) + \left(\int_0^\infty \left\{\frac{1}{R}\int_0^R v(r)\,dr\right\}\right. \\
&\qquad \left. \times \left\{\frac{1}{R}\int_0^R |S_r^2(f;x) - S_r^1(f;x)|^2 v(r)\,dr\right\}\frac{dR}{R}\right)^{1/2}.
\end{aligned}$$

Observing (cf. [1], [4]) that $R^{-1}\int_0^R v(r)\,dr \leq cB$, choose $u(r) := (v(r) + B)/B$. Then u satisfies (1.2) and an interchange of the integration order gives

$$\begin{aligned}
g_1^*(h)(x) &\leq Bg_1^*(f)(x) + cB\left(\int_0^\infty |S_r^2(f;x) - S_r^1(f;x)|^2 u(r)\frac{dr}{r}\right)^{1/2} \\
&\leq cBg_1(f)(x),
\end{aligned}$$

which completes the proof.

Remarks. 1. The differentiability-growth condition on m in Theorem 1 is equivalent to

$$\sup_{r>0}|m(r)| + \sup_{j\in Z}\int_{2^{j-1}}^{2^j} r\,|dm'(r)| < \infty$$

(see [8; p. 109]). Applying this to $(1 - |\xi|)_+$, it follows that $(1 - |\xi|)_+ \in M_p(\mathbf{R}^n)$ if $2n/(n+3) < p < 2n/(n-3)$, $n \geq 3$. On the other hand, it is well known (see [3], [4], [9]) that these p-bounds are necessary and sufficient for $(1 - |\xi|)_+$ to be a bounded multiplier.

2. Let us mention that we may interpolate between Theorem 1 and a result due to Bonami and Clerc **[1]** and Gasper and Trebels **[5]** to obtain sharp Marcinkiewicz criteria in the range $1 < p \le 2n/(n+3)$ and $2n/(n-3) \le p < \infty$. In particular it will be shown that Theorem 1 already implies an improvement of the following result of Igari and Kuratsubo. Let $m(r)$ be an absolutely continuous function on $(0, \infty)$ satisfying

$$\sup_{r>0} |m(r)| + \sup_{j \in Z} \left(\int_{2^{j-1}}^{2^j} r |m'(r)|^2 \, dr \right)^{1/2} < \infty.$$

Then $m(|\xi|) \in M_p(\mathbf{R}^n)$ if $2n/(n+1) < p < 2n/(n-1)$.

3. Modifications of the above techniques lead to: Let $\{r_j\}$ be any sequence of positive real numbers, $\{f_j\}$ any sequence in S. Then with λ, p as in Theorem 2 there holds

$$\left\| \left(\sum_{j \in \mathbf{N}} |S_{r_j}^{\lambda}(f_j; \cdot)|^2 \right)^{1/2} \right\|_p \le C \left\| \left(\sum_{j \in \mathbf{N}} |f_j(\cdot)|^2 \right)^{1/2} \right\|_p,$$

where C depends only on λ, p and the dimension n.

Acknowledgement. With greatest pleasure I acknowledge Professor W. Trebels for his mathematical and moral support during the whole work.

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Received March 26, 1981 and in revised form January 6, 1982.

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