PACIFIC JOURNAL OF MATHEMATICS
Vol. 111, No. 2, 1984

A NUMBER THEORETIC SERIES OF I. KASARA

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The series

$$S(x) = 1 + \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{n_1 n_2 \cdots n_k \leq x \\ n_1, n_2, \ldots, n_k > 1}} \frac{1}{\log n_1 \log n_2 \cdots \log n_k}$$

is interpreted as a statement about Beurling generalized prime numbers and is estimated by means of Beurling theory.

This series was considered by I. Kasara in [**5**], in which he asserted that

$$\text{(1)} \qquad \text{“}S(x) = x + O(x/\log x).\text{”}$$

This assertion is not correct as it stands. We shall show that

$$\text{(2)} \qquad S(x) = cx + O\left\{x \exp\left(-(\log x)^{1/2-\varepsilon}\right)\right\},$$

where $c \doteq 1.24292$.

We begin by giving the heuristic argument. Each integer in $(1, x]$ is uniquely expressible as a product of a certain number of primes. Thus we have

$$\text{(3)} \qquad [x] = 1 + \pi_1(x) + \pi_2(x) + \cdots$$

for $x \geq 1$, where

$$\pi_k(x) = \#\{n \leq x \colon n \text{ has exactly } k \text{ prime factors}\}$$

with repetitions allowed.

An estimate from prime number theory [**4**, §22.18] and a small calculation give, for each fixed k,

$$\text{(4)} \qquad \pi_k(x) \sim x(\log\log x)^{k-1} / \{(k-1)! \log x\}$$
$$\sim \frac{1}{k!} \sum_{\substack{n_1 n_2 \cdots n_k \leq x \\ n_1, n_2, \ldots, n_k > 1}} \frac{1}{\log n_1 \log n_2 \cdots \log n_k}.$$

This relation and (3) suggest formula (1). However, (4) does not hold uniformly in k, so this argument does not even show that $S(x) \sim cx$.

Define arithmetic functions

$$f(n) = \begin{cases} 1/\log n, & n \geq 2, \\ 0, & n < 2, \end{cases}$$

and

$$e(n) = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

For g and h arithmetic functions, define the convolution $g * h$ by

$$g * h(n) = \sum_{ij=n} g(i)h(j).$$

Finally, define an arithmetic function s by setting $s(n) = S(n) - S(n-1)$. The formula defining S can now be rewritten as

$$s = e + f + f * f/2! + f * f * f/3! + \cdots = \exp f. \tag{5}$$

The last formula is of the type that appears in the theory of Beurling generalized prime numbers, with

$$\sum_{n \leq x} f(n) \leftrightarrow \Pi(x), \qquad S(x) \leftrightarrow [x].$$

Viewed from this perspective, (1) is suspicious, because special conditions are required in order that a Beurling generalized number system should have density exactly 1.

We prove (2) with the aid of Theorem 3.3b of [**3**]: *Suppose f and s satisfy* (5) *and*

$$\sum_{n \leq x} \frac{f(n)}{n} = \int_1^x \frac{1 - t^{-1}}{t \log t} dt + \log c + O\{\exp(-\log^a x)\}$$

for some $c > 0$ and $a \in (0, 1)$. Then

$$S(x) = cx + O\left\{x \exp\left(-[\log x \log\log x]^{a'}\right)\right\},$$

where $a' = a/(1 + a)$.

Here we have

$$\sum_{n \leq x} \frac{f(n)}{n} = \sum_{2 \leq n \leq x} \frac{1}{n \log n} = \int_2^x \frac{dt}{t \log t} + \gamma' + O\left(\frac{1}{x \log x}\right),$$

where $\gamma' \doteq .428166$ [**2**, p. 244, Table 2], and

$$\int_1^x \frac{1 - t^{-1}}{t \log t} dt = \log\log x + \gamma + O\left(\frac{1}{x \log x}\right),$$

where $\gamma \doteq .577216$ [**1**, p. 228, footnote 3].

It follows that

$$\sum_{n \le x} \frac{f(n)}{n} - \int_1^x \frac{1 - t^{-1}}{t \log t}\, dt = \gamma' - \gamma - \log\log 2 + O\left(\frac{1}{x \log x}\right).$$

Thus, $c = \exp(\gamma' - \gamma - \log\log 2) \doteq 1.24292$, and we can take a to be any number less than 1 in Theorem 3.3b of [3]. This proves (2).

We note in conclusion that if it is assumed that $S(x) \sim cx$, then the constant c can be evaluated by an Abelian argument. We use the formula

$$\int_1^\infty x^{-\sigma}\, dS(x) = \exp \sum_{n \ge 2} \frac{1}{n^\sigma \log n},$$

valid for $\sigma > 1$, and evaluate each side. On the one hand,

$$\int_1^\infty x^{-\sigma}\, dS(x) = \sigma \int_1^\infty x^{-\sigma-1} S(x)\, dx \sim \frac{c}{\sigma - 1}$$

as $\sigma \to 1+$. On the other hand, as $\sigma \to 1+$,

$$\begin{aligned}
&\sum_{n \ge 2} \frac{1}{n^\sigma \log n} \\
&\quad = \log \frac{\sigma}{\sigma - 1} + (\sigma - 1) \int_1^\infty t^{-\sigma} \left\{ \sum_{n \le t} \frac{f(n)}{n} - \int_1^t \frac{1 - u^{-1}}{u \log u}\, du \right\} dt \\
&\quad = \log \frac{\sigma}{\sigma - 1} + (\sigma - 1) \int_1^\infty t^{-\sigma} \{\gamma' - \gamma - \log\log 2 + o(1)\}\, dt \\
&\quad = \log \frac{1}{\sigma - 1} + \gamma' - \gamma - \log\log 2 + o(1).
\end{aligned}$$

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Received September 8, 1982. Research supported in part by a grant from the National Science Foundation.

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