

ON ASYMPTOTIC PRIME DIVISORS

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Several results are proved concerning the set $\hat{A}^*(I) = \{P \in \text{Spec } R; P \text{ is a prime divisor of the integral closure } (I^i)_a \text{ of } I^i \text{ for all large } i\}$, where I is an ideal in a Noetherian ring R . Among these are: if P is a prime divisor of $(I^i)_a$ for some $i \geq 1$, then P is a prime divisor of $(I^n)_a$ for all $n \geq i$; a characterization of Cohen-Macaulay rings and of altitude two local UFDs in terms of $\hat{A}^*(I)$; and, some results on the relationship of $\hat{A}^*(I)$ to $\hat{A}^*(IS)$ with S a flat R -algebra and to $\hat{A}^*((I+z)/z)$ with z a minimal prime ideal in R .

1. Introduction. The sets $\text{Ass}(R/I^i)$ and $\text{Ass}(R/(I^i)_a)$ have been studied in several recent papers, where I is an ideal in a Noetherian ring R . For example, in [14, (2.5)] and [5, Prop. 7] it was shown that if $\text{height } I \geq 1$, then the sets $\text{Ass}(R/(I^i)_a)$ are equal for all large i . Calling this set $\hat{A}^*(I)$, it was shown in [14, (2.6.1)] that $\hat{A}^*(I)$ is contained in the sets $\text{Ass}(R/I^i)$ for all large i . Also, in [1] and in [5, Corol. 5] it was shown that these latter sets are also equal for all large i , even without the assumption that $\text{height } I \geq 1$. Further, in [15, Thm. 1] it was shown that there exists a positive integer i such that the maximal ideal M in a local domain R is in $\text{Ass}(R/(I_a))$ for all nonzero ideals $I \subseteq M^i$ (and hence $M \in \hat{A}^*(I)$ for all nonzero ideals $I \subseteq M$) if and only if there exists a height one maximal ideal in the integral closure of R , and in [22, Corol. 2.8] it was shown that if $M \in \hat{A}^*(I)$ for a given ideal I in R , then $M \in \hat{A}^*(IJ)$ for all nonzero ideals J in R . Numerous other such results could be mentioned here, but instead we refer the reader to [7] and to [18] where most of the known results in this area are summarized.

The purpose of this paper is to add several new results to this area and also to strengthen several of the known results. In §2 we remove the restriction on $\text{height } I \geq 1$ in the above mentioned results, and also show that the sets $\text{Ass}(R/(I^i)_a)$ are monotonically increasing (and eventually constant). In §3 the monotonicity of the sets $\text{Ass}(R/I^i)$ is considered, and §§4 and 5 contain characterizations in terms of $\hat{A}^*(I)$ of Cohen-Macaulay rings and of altitude two local UFDs, respectively. Some results showing the relation between the sets $\hat{A}^*(I)$ and the analogous sets in S and in R/z are given in §6, where S is a flat R -algebra and z is a minimal prime ideal in R , and the final section, §7, contains some results on these sets and the analytic spread of the ideal I .

I think results on asymptotic prime divisors are quite interesting in their own right, and it turns out that they are also closely connected to some other areas of current research interest, such as: going-down between prime ideals; asymptotic sequences; and the catenary chain conjectures. (See [7] and [18].) Thus knowledge about such prime divisors is of some interest and importance.

2. Basic properties of $\hat{A}^*(I)$. In this section it is shown that the basic results concerning $\hat{A}^*(I)$ which are known to hold for ideals I such that height $I \geq 1$ actually hold for all ideals I in all Noetherian rings. Then it is shown that the sets $\text{Ass}(R/(I^i)_a)$ are monotonically increasing (and eventually constant).

We begin by listing a few notational conventions and definitions.

(2.1) **DEFINITIONS.** Throughout R is a Noetherian ring (commutative with identity) and I is an ideal in R .

(2.1.1) The *integral closure* of R in its total quotient ring will be denoted R' .

(2.1.2) If R is local with maximal ideal M , then the M -adic completion of R will be denoted R^* .

(2.1.3) If B is a finite R -module, then $\text{Ass}(B)$ denotes the set of associated primes of B . In particular, $\text{Ass}(R)$ is the set of prime divisors of zero in R , and $\text{mAss } R$ denotes the subset of minimal prime ideals in R .

(2.1.4) The *integral closure* of I in R , I_a , is the set of all $x \in R$ satisfying an equation of the form $X^n + b_1 X^{n-1} + \cdots + a_n = 0$ where $a_i \in I^i$. Therefore I_a is an ideal in R and $I \subseteq I_a \subseteq \text{Rad } I$.

(2.1.5) $\hat{A}^*(I) = \text{Ass}(R/(I^i)_a)$ for all large i . (It is shown in (2.7) that this set is well defined for all ideals I .)

(2.1.6) $A^*(I) = \text{Ass}(R/I^i)$ for all large i . This set is well defined for all ideals I by [1] and [5, Corol. 5].

(2.1.7) $\mathfrak{R} = \mathfrak{R}(R, I)$ denotes the *Rees ring* of R with respect to I . Therefore \mathfrak{R} is the graded subring $R[u, tI]$ of $R[u, t]$, where t is an indeterminate and $u = 1/t$.

Concerning (2.1.7), note that \mathfrak{R} is a Noetherian ring, u is a regular element in \mathfrak{R} , and $u^i \mathfrak{R} \cap R = I^i$ for all $i \geq 1$.

The following lemma and its corollary will be quite useful in this section. Concerning (2.2), it is shown in [12, Corol. 2.12(1)] that if b is a regular nonunit in a Noetherian ring R , then bR' is a finite intersection of height one primary ideals.

(2.2) LEMMA. *Let I be an ideal in a Noetherian ring R , let $\mathfrak{R} = \mathfrak{R}(R, I)$, let p' be a (height one) prime divisor of $u\mathfrak{R}'$, and let $Q = p' \cap R$. Then $tI \subseteq p'$ if and only if $\text{height } Q = 0$.*

Proof. Let z' be a minimal prime ideal in \mathfrak{R}' that is contained in p' . Then $z = z' \cap R$ is a minimal prime ideal in R that is contained in Q and $z' = zT \cap \mathfrak{R}'$, where T is the total quotient ring of \mathfrak{R}' (and of $R[t]$). Now $\mathfrak{R}(R/z, (I+z)/z) \cong \mathfrak{R}/(z' \cap \mathfrak{R})$, by [20, Lemma 1.1], and \mathfrak{R}'/z' is an integrally dependent overring of $\mathfrak{R}/(z' \cap \mathfrak{R})$. We consider two cases: (a) $\text{height } Q = 0$; and (b) $\text{height } Q \geq 1$.

If (a) holds then $Q = z$ and $I \subseteq u\mathfrak{R}' \cap R \subseteq Q = z$, so $tI \subseteq zT \cap \mathfrak{R}' = z' \subset p'$. Therefore $\mathfrak{R}/(z' \cap \mathfrak{R}) = (R/z)[u]$, so it readily follows that $p' = (z', u)\mathfrak{R}'$.

If (b) holds let $\mathfrak{S} = \mathfrak{R}(R/z, (I+z)/z) \cong \mathfrak{R}/(z' \cap \mathfrak{R})$, and to simplify notation identify these rings. Then $\mathfrak{S} \subseteq \mathfrak{R}'/z' \subseteq \mathfrak{S}'$, so there exists a height one prime divisor P' of $u\mathfrak{S}'$ that lies over p'/z' . Since $Q/z = (p'/z') \cap (R/z) \neq (0)$, it follows from what was shown in the preceding paragraph that $I \not\subseteq z$. Therefore $\text{height}(I+z)/z \geq 1$, so, by the proof of [10, Lemma 3.2(1)] (from the third sentence on), $t^* = \text{trd}(\mathfrak{S}'/P')/(R/Q) \geq 1$, hence $t((I+z)/z) \not\subseteq P'$, since $u \in P'$. Therefore it follows that $tI \not\subseteq p'$. \square

(2.3) COROLLARY. *With R, I , and \mathfrak{R} as in (2.2), if P is a prime divisor of $(u^i\mathfrak{R})_a$ for some $i \geq 1$, then either $\text{height } P \cap R = 0$ and $P = ((P \cap R)R[t, u] \cap \mathfrak{R}, u)\mathfrak{R} = (u, P \cap R, tI)\mathfrak{R}$ or $\text{height } P \cap R \geq 1$ and $tI \not\subseteq P$.*

Proof. If P is a prime divisor of $(u^i\mathfrak{R})_a$ for some $i \geq 1$, then there exists a prime divisor p' of $u^i\mathfrak{R}'$ (and of $u\mathfrak{R}'$) such that $p' \cap \mathfrak{R} = P$, since $(u^i\mathfrak{R})_a = u^i\mathfrak{R}' \cap \mathfrak{R}$, so the conclusion readily follows from (2.2) and the second paragraph of its proof. \square

It was shown in [14, (2.5)] that if I is an ideal in a Noetherian ring R such that $\text{height } I \geq 1$ and if P is a prime divisor of $(I^i)_a$ for some $i \geq 1$, then P is a prime divisor of $(I^n)_a$ for all large n . (2.4) is a considerable extension of this result. In the proof of (2.4), the following readily shown fact is used, where \mathfrak{R} is the Rees ring of R with respect to I : if $bt^k \in \mathfrak{R}$, then $(u^i\mathfrak{R})_a: bt^k\mathfrak{R} = (u^{i+k}\mathfrak{R})_a: b\mathfrak{R}$.

(2.4) THEOREM. (Cf. [14, (2.5)].) *Let I be an ideal in a Noetherian ring R and let Q be a prime divisor of $(I^i)_a$ for some $i \geq 1$. Then Q is a prime divisor of $(I^n)_a$ for all $n \geq i$.*

Proof. By hypothesis there exists $b \in R$ such that $(I^i)_a : bR = Q$. Let $\mathcal{R} = \mathcal{R}(R, I)$. Then $Q\mathcal{R} = ((I^i)_a : bR)\mathcal{R} \subseteq (I^i)_a\mathcal{R} : b\mathcal{R} \subseteq (u^i\mathcal{R})_a : b\mathcal{R}$ and $((u^i\mathcal{R})_a : b\mathcal{R}) \cap R = (I^i)_a : bR = Q$, by [23, p. 220], since $(u^i\mathcal{R})_a \cap R = (I^i)_a$. Therefore there exists a prime divisor P of $(u^i\mathcal{R})_a$ such that $(u^i\mathcal{R})_a : b\mathcal{R} \subseteq P$ and $P \cap R = Q$. By (2.3), either $\text{height } Q = 0$ or $\text{height } Q \geq 1$ and $tI \not\subseteq P$. If $\text{height } Q = 0$, the conclusion is clear, so it may be assumed $\text{height } Q \geq 1$, so there exists $c \in I$ such that $ct \notin P$. Therefore for all $j \geq 0$ it holds that $(u^{i+j}\mathcal{R})_a : bc^j\mathcal{R} = (u^i\mathcal{R})_a : bc^jt^j\mathcal{R} \subseteq P$, and contracting to R it follows that $(I^{i+j})_a : bc^jR = Q$, so Q is a prime divisor of $(I^n)_a$ for all $n \geq i$. \square

(2.5) was proved in [11, Prop. 3.5] for local domains. It is important to extend this result to the local ring case, because its corollary, (2.6), plays an important role in the deeper results in §6.

(2.5) PROPOSITION. *Let R be a local ring. If there exists a height one maximal ideal in R' , then there exists a depth one minimal prime ideal in the completion R^* of R .*

Proof. If there exists a height one maximal ideal in R' , then there exists an element x in R' such that $R[x]$ has a height one maximal ideal, so $R^*[x] = (R[x])^*$ has a height one maximal ideal, say N^* . Then if z is a minimal prime ideal in $R[x]^*$ that is contained in N^* , then $\text{depth } z = \text{altitude } R[x]^*/z = \text{height } N^*/z = 1$, since $R[x]^*/z$ is a complete local domain. Finally, $z \cap R^*$ is minimal and $\text{depth } z = \text{depth } z \cap R^*$ by integral dependence. \square

It is shown in [15, Thm. 1] that if R is a local ring and there exists a depth one $z \in \text{mAss } R^*$, then there exists a positive integer n such that M is a prime divisor of I_a for all ideals $I \subseteq M^n$ such that $\text{height } I \geq 1$, and conversely. This, together with (2.5), is used in the next two results to extend known results to more general cases.

(2.6) COROLLARY. (Cf. [15, Corol. 2].) *Let b be a regular element in a Noetherian ring R . Then the following statements hold:*

(2.6.1) *If Q' is a prime divisor of bR' , then $Q' \cap R$ is a prime divisor of $(b^iR)_a$ for all large i .*

(2.6.2) *If Q is a prime divisor of $(b^i R)_a$ for some $i \geq 1$, then there exists an integer n such that Q is a prime divisor of I_a for all ideals $I \subseteq Q^{(n)} = Q^n R_Q \cap R$ such that $\text{height } I_Q \geq 1$, hence $Q \in \hat{A}^*(I)$ for all ideals $I \subseteq Q$ such that $\text{height } I_Q \geq 1$.*

Proof. The proof is essentially the same as the proof of [15, Corol. 2], but use (2.5) and [15, (1.1) \Rightarrow (1.2)] in place of [15, (1.3) \Rightarrow (1.2)]. \square

In regard to (2.6.1), it should be noted that $Q' \cap R$ may not be a prime divisor of $(bR)_a$. For example, let (R, Q) be as in [8, Example 2, pp. 203–205] in the case $r = 1$ and $m = 0$, so altitude $R = 2$ and there exists a height one maximal ideal Q' in R' . Let x be as in [8] and let $b = x^2 - x$. Then $(bR)_a = (b, bx)R$ is the height one prime ideal $(x - 1)R' \cap R$, so Q is not a prime divisor of $(bR)_a$, even though Q' is a prime divisor of bR' .

It was shown in (2.4) that the sets $\text{Ass}(R/(I^i)_a)$ are increasing. They are also eventually all equal, as is shown in (2.7). (2.7) is a considerable extension of the first two referenced results, and it also extends the third referenced result from the Noetherian domain case.

(2.7) THEOREM. (Cf. [5, Props. 7 and 18], [14, (2.5)], and [15, Corol. 3].) *If I is an ideal in a Noetherian ring R , then the sets $\text{Ass}(R/(I^i)_a)$ are equal for all large i . In fact, for all large i ,*

$$\begin{aligned} \text{Ass}(R/(I^i)_a) &= \{p' \cap R; p' \text{ is a (height one) prime divisor of } u\mathfrak{R}(R, I)'\} \\ &= \{P \cap R; P \text{ is a prime divisor of } (u^n \mathfrak{R})_a \text{ for some } n \geq 1\}. \end{aligned}$$

Proof. Let $\mathfrak{R} = \mathfrak{R}(R, I)$. Then $u^i \mathfrak{R}' \cap R = (I^i)_a$ for all $i \geq 1$. Also, since u is a regular element, the prime divisors of $u^i \mathfrak{R}'$ are the prime divisors of $u\mathfrak{R}'$. Therefore

$$\begin{aligned} \text{Ass}(R/(I^i)_a) &\subseteq \{p' \cap R; p' \text{ is a (height one)} \\ &\quad \text{prime divisor of } u\mathfrak{R}'\}, \end{aligned}$$

and since \mathfrak{R}' is the integral closure of a Noetherian ring, $u\mathfrak{R}'$ has only finitely many prime divisors. Therefore the sets $\text{Ass}(R/(I^i)_a)$ are constant for all large i , by (2.4), and are contained in $\{p' \cap R; p' \text{ is a prime divisor of } u\mathfrak{R}'\}$.

Now if p' is a prime divisor of $u\mathfrak{R}'$, then $p' \cap \mathfrak{R}$ is a prime divisor of $(u^i \mathfrak{R})_a$ for all large i , by (2.6.1). Also, if P is a prime divisor of $(u^i \mathfrak{R})_a$ for

some $i \geq 1$, then there exists a prime divisor p'' of $u^i \mathfrak{R}'$ (and hence of $u \mathfrak{R}'$) that lies over P , since $u^i \mathfrak{R}' \cap \mathfrak{R} = (u^i \mathfrak{R})_a$. Therefore

$$\begin{aligned} & \{P; P \text{ is a prime divisor of } (u^i \mathfrak{R})_a \text{ for some } i \geq 1\} \\ &= \{p' \cap \mathfrak{R}; p' \text{ is a prime divisor of } u \mathfrak{R}'\}. \end{aligned}$$

Finally, if p' is a prime divisor of $u \mathfrak{R}'$, then either $\text{height } p' \cap R = 0$, and so clearly $p' \cap R$ is a prime divisor of $(I^i)_a$ for all $i \geq 1$, or $\text{height } p' \cap R > 0$ and $tI \not\subseteq P = p' \cap \mathfrak{R}$, by (2.2). Also, P is a prime divisor of $(u^i \mathfrak{R})_a$ for some $i \geq 1$, by (2.6.1), so $(u^i \mathfrak{R})_a : bt^k \mathfrak{R} = P$ for some homogeneous element bt^k in \mathfrak{R} , since $(u^i \mathfrak{R})_a$ is homogeneous. Then with $ct \in \mathfrak{R} - P$, it follows as in the proof of (2.4) that $p' \cap R = P \cap R$ is a prime divisor of $(I^{i+k+j})_a$ for all $j \geq 0$. Therefore, by the preceding two paragraphs,

$$\begin{aligned} \text{Ass}(R/(I^i)_a) &= \{p' \cap R; p' \text{ is a prime divisor of } u \mathfrak{R}'\} \\ &= \{P \cap R; P \text{ is a prime divisor of } (u^n \mathfrak{R})_a \text{ for some } n \geq 1\}. \quad \square \end{aligned}$$

This section will be closed with the following extension of [14, (2.6.1)] to all ideals in all Noetherian rings.

(2.8) THEOREM. (Cf. [14, (2.6.1)].) *If I is an ideal in a Noetherian ring R , then $\hat{A}^*(I) \subseteq A^*(I)$.*

Proof. Let $Q \in \hat{A}^*(I)$. If $\text{height } Q = 0$, then clearly $Q \in A^*(I)$. If $\text{height } Q \geq 1$, then there exists a prime divisor p' of $u \mathfrak{R}'$ such that $tI \not\subseteq p'$ and $p' \cap R = Q$, by (2.7) and (2.2), where $\mathfrak{R} = \mathfrak{R}(R, I)$. Let $P = p' \cap \mathfrak{R}$. Then P is a prime divisor of $u \mathfrak{R}$, by [12, Thm. 2.15], and $tI \not\subseteq P$. Therefore $P = u \mathfrak{R} : bt^k \mathfrak{R}$ for some homogeneous element $bt^k \in \mathfrak{R}$, and $ct \notin P$ for some $c \in I$. Therefore

$$P = u^{k+j+1} \mathfrak{R} : bc^j \mathfrak{R} = u \mathfrak{R} : bt^k c^j t^j \mathfrak{R},$$

so contracting to R it follows that $I^{k+j+1} : bc^j R = Q$ for all $j \geq 0$, so $Q \in A^*(I)$. \square

3. On the monotonicity of $\text{Ass}(R/I^i)$. It was shown in (2.4) that the sets $\text{Ass}(R/(I^i)_a)$ are increasing, and it was shown in [1] that this does not hold for the sets $\text{Ass}(R/I^i)$. However, the following is still an open problem:

(3.1) *Question.* If I is an ideal in a Noetherian ring R and $Q \in A^*(I)$ is a prime divisor of I^i for some $i \geq 1$, is $Q \in \text{Ass}(R/I^n)$ for all $n \geq i$?

Of course, (3.1) holds for ideals of the form $I = J^i$ with J an ideal and i large, by the existence and definition of $A^*(J)$. (3.2) gives several equivalent sufficient conditions for (3.1) to hold for a given ideal I .

(3.2) THEOREM. *Let I be an ideal in a Noetherian ring R and assume any of the following equivalent conditions hold:*

(3.2.1) $I^{i+1} : I = I^i$ for all $i \geq 1$.

(3.2.2) *There exists an integer k and an element $b \in I^k$ such that $I^{k+i} : bR = I^i$ for all $i \geq 1$.*

(3.2.3) $I^{i+1} = I^i \cap (I^{i+1} : Q) \cap (I^{i+2} : I)$ for all $i \geq 0$ and for all $Q \in \text{Spec } R$ that contain I .

Then if Q is a prime divisor of I^i for some $i \geq 1$, then Q is a prime divisor of I^n for all $n \geq i$.

Proof. It is shown in [14, (3.6)] that (3.2.1)–(3.2.3) are equivalent and each is equivalent to: every prime divisor of $u^{\mathcal{R}}(R, I)$ is relevant. Therefore the proof of this theorem is essentially the same as the proof of (2.4). \square

The following corollary is well known, but it is included here because it follows quite readily from (3.2).

(3.3) COROLLARY. (Cf. [19, Thm. 2.3].) *Let b_1, \dots, b_g be an R -sequence in a Noetherian ring R and let $I = (b_1, \dots, b_g)R$. If Q is a prime divisor of I^i for some $i \geq 1$, then Q is a prime divisor of I^n for all $n \geq 1$.*

Proof. In [19, Thm. 2.3] it is shown that if $Q \in \text{Ass}(R/I^i)$ for some $i \geq 1$, then $\text{grade } Q = g$. Therefore, since the images of b_1, \dots, b_g are an R_Q -sequence and Q_Q is a prime divisor of $I^i R_Q$, it follows that $\text{grade } Q_Q = g$. Therefore Q_Q must be a prime divisor of I_Q , so Q is a prime divisor of I . Finally, it is well known that $I^{i+1} : b_1 R = I^i$ for all $i \geq 1$, so it follows from (3.2) that Q is a prime divisor of I^n for all $n \geq 1$. \square

An important special case of (3.3) is considered in (3.4). For (3.4) recall that an ideal I is of *principal class* in case I can be generated by $h = \text{height } I$ elements.

(3.4) COROLLARY. (Cf. [24, Lemma 5, p. 401].) *Let I be an ideal of principal class in a Cohen-Macaulay ring R , say $\text{height } I = h$. Then every prime divisor of I^i has height h for all $i \geq 1$.*

Proof. R satisfies the Unmixedness Theorem, by [8, (25.6)], so every prime divisor of I has height h . Also, ideals of the principal class in R are generated by R -sequences, since R is Cohen-Macaulay, so the conclusion follows from (3.3). \square

This section will be closed with the following result which is closely related to (3.2) and which gives a sufficient condition for (3.1) to hold for a specified $Q \in A^*(I)$.

(3.5) **PROPOSITION.** *Let I be an ideal in a Noetherian ring R and let $Q \in \text{Ass}(R/I^i)$ for some $i \geq 1$. If $I^{j+1} = I^j \cap (I^{j+1} : Q) \cap (I^{j+2} : I)$ for all $j \geq 0$, then Q is a prime divisor of I^n for all $n \geq i$.*

Proof. By [14, (3.3)], the hypothesis implies $(QR[t, u] \cap \mathcal{R}, u)\mathcal{R}$ ($= (u, Q, tI)\mathcal{R}$) is not contained in any (necessarily irrelevant) prime divisor of $u\mathcal{R}$, so it follows that no irrelevant prime divisor of $u\mathcal{R}$ lies over Q . Therefore the proof is essentially the same as the proof of (2.4). \square

4. A characterization of Cohen-Macaulay rings in terms of $\hat{A}^*(I) = A^*(I)$. During the summer of 1980, Steve McAdam and I both spent a few days investigating what we considered a very nice generalization of R -sequences (see the condition in (4.1)). After we had proved a few results about such sequences, McAdam's student, Dan Katz, showed us the result in (4.1), and so our new sequences were, in fact, just R -sequences. But at least we had found a new (to us) equivalence of the definition of an R -sequence, and this new equivalence is fairly useful, as is shown by the remaining results in this section.

(4.1) **LEMMA.** *Let b_1, \dots, b_g be elements in a Noetherian ring R , let $I_j = (b_1, \dots, b_j)R$ for $j = 0, 1, \dots, g$, and assume $I_g \neq R$. Then b_1, \dots, b_g are an R -sequence if and only if $I_{j-1} : b_j R = I_{j-1}^i$ for $j = 1, \dots, g$ and for all large i .*

Proof. The necessity of the condition follows immediately from the definition of an R -sequence and (3.3), so assume the condition holds. Then it follows that b_1 is a regular element, so b_1 is an R -sequence. Therefore $b_1 R$ and $b_1^i R$ have the same prime divisors for all $i \geq 1$, by (3.3), so the condition implies $b_1 R : b_2 R = b_1 R$, hence b_1, b_2 are an R -sequence. Therefore the condition together with (3.3) imply b_1, b_2, b_3 are an R -sequence, and a repetition of this shows that b_1, \dots, b_g are an R -sequence. \square

(4.1) yields an easy proof of the following characterization of Cohen-Macaulay rings.

(4.2) PROPOSITION. *A Noetherian ring R is Cohen-Macaulay if and only if for all ideals I of the principal class in R it holds that I^i has no imbedded prime divisors for all large i .*

Proof. If R is Cohen-Macaulay, then the condition holds by (3.4). Conversely, let M be a maximal ideal in R , say height $M = h > 0$, and let b_1, \dots, b_h in M such that height $I_j = j$, where $I_j = (b_1, \dots, b_j)R$ ($j = 0, 1, \dots, h$). Then b_j is not in any minimal prime divisor of I_{j-1} , so $I'_{j-1} : b_j R = I'_{j-1}$ for all large i , by hypothesis. Therefore b_1, \dots, b_h are an R -sequence, by (4.1), so their images in R_M are an R_M -sequence, hence R_M is Cohen-Macaulay, and so it follows that R is Cohen-Macaulay. \square

In 1980 in [15, Thm. 4] I essentially characterized Noetherian rings R such that $\hat{A}^*(I) = A^*(I)$ for all ideals I in R . Since this is a useful property, it is somewhat disappointing that this class of such rings is quite small (for example, if altitude $R \geq 4$, then R is not in this class). However, just asking that this property hold for certain types of ideals, such as ideals of the principal class, leads to more interesting classes. (4.3) shows that this property can be used to characterize Cohen-Macaulay rings. For (4.3), recall that a Noetherian ring R is *locally quasi-unmixed* in case for each $P \in \text{Spec } R$ all minimal prime ideals in $(R_P)^*$ have the same depth.

(4.3) THEOREM. *Let R be a locally quasi-unmixed Noetherian ring. Then R is Cohen-Macaulay if and only if $\hat{A}^*(I) = A^*(I)$ for all ideals I of the principal class in R .*

Proof. If R is a Cohen-Macaulay ring and I is an ideal of the principal class in R , then I^i has no imbedded prime divisors for all $i \geq 1$, by (3.4). Therefore $A^*(I) \subseteq \hat{A}^*(I)$, since it is clear that minimal prime divisors of I are in $\hat{A}^*(I) \cap A^*(I)$. However, it is always true that $\hat{A}^*(I) \subseteq A^*(I)$, by (2.8), so $\hat{A}^*(I) = A^*(I)$.

Conversely, let I be an ideal of the principal class in R . Then $(I^i)_a$ has no imbedded prime divisors for all $i \geq 1$, by [13, Thm. 2.12], since R is locally quasi-unmixed, so I^i has no imbedded prime divisors for all large i , by hypothesis, and so R is Cohen-Macaulay, by (4.2). \square

Concerning (4.3), it should be noted that the quasi-unmixed assumption is necessary. For example, let (R, M) be as in [8, Example 2, pp.

203–205] in the case $m = 0$ and $r = 1$. Then R is a local domain of altitude two and $\hat{A}^*(I) = A^*(I)$ for all ideals I in R , by [15, (4.2.2)]. But since there exist a height one maximal ideal in R' , $M \in \hat{A}^*(bR) = A^*(bR)$ for each nonzero b in M , by (2.6.1), hence R is not Cohen-Macaulay.

Even though the condition in (4.1) was not a generalization of an R -sequence, the same condition with $(I'_{j-1})_a$ replacing I'_{j-1} is a very useful generalization. Specifically:

(4.4) DEFINITION. Let b_1, \dots, b_g be elements in a Noetherian ring R and let $I_j = (b_1, \dots, b_j)R$ ($j = 0, 1, \dots, g$). Then b_1, \dots, b_g are an *asymptotic sequence* in case $I_g \neq R$ and $(I'_{j-1})_a : b_j R = (I'_{j-1})_a$ for $j = 1, \dots, g$ and for all large i .

It is shown in [17] that most of the basic properties of R -sequences have valid analogues for asymptotic sequences. It is also shown in [17, (2.3.5)] that an R -sequence is an asymptotic sequence, but not conversely. (4.5) gives a sufficient condition for the converse to hold.

(4.5) PROPOSITION. Let R be a Noetherian ring such that $\hat{A}^*(I) = A^*(I)$ for all ideals I of the principal class. Then elements b_1, \dots, b_g in R are an R -sequence if and only if they are an asymptotic sequence.

Proof. Let $I_j = (b_1, \dots, b_j)R$ ($j = 0, 1, \dots, g$). If b_1, \dots, b_g are an R -sequence, then for $j = 1, \dots, g$ it holds that $b_j \notin \bigcup \{Q; Q \in A^*(I_{j-1})\} = \bigcup \{Q; Q \in \hat{A}^*(I_{j-1})\}$, by (3.3) and hypothesis, so b_1, \dots, b_g are an asymptotic sequence, by (4.4).

Conversely, $b_j \notin \bigcup \{Q; Q \in \hat{A}^*(I_{j-1})\} = \bigcup \{Q; Q \in A^*(I_{j-1})\}$, by definition and hypothesis (it follows immediately from (4.4) that an ideal generated by an asymptotic sequence is an ideal of the principal class). Therefore $I'_{j-1} : b_j R = I'_{j-1}$ for $j = 1, \dots, g$ and for all large i , so b_1, \dots, b_g are an R -sequence, by (4.1). \square

This section will be closed with the following important special case of (4.5).

(4.6) COROLLARY. Elements b_1, \dots, b_g in a Cohen-Macaulay ring R are an R -sequence if and only if they are an asymptotic sequence.

Proof. $\hat{A}^*(I) = A^*(I)$ for all ideals I of the principal class in R , by the first paragraph of the proof of (4.3), so the conclusion follows from (4.5). \square

5. A characterization of an altitude two local UFD in terms of $\hat{A}^*(I)$. In 1979 in [5, Prop. 24] it was shown that if R is a Noetherian domain of altitude two which is locally a UFD, then $\text{Ass}(R/I^i) = \text{Ass}(R/(I^i)_a)$ for all ideals I in R and for all positive integers i and j . In this brief section it is shown that this condition characterizes local UFDs of altitude two.

(5.1) **THEOREM.** *Let (R, M) be a Noetherian domain such that altitude $R \geq 2$. Then altitude $R = 2$ and R is a UFD if and only if $\text{Ass}(R/I^i) = \text{Ass}(R/(I^i)_a)$ for all ideals I in R and for all positive integers i and j .*

Proof. The necessity of the condition was proved in [5, Prop. 24], so assume the condition holds. Assume first that height $M = 2$. Now it is clear that $\hat{A}^*(I) = A^*(I)$ for all ideals I in R , so either $R = R'$ or there exists a height one maximal ideal in R' , by [15, (4.2.2)]. Thus if $R \neq R'$, then $M \in \hat{A}^*(I)$ for all nonzero ideals I in R , by (2.6), so if p is a height one prime ideal in R , then $M \in \hat{A}^*(p)$. However, this and the condition imply M is a prime divisor of p , so it follows that $R = R'$.

Let p be a height one prime ideal in R and suppose p is not principal. Then $M \in \hat{A}^*(p)$, by [5, Prop 21], and this and the condition then imply the contradiction that M is a prime divisor of p . Therefore, if altitude $R = 2$, R is a UFD.

Now suppose altitude $R > 2$. Then, by [15, Thm. 4], altitude $R = 3$ and either: (a) $R = R'$ and R is not quasi-unmixed; or, (b) there exists a height one maximal ideal in R' . If (a) holds, there exists a minimal prime ideal z in R^* such that $\text{depth } z < 3$. Since $R = R'$, $\text{depth } z > 1$, by [15, (1.1) \Rightarrow (1.3)], so $\text{depth } z = 2$. Then there exist at most finitely many $P^* \in \text{Spec } R^*$ such that $z \subset P^*$ and $\text{height } P^* = 2$, by [4, Thm. 1]. Therefore there exists a height two prime ideal P in R such that $z + PR^*$ is M^* -primary, so $M \in \hat{A}^*(P)$, by (6.6). But this and the condition lead to a contradiction. Therefore (b) holds. However, this implies $M \in \hat{A}^*(I)$ for all nonzero ideals I in R , by (2.6), and this and the condition again lead to a contradiction. Therefore R is an altitude two UFD. \square

(5.2) **REMARK.** (5.2.1) Since $\hat{A}^*(I) \subseteq A^*(I)$ always holds, by (2.8), it follows from the proof of (5.1) that the following conditions are equivalent for a local domain R such that altitude $R \geq 2$: (a) Altitude $R = 2$ and R is a UFD; (b) for each ideal I in R the sets $\text{Ass}(R/I^i)$ are equal for all $i \geq 1$; and, (c) for each ideal I in R the sets $\text{Ass}(R/(I^i)_a)$ are equal for all $i \geq 1$.

(5.2.2) It follows immediately from (5.1) and [5, Prop. 24] that a Noetherian domain which is not a field satisfies the condition in (5.1) if and only if for each maximal ideal M in R , either $\text{height } M = 1$ or R_M is an altitude two UFD.

6. $\hat{A}^*(I)$ in flat extension rings and certain factor rings. In this section it is shown in (6.3) that if I is an ideal in a Noetherian ring R , then the sets $\hat{A}^*((I+z)/z)$ (with $z \in \text{mAss } R$) are known once $\hat{A}^*(I)$ is known, and conversely. And in (6.9) and (6.5) it is shown that if S is a Noetherian ring which is a faithfully flat R -module, then $\hat{A}^*(I)$ is known when $\hat{A}^*(IS)$ is, and knowledge about $\hat{A}^*(I)$ provides considerable information about $\hat{A}^*(IS)$. These results are quite technical, but they are also very useful, as is shown in [17].

We begin with two lemmas, the first of which is closely related to (2.6.1).

(6.1) **LEMMA.** *Let R be a Noetherian ring, let A be a ring between R and R' , and let b be a regular element in A . Then if $P \in \text{Ass}(A/(b^i)_a)$ for some $i \geq 1$, there exists a height one prime divisor p' of bR' that lies over P . And, if p'' is a prime divisor of bR' , then $p'' \cap A$ is a prime divisor of $(b^iA)_a$ for all large i .*

Proof. Since $(b^iA)_a = b^iR' \cap A$, the first statement is clear. To complete the proof let p'' be a prime divisor of bR' and note that there exists a finite extension ring B of R such that $B \subseteq A$, $b \in B$, and $p'' \cap A$ is the only prime ideal in A that lies over $p'' \cap B$. Therefore, since $(b^iB)_a = b^iR' \cap B = b^iR' \cap A \cap B = (b^iA)_a \cap B$ and $p'' \cap B \in \hat{A}^*(bB)$, by (2.6.1), it follows that $p'' \cap A$ is a prime divisor of $(b^iA)_a$ for all large i . \square

(6.2) **LEMMA.** *Let R be a Noetherian ring, let A be a ring between R and R' , let b be a regular element in A , such that $bA = (bA)_a$, let $z \in \text{mAss } A$, and let $\bar{}$ denote residue class modulo z . Then there is a one-to-one correspondence between the sets $\{P; P \text{ is a prime divisor of } bA \text{ and } z \subset P\}$, $\{p; p \text{ is a prime divisor of } (b^iA)_a \text{ for some } i \geq 1\}$, and $\{Q; Q \text{ is a prime divisor of } \overline{bA'}\}$ given by $A_P/zA_P = \overline{A_P} = (\overline{A'})_Q$, and these rings are discrete valuation rings.*

Proof. Let $Z = \text{Rad } A$, so $Z \subseteq bA = (bA)_a$, so there exists a one-to-one correspondence between the prime divisors of bA and of $(bA)/Z$ such that $A_P/Z A_P = (A/Z)_{P/Z}$. (By [12, Corol. 2.11], bA is a finite intersection of height one primary ideals.) Also, $(bA)/Z = ((bA)/Z)_a$, so there exists

a one-to-one correspondence between the prime divisors P/Z of $(bA)/Z$ and the prime divisors P' of $(b + Z)(A/Z)'$ given by $(A/Z)_{P/Z} = ((A/Z)')_{P'}$ and these rings are discrete valuation rings, by [12, Corol. 2.11, Prop. 2.7, and Thm. 2.15]. Let z' be the minimal prime ideal in $(A/Z)'$ that lies over z/Z . Then $\bar{A}' = (A/Z)'/z'$ is a direct summand of $(A/Z)'$, so the prime divisors of $\bar{b}\bar{A}'$ are the ideals P'/z' with P' a prime divisor of $(b + Z)(A/Z)'$ that contains z' , and these lie over the ideals \bar{P} in \bar{A} with P a prime divisor of bA that contains z , by the one-to-one correspondence. But $(\bar{b}'\bar{A})_a = \bar{b}'\bar{A}' \cap \bar{A}$, so the conclusion follows from (6.1). \square

The first of the main results in this section shows the nice behavior of $\hat{A}^*(I)$ between R and the rings R/z with $z \in \text{mAss } R$.

(6.3) THEOREM. *The following statements hold for an ideal I in a Noetherian ring R :*

(6.3.1) *If $P \in \hat{A}^*(I)$, there exists $z \in \text{mAss } R$ such that $z \subseteq P$ and $P/z \in \hat{A}^*((I + z)/z)$.*

(6.3.2) *if $z \in \text{mAss } R$ and $Q \in \hat{A}^*((I + z)/z)$, there exists $P \in \hat{A}^*(I)$ such that $z \subseteq P$ and $P/z = Q$. Moreover, $P \in \hat{A}^*(I + z)$.*

Proof. (6.3.1) Let $P \in \hat{A}^*(I)$ and let $\mathcal{R} = \mathcal{R}(R, I)$. Then there exists a (height one) prime divisor p' of $u\mathcal{R}'$ such that $p' \cap R = P$, by (2.7). Let z' be the minimal prime ideal in \mathcal{R}' which is contained in p' (see [12, Props. 2.13 and 2.7]) and let $z = z' \cap R$. Then $z \subseteq P$, $z \in \text{mAss } R$, and $z' = zT \cap \mathcal{R}'$, where T is the total quotient of \mathcal{R} . Also, $R/z \subseteq \mathcal{R}/(z' \cap \mathcal{R}) \subseteq \mathcal{R}'/z' \subseteq (\mathcal{R}/(z' \cap \mathcal{R}))'$ and p'/z' is a height one prime ideal in \mathcal{R}'/z' that lies over P/z , so there exists a height one prime divisor of $u + z'$ in $(\mathcal{R}'/z')'$ that lies over p'/z' . Therefore, since $\mathcal{R}/(z' \cap \mathcal{R}) \cong \mathcal{R}(R/z, (I + z)/z) = (\text{say}) \mathcal{R}_0$, by [20, Lemma 1.1], there exists a height one prime divisor of $u(\mathcal{R}_0)'$ that lies over P/z , hence $P/z \in \hat{A}^*((I + z)/z)$, by (2.7).

(6.3.2) Let $Q \in \hat{A}^*((I + z)/z)$, let \mathcal{R} , \mathcal{R}_0 , and T be as in the preceding paragraph, and let $z^* = zT \cap \mathcal{R}'$, so $\mathcal{R}/(z^* \cap \mathcal{R}) \cong \mathcal{R}_0$. Since $Q \in \hat{A}^*((I + z)/z)$, there exists a height one prime divisor q' of $u\mathcal{R}'_0$ that lies over Q , by (2.7). Let \mathcal{S} be the ring such that $\mathcal{R}_0 \subseteq \mathcal{S} \subseteq \mathcal{R}'_0$ and $\mathcal{S} \cong \mathcal{R}'/z^*$ and let $q = q' \cap \mathcal{S}$, so q is a prime divisor of $(u'\mathcal{S})_a$ for some $i \geq 1$, by (6.1). Let $p' \in \text{Spec } \mathcal{R}'$ such that $z^* \subset p'$ and p'/z^* corresponds to q . Then p'/z^* is a prime divisor of $((u^i + z^*)(\mathcal{R}'/z^*))_a$, by the isomorphism, so (6.2) implies p' is a (height one) prime divisor of $u\mathcal{R}'$. Let $P = p' \cap \mathcal{R}$. Then $z \subseteq P$ and $P/z = Q$, since $q' \cap (R/z) = Q$, and

$P \in \hat{A}^*(I)$, by (2.7). Finally, applying what was just shown to the ideal $I + z$ in place of I , it follows quite readily (since $\mathcal{R}' \subseteq \mathcal{Q}' = \mathcal{R}(R, I + z)'$ and $\mathcal{R}'/z^* \cong \mathcal{Q}'/(zT \cap \mathcal{Q}')$) that $P \in \hat{A}^*(I + z)$. \square

To prove the second of the main results in this section we need the following lemma. (6.4) is known, but I know no reference for it, so it is given here together with a brief sketch of its proof.

(6.4) LEMMA. *Let $R \subseteq S$ be Noetherian rings such that S is a faithfully flat R -module, let X_1, \dots, X_n be indeterminates, and let f_1, \dots, f_k be elements in the total quotient ring of $R[X_1, \dots, X_n]$. Then $S[X_1, \dots, X_n, f_1, \dots, f_k]$ is a faithfully flat $R[X_1, \dots, X_n, f_1, \dots, f_k]$ -module.*

Proof. $B^* = S[X_1, \dots, X_{n+k}]$ is a faithfully flat $B = R[X_1, \dots, X_{n+k}]$ -module. Let K^* and K be the kernels of the natural homomorphisms from B^* onto $S[X_1, \dots, X_n, f_1, \dots, f_k]$ and from B onto $R[X_1, \dots, X_n, f_1, \dots, f_k]$, respectively. Then it is quite straightforward to see that $K^* = KB^*$, since the f_i are in the total quotient ring of $R[X_1, \dots, X_n]$ and since $S[X_1, \dots, X_n]$ is a faithfully flat $R[X_1, \dots, X_n]$ -module, and the conclusion clearly follows from this. \square

(6.5) and (6.8) consider the relationship between $\hat{A}^*(I)$ and $\hat{A}^*(IS)$, where I is an ideal in a Noetherian ring R and S is a flat R -algebra.

(6.5) THEOREM. *Let $R \subseteq S$ be Noetherian rings such that S is a flat R -module, let I be an ideal in R , let $P \in \hat{A}^*(I)$, and let P^* be a minimal prime divisor of PS . Then $P^* \in \hat{A}^*(IS)$.*

Proof. S_{p^*} is a faithfully flat R_p -module, $P_p \in \hat{A}^*(I_p)$, and $P_{p^*}^* \in \hat{A}^*(IS_{p^*})$ if and only if $P^* \in \hat{A}^*(IS)$. Therefore it may be assumed to begin with that R and S are local with maximal ideals P and P^* , respectively, and that PS is P^* -primary. Let $\mathcal{R} = \mathcal{R}(R, I)$ and $\mathcal{S} = \mathcal{R}(S, IS)$, and by (2.7) and the hypothesis let p' be a height one prime divisor of $u\mathcal{R}'$ that lies over P . Let $f \in p'$ such that f is not in any other prime ideal in \mathcal{R}' that lies over $p' \cap \mathcal{R}$, so $p' \cap \mathcal{R}[f]$ is a height one prime divisor of $u\mathcal{R}[f]$. Now $\mathcal{R}[f] \subseteq \mathcal{S}[f]$ are Noetherian rings such that $\mathcal{S}[f]$ is a faithfully flat $\mathcal{R}[f]$ -module, by (6.4), and $\mathcal{S}[f] \subseteq \mathcal{S}'$, since $f \in \mathcal{R}' \subseteq \mathcal{S}'$. Let p^* be a minimal prime divisor of $(p' \cap \mathcal{R}[f])\mathcal{S}[f]$. Then height $p^* = 1$ and p^* is a prime divisor of $u\mathcal{S}[f]$, so there exists a height one prime divisor of $u\mathcal{S}'$ that lies over p^* . Hence, since $p \cap R = P$ and PS is P^* -primary, it follows from (2.7) that $P^* = p^* \cap S \in \hat{A}^*(IS)$. \square

To prove the last of the main results in this section we need one further lemma. This lemma was proved in [17, (2.4)], but it is included here for the sake of completeness.

(6.6) LEMMA. *Let R and S be Noetherian rings such that S is a flat R -module and let P be a minimal prime divisor of $(I, z)S$, where I is an ideal in R and $z \in \text{mAss } S$. Then $P \in \hat{A}^*(IS)$, $\varphi^{-1}(P) \in \hat{A}^*(I)$, and P is a prime divisor of $(I^i)_a S$ for all large i , where φ is the natural homomorphism from R to S .*

Proof. The proof is essentially the same as the proof of [16, (2.9.2)], where the lemma is proved under the assumption that $S_p/(\text{Rad } S_p)$ is analytically unramified. This assumption was used in [16] to invoke a result in the proof of [15, Thm. 1] to show that $\bigcap \{(P^i S_p)_a; i \geq 1\} = \text{Rad } S_p$. However, it was shown in [3] that this holds in all Noetherian rings S , so use [3] in place of the result in [15, Thm. 1] in the proof given in [16]. \square

The following corollary of (6.6) is often useful.

(6.7) COROLLARY. *Let R , S , I , and z be as in (6.6) and assume that a prime ideal P in S is a minimal prime divisor of $(I, z)S$. Then P is a prime divisor of $\varphi^{-1}(P)S$.*

Proof. By (6.6), P is a prime divisor of $(I^i)_a S$, so P is a prime divisor of pS for some prime divisor p of $(I^i)_a$, by [8, (18.11)], and then necessarily $\varphi^{-1}(P) = p$. \square

(6.8) contains another useful result concerning $\hat{A}^*(I)$ and $\hat{A}^*(IS)$.

(6.8) THEOREM. *Let $R \subseteq S$ be Noetherian rings such that S is a flat R -module, let I be an ideal in R , and let $P^* \in \hat{A}^*(IS)$. Then $P^* \cap R \in \hat{A}^*(I)$.*

Proof. Let $P = P^* \cap R$, so S_{P^*} is a faithfully flat R_P -module. Also, $P_{P^*}^* \in \hat{A}^*(IS_{P^*})$, and $P \in \hat{A}^*(I)$ if and only if $P_P \in \hat{A}^*(I_P)$, so it may be assumed to begin with that S is a faithfully flat R -module and that P^* and P are the unique maximal ideals in S and R , respectively. Let S^* be the completion of S , so $R \subseteq S^*$ satisfy the same conditions as $R \subseteq S$. Also, if N is the maximal ideal in S^* , then $N \in \hat{A}^*(IS^*)$, by (6.5), so it may be assumed that S is complete. Let $\mathfrak{R} = \mathfrak{R}(R, I)$ and $\mathfrak{S} = \mathfrak{R}(S, IS)$, so $\mathfrak{R} \subseteq \mathfrak{S}$ are Noetherian rings and \mathfrak{S} is a faithfully flat \mathfrak{R} -module, by (6.4).

By hypothesis and (2.7), there exists a height one prime divisor p^{**} of $u\mathfrak{S}'$ that lies over P^* . Let $p^* = p^{**} \cap \mathfrak{S}$, and let z be the minimal prime ideal in \mathfrak{S}' that is contained in p^{**} . Then $S/(z \cap S)$ is a complete local domain, since S is complete, so by the altitude formula it follows that $(p^{**}/z) \cap (\mathfrak{S}/(z \cap \mathfrak{S}))$ is a height one prime ideal, hence p^* is a minimal prime divisor of $(u, z \cap \mathfrak{S})\mathfrak{S}$. Therefore, since $z \cap \mathfrak{S}$ is a minimal prime ideal, $p = p^* \cap \mathfrak{R} \in \hat{A}^*(u\mathfrak{R})$, by (6.6). Thus there exists a height one prime divisor of $u\mathfrak{R}'$ that lies over p , since $u^n\mathfrak{R}' \cap \mathfrak{R} = (u^n\mathfrak{R})_a$ for all $n \geq 1$, and so $P = p \cap R \in \hat{A}^*(I)$, by (2.7). \square

This section will be closed with the following corollary of (6.5) and (6.8).

(6.9) COROLLARY. *Let $R \subseteq S$ be Noetherian rings such that S is a faithfully flat R -module and let I be an ideal in R . Then $\hat{A}^*(I) = \{P^* \cap R; P^* \in \hat{A}^*(IS)\}$.*

Proof. If $P \in \hat{A}^*(I)$, then $PS \neq S$ and each prime divisor of PS lies over P , by faithful flatness, so the conclusion follows from (6.5) and (6.8). \square

7. On $\hat{A}^*(I)$ and $l(I)$. In this section several necessary and sufficient conditions are given for the maximal ideal in a local ring R to be in $\hat{A}^*(I)$ for a given ideal I in R . We begin by recalling the following definition.

(7.1) DEFINITION. If I is an ideal in a local ring (R, M) , then the *analytic spread* of I , denoted $l(I)$, is defined to be the depth of the ideal $(M, u)\mathfrak{R}$ in the ring $\mathfrak{R} = \mathfrak{R}(R, I)$.

(7.2) contains a couple of useful results concerning $l(I)$.

(7.2) LEMMA. *Let (R^*, M^*) be the completion of a local ring (R, M) and let I be an ideal in R . Then:*

(7.2.1) $l(I) \geq l((I + w)/w)$ for all $w \in \text{mAss } R$ and equality holds for some w .

(7.2.2) $l(I) = l(IR^*)$.

Proof. Let $\mathfrak{R} = \mathfrak{R}(R, I)$ and $\mathfrak{S} = \mathfrak{R}(R^*, IR^*)$. Then $\mathfrak{R}/(M, u)\mathfrak{R} = \mathfrak{S}/(M^*, u)\mathfrak{S}$, so $l(I) = l(IR^*)$. Let $w \in \text{mAss } R$ and let $w' = wR[t, u] \cap \mathfrak{R}$, so $w' \in \text{mAss } \mathfrak{R}$ and $\mathfrak{R}/w' \cong \mathfrak{R}(R/w, (I + w)/w)$, by [20, Lemma 1.1]. Therefore, if P is a prime divisor of $(M/w, u)(\mathfrak{R}/w')$ such that

$\text{depth } P = l((I + w)/w)$ and Q is the preimage of P in \mathcal{R} , then $(M, u)\mathcal{R} \subseteq Q$, so

$$l(I) = \text{depth}(M, u)\mathcal{R} \geq \text{depth } Q = l((I + w)/w).$$

Finally, let Q be a minimal prime divisor of $(M, u)\mathcal{R}$ such that $\text{depth } Q = l(I)$ and let $z' \in \text{mAss } \mathcal{R}$ such that $z' \subseteq Q$. Then $z = z' \cap R \in \text{mAss } R$ and $z' = zR[t, u] \cap \mathcal{R}$, so $\mathcal{R}/z' \cong \mathcal{R}(R/z, (I + z)/z)$ and $(M/z, u)(\mathcal{R}/z') \subseteq Q/z'$, so it follows that $l((I + z)/z) = l(I)$. \square

In [21, Lemma 1.1] it was shown that if I is an ideal in a local ring (R, M) and if $l(I) = \text{altitude } R$, then there exists a height one prime divisor P of $u\mathcal{R}(R, I)$ such that $P \cap R = M$, and that the converse holds when R is quasi-unmixed. Actually, somewhat more can be said in this case, as is shown in (7.3).

(7.3) THEOREM. *Let (R, M) be a local ring, let (R^*, M^*) be the completion of R , and let I be an ideal in R . Then, for the following statements, (7.3.2)–(7.3.5) are equivalent and are implied by (7.3.1), and (7.3.1)–(7.3.5) are equivalent if R is quasi-unmixed.*

(7.3.1) $l(I) = \text{altitude } R$.

(7.3.2) $M \in \hat{A}^*(I)$.

(7.3.3) $M^* \in \hat{A}^*(IR)$.

(7.3.4) *There exists $z \in \text{mAss } R^*$ such that $M^*/z \in \hat{A}^*((IR^* + z)/z)$.*

(7.3.5) *There exists $z \in \text{mAss } R^*$ such that $\text{depth } z = l((IR^* + z)/z)$.*

Proof. Let $\mathcal{R} = \mathcal{R}(R, I)$ and $\mathcal{S} = \mathcal{R}(R^*, IR^*)$. Then, by (7.2), $l(I) = l(IR^*) = l((IR^* + z)/z)$ for some $z \in \text{mAss } R^*$. Therefore, if (7.3.1) holds, then

$$\text{altitude } R = l(I) = l((IR^* + z)/z) \leq \text{altitude } R^*/z,$$

by [2], and

$$\text{altitude } R^*/z = \text{depth } z \leq \text{altitude } R^*,$$

so (7.3.1) \Rightarrow (7.3.5). Now R^*/z satisfies the altitude formula, so $M^*/z \in \hat{A}^*((IR^* + z)/z)$ if and only if $l((IR^* + z)/z) = \text{depth } z$, by [6, Thm. 3], so (7.3.5) \Leftrightarrow (7.3.4). Also, (7.3.4) \Leftrightarrow (7.3.3), by (6.3), and (7.3.3) \Leftrightarrow (7.3.2), by (6.5) and (6.8). Finally, if R is quasi-unmixed, then $\text{depth } w = \text{altitude } R$, and R/w satisfies the altitude formula for all $w \in \text{mAss } R$. Also, $M \in \hat{A}^*(I)$ implies $M/w \in \hat{A}^*((I + w)/w)$ for some $w \in \text{mAss } R$, by (6.3.1). Therefore $l((I + w)/w) = \text{depth } w = \text{altitude } R$, by [6, Thm. 3], and $\text{altitude } R \geq l(I) \geq l((I + w)/w)$, by [2] and (7.2), so (7.3.2) \Rightarrow (7.3.1). \square

(7.4) sharpens [6, Thm. 3], proves the converse of that result, and also extends the result from the Noetherian domain case.

(7.4) COROLLARY. *Let $I \subseteq P$ be ideals in a Noetherian ring R such that P is prime. If $l(I_P) = \text{height } P$, then $P \in \hat{A}^*(I)$. The converse holds if R is locally quasi-unmixed.*

Proof. If $l(I_P) = \text{height } P$, then $P_P \in \hat{A}^*(I_P)$, by (7.3.1) \Rightarrow (7.3.2), so $P \in \hat{A}^*(I)$. If R is locally quasi-unmixed and $P \in \hat{A}^*(I)$, then $P_P \in \hat{A}^*(I_P)$ and R_P is quasi-unmixed, so $l(I_P) = \text{height } P$, by (7.3.2) \Rightarrow (7.3.1). \square

This paper will be closed by using (7.3) to give another proof of part of [15, Thm. 1].

(7.5) COROLLARY. (Cf. [15, Thm. 1].) *Let (R, M) be a local ring such that $\text{altitude } R \geq 1$. Then there exists a depth one $z \in \mathfrak{m}\text{Ass } R^*$ if and only if $M \in \hat{A}^*(I)$ for all ideals I in R such that $\text{height } I \geq 1$.*

Proof. Assume first that there exists a depth one $z \in \mathfrak{m}\text{Ass } R^*$. Now for all ideals I in all local rings L it holds that $\text{altitude } L \geq l(I) \geq \text{height } I$, by [2] and [9, Lemma 4, p. 151], so the conclusion follows immediately from (7.3.5) \Rightarrow (7.3.2).

Conversely, if $M \in \hat{A}^*(bR)$ for some height one principal ideal bR , then (7.3.2) \Rightarrow (7.3.5) implies that there exists a depth one $z \in \mathfrak{m}\text{Ass } R^*$, since $(bR^* + z)/z$ is principal and $l(I) \leq v(I) = \text{the minimal number of generators of } I$, by [9, Lemma 4, p. 151]. \square

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