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CHARACTERIZATIONS OF (H)PI EXTENSIONS

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A generalization of I. U. Bronstein's characterization for PD extensions is given and, exploiting similar ideas, HPI extensions are characterized intrinsically.

I. Introduction. Among the most important features in topological dynamics in order to classify the minimal flows are the PI and HPI towers for homomorphisms (extensions) of minimal flows ([**EGS 75**], [**AG 77**] and [**V 77**]). The techniques involved depend on transfinite induction, on hyperspaces of transfinite degree and also on rather uncontrollable algebraic features in subgroups of the universal minimal flow.

Although this theory is elegant and powerful, it would be more satisfactory if these techniques and properties were related to some internal structure, only using (finite powers of) the flow itself or the fibered product (powers).

In [**B 77**] such an intrinsic description was given for PI extensions of metric minimal flows. We shall prove a nonmetric version in §3, and we relate this characterization to the one given in [**MN 80**] (the relativized version there of).

In §4 we shall give a similar kind of characterization for HPI extensions.

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In the following we establish some notations, definitions and facts; for more detailed discussions the reader is referred to [**G 76**] and [**Wo 82**].

A *topological transformation group* (*ttg* or *flow*) $\mathscr{X}$ is a triple $\mathscr{X} = \langle T, X, \pi \rangle$, where T is a T_2 topological group, X is a compact T_2 space with unique uniformity $\mathscr{U}_X$ and π is a jointly continuous action $\pi\colon T \times X \to X$. We shall consider ttgs for a fixed (but arbitrary) topological group T and we shall suppress the action symbol, writing the action as a (left) multiplication. A ttg will then be denoted by $\mathscr{X}$ or by its phase space X only. If $x \in X$, then Tx ($\overline{Tx}$) is the *orbit* (*closure*) of x and a subset $A \subseteq X$ is *invariant* if $Tx \subseteq A$ for every $x \in A$, i.e. $TA \subseteq A$. A ttg $\mathscr{X}$ is *minimal* if $\overline{Tx} = X$ for every $x \in X$ or, equivalently, if X does not contain proper closed invariant subsets. If $x \in X$ has a minimal orbit closure, x is called an *almost periodic point*. Note that $\mathscr{X}$ is minimal if for every nonempty open $U \subseteq X$ there is a finite subset $F \subseteq T$ such that $X = FU = TU$. A ttg $\mathscr{X}$ is called *ergodic* if X does not contain proper closed invariant subsets with nonempty interior or, equivalently, if $X = \overline{TU}$ for every nonempty open $U \subseteq X$.

A *homomorphism* (*extension*) of ttgs ϕ: $\mathscr{X} \to \mathscr{Y}$ is a continuous map ϕ: $X \to Y$ that commutes with the action of T (equivariance), i.e. $\phi(tx) = t\phi(x)$ for $t \in T$, $x \in X$. Unless stated otherwise, a homomorphism of ttgs will be assumed to be a surjection. Let ϕ: $\mathscr{X} \to \mathscr{Y}$ and ψ: $\mathscr{Z} \to \mathscr{Y}$ be homomorphisms of ttgs. Then define:

$R_{\phi\psi} := \{ (x, z) \in X \times Z \,|\, \phi(x) = \psi(z) \}$, the fibered product of ϕ and ψ;

$R_\phi := R_{\phi\phi} = \{ (x_1, x_2) \in X \times X \,|\, \phi(x_1) = \phi(x_2) \}$;

$R^n_\phi := \{ (x_1, \ldots, x_n) \in X^n \,|\, \phi(x_1) = \cdots = \phi(x_n) \}$,
the fibered n-power of ϕ;

$P_\phi := \bigcap \{ T\alpha \cap R_\phi \,|\, \alpha \in \mathscr{U}_X \}$, the *relative proximal relation* for ϕ;

$Q_\phi := \bigcap \{ \overline{T\alpha \cap R_\phi} \,|\, \alpha \in \mathscr{U}_X \}$, the *relative regionally proximal relation* for ϕ and

E_ϕ, the smallest closed invariant equivalence relation containing Q_ϕ, the *equicontinuous structure relation*.

Note that the action of T on powers of X is coordinatewise.

The homomorphism ϕ is called *almost periodic* if $Q_\phi = \Delta_X$ and *proximal* if $P_\phi = R_\phi$. The map θ: $X/E_\phi \to Y$ is the maximal almost periodic factor of ϕ, so θ is nontrivial iff $R_\phi \neq E_\phi$. A homomorphism ϕ: $\mathscr{X} \to \mathscr{Y}$ of minimal ttgs is called *highly proximal* (*hp*) if for some $y \in Y$, the whole fiber $\phi^{\leftarrow}(y)$ of y under ϕ shrinks to a point (singleton set) under the action of T on the hyperspace 2^X of X endowed with the usual Vietoris topology. I.e., for some (any) $y \in Y$ there is a net $\{t_i\}_i$ in T such that $t_i\phi^{\leftarrow}(y) \to \{x\}$ in 2^X. Clearly, a highly proximal extension is proximal. One can show that ϕ is hp iff ϕ is an *irreducible* map (if $A = \overline{A}$ and $\phi[A] = Y$ then $A = X$), e.g. see **[AG 77]**.

Proximal, highly proximal and almost periodic extensions are considered to be basic building blocks in determining the structure of minimal ttgs. For instance in **[V 77]** it was shown that every extension is built up by proximal and almost periodic extensions up to some weakly mixing "junk".

A homomorphism $\phi: \mathscr{X} \to \mathscr{Y}$ is a *strictly* PI *extension* if there is an ordinal ν and a collection $\{\phi_\alpha^\beta: \mathscr{X}_\beta \to \mathscr{X}_\alpha | \alpha \leq \beta \leq \nu\}$ of homomorphisms of minimal ttgs such that

(a) $\mathscr{X}_0 = \mathscr{Y}$; $\mathscr{X}_\nu = \mathscr{X}$; $\phi_0^\nu = \phi$;

(b) $\phi_\alpha^\gamma \circ \phi_\gamma^\beta = \phi_\alpha^\beta$ for $\alpha \leq \gamma \leq \beta \leq \nu$;

(c) if β is limit ordinal then $\phi_\alpha^\beta = \operatorname{inv}\lim\{\phi_\alpha^\gamma | \alpha \leq \gamma < \beta\}$;

(d) $\phi_\alpha^{\alpha+1}$ is either proximal or almost periodic for every $\alpha < \nu$. The collection $\{\phi_\alpha^{\alpha+1}: \mathscr{X}_{\alpha+1} \to \mathscr{X}_\alpha | \alpha < \nu\}$ is called the *strictly* PI *tower* for ϕ.

A homomorphism $\phi: \mathscr{X} \to \mathscr{Y}$ is called a *strictly* HPI *extension* if it is a strictly PI extension where every proximal $\phi_\alpha^{\alpha+1}$ is highly proximal. The map ϕ is called an (H)PI-*extension* if there is a strictly (H)PI-extension ψ: $\mathscr{Z} \to \mathscr{Y}$ and a (highly) proximal $\theta: \mathscr{Z} \to \mathscr{X}$ with $\psi = \phi \circ \theta$.

One can show that factors of (H)PI extensions are (H)PI extensions again; i.e.: if $\phi: \mathscr{X} \to \mathscr{Y}$ is an (H)PI extension and $\phi = \mu \circ \lambda$ then μ is an (H)PI extension.

A last remark in this introduction is concerned with some maximal ttgs.

There exists a *universal minimal ttg* $\mathscr{M}$ for T with phase space M. M may be considered as a minimal left ideal in S_T, the universal ambit (point transitive flow with distinguished point) for T. M inherits a semigroup structure and we write $M = JG$ where G is a maximal subgroup of M and J is the collection of idempotents in M. The semigroup M acts on every ttg $\mathscr{X}$ as the "limit action" of T. Note that $x \in X$ is an almost periodic point iff $x = ux$ for some $u \in J$.

If $\mathscr{X}$ is a minimal ttg, then there exists a *maximal* minimal *proximal extension* $\mathfrak{A}(\mathscr{X})$ of $\mathscr{X}$; i.e., $\mathfrak{A}(\mathscr{X}) \to \mathscr{X}$ is proximal and has every proximal extension of $\mathscr{X}$ as a factor. The ttg $\mathfrak{A}(\mathscr{X})$ can be characterized as a certain quasifactor of $\mathscr{M}$; i.e. a minimal sub ttg of the hyper ttg $2^{\mathscr{M}}$ (e.g. see [**EGS 75**] or [**G 76**]), as follows: let $G = uM$, a maximal subgroup of M; here $u \in J$. For $x \in X$ define the Ellis group $H = \mathfrak{G}(\mathscr{X}, ux) = \{g \in G | gx = ux\}$. Then $\mathfrak{A}(\mathscr{X}) := \mathscr{Q}\mathscr{F}(u \circ H, \mathscr{M})$, mostly denoted by $\mathfrak{A}(H)$, is the quasifactor of $\mathscr{M}$ which is the minimal orbit closure of the closed subset (point in $2^{\mathscr{M}}$) $u \circ H$. Here $u \circ H = \lim_{2^M} t_i \overline{H}$, where $t_i \to u$ in the universal ambit S_T. I.e., the "*circle operation*" refers to the action of M on hyperspaces.

If $\phi: \mathscr{X} \to \mathscr{Y}$ is a homomorphism of minimal ttgs then ϕ can be lifted to $\mathfrak{A}(\phi): \mathfrak{A}(\mathscr{X}) \to \mathfrak{A}(\mathscr{Y})$ which is a RIC extension. RIC extensions will be very important in this paper. An extension $\psi: \mathscr{W} \to \mathscr{Z}$ of minimal ttgs is called a RIC *extension* if $R_{\psi\theta}$ is minimal for every proximal extension θ: $\mathscr{W}' \to \mathscr{Z}$. Another, equivalent, definition could be ψ is a RIC extension iff $\psi^{\leftarrow}(z) = u \circ u\psi^{\leftarrow}(z)$ for every $z \in \mathscr{Z}$, $u \in I_z = \{w \in I | wz = z\}$.

For a minimal ttg $\mathscr{X}$ there also exists a *maximal highly proximal extension* $\mathscr{X}^*$ of $\mathscr{X}$ and every homomorphism ϕ: $\mathscr{X} \to \mathscr{Y}$ can be lifted to a map ϕ^*: $\mathscr{X}^* \to \mathscr{Y}^*$ which is an open extension. The ttg $\mathscr{X}^*$ can also be viewed as a quasifactor of $\mathscr{M}$. Moreover, $\mathscr{X}^*$ may be characterized as follows: $\mathscr{X} = \mathscr{X}^*$ iff every extension which values in $\mathscr{X}$ is open. For more details on $\mathscr{X}^*$ and hp extensions see [**AG 77**] and [**AW 81**].

For the remainder of this paper it might be useful to remember that ϕ: $\mathscr{X} \to \mathscr{Y}$ is a *weakly mixing extension* iff R_ϕ is ergodic and that ϕ is said to satisfy the *Bronstein condition* (Bc) iff R_ϕ has a dense set of almost periodic points ($R_\phi = \overline{JR_\phi}$). Examples of Bc extensions are almost periodic extensions and RIC extensions. Examples of weakly mixing extensions are Bc extensions with $R_\phi = E_\phi$ ([**V 77**] 2.6.2) and open proximal extensions ([**Wo 82**] VII.2.14).

II. Some basic tools. In this section we gather several useful facts, lemmas and theorems, mainly for the sake of reference and comparison. The cohesive theme is the question about properties of the domain of a homomorphism if the range is ergodic or minimal.

2.1. THEOREM. *Let ϕ: $\mathscr{X} \to \mathscr{Y}$ be an almost periodic homomorphism of ttgs with $\mathscr{Y}$ minimal. Then $\mathscr{X}$ is ergodic iff $\mathscr{X}$ is minimal* ([**AMWW 83**] 2.15).□

2.2. THEOREM. *Let ϕ: $\mathscr{X} \to \mathscr{Y}$ be a proximal homomorphism of ttgs with $\mathscr{Y}$ ergodic. If X has a dense set of almost periodic points, then $\mathscr{X}$ is ergodic* ([**AMWW 83**] 2.10). □

2.3. THEOREM. *Let ϕ: $\mathscr{X} \to \mathscr{Y}$ be a homomorphism of ttgs such that ϕ: $X \to Y$ is an irreducible map. Then $\mathscr{X}$ is ergodic (minimal) iff $\mathscr{Y}$ is ergodic (minimal).*

Proof. The statement about minimality is obvious.

Suppose $\mathscr{Y}$ is ergodic and let $A = \overline{TA} \subseteq X$ have a nonempty interior. As either $\phi[A]$ or $\phi[X \setminus A^\circ]$ has a nonempty interior in Y, ergodicity of $\mathscr{Y}$ implies either $Y = \phi[A]$ or $Y = \phi[X \setminus A^\circ]$. By irreducibility of φ, $X = A$ or $X = X \setminus A^\circ$; but as $A^\circ \neq \varnothing$ it follows that $X = A$. □

In the fourth section, the notion of semi-openness is very important. Remember that a map f: $X \to Y$ is *semi-open* if for every open $U \subseteq X$ the image $f[U]$ has a nonempty interior in Y.

The following remark gives some useful examples of semi-open maps:

2.4. Remark. Let $\phi: \mathscr{X} \to \mathscr{Y}$ be a homomorphism of ttgs.

(a) If ϕ is irreducible then ϕ is semi-open; even: for every open $U \subseteq X$ there is an open $U' \subseteq U$ with $\overline{U'} = \overline{U}$ such that $\phi^{\leftarrow}\phi[U'] = U'$, and $\phi[U']$ is open in Y.

(b) If $\mathscr{Y}$ is minimal and if X has a dense set of almost periodic points, then ϕ is semi-open. In particular, every homomorphism of minimal ttgs is semi-open.

(c) If $\phi: \mathscr{X} \to \mathscr{Y}$ is open, then for every $n \in N$, $n \geq 2$ and for every $m \in \mathbf{N}$ with $m \leq n$ the projection $\pi_m: \mathscr{R}_\phi^n \to \mathscr{X}$ onto the mth coordinate is semi-open. □

The following result deals with the semi-open lifting of ergodicity. It is the "ergodic" counterpart of the fact that the preimage of an almost periodic point contains an almost periodic point.

2.5. Theorem. *Let $\phi: \mathscr{X} \to \mathscr{Y}$ be a homomorphism of ttgs with $\mathscr{Y}$ ergodic. Then there exists an ergodic subttg $\mathscr{A}$ of $\mathscr{X}$ such that $\phi|_A: \mathscr{A} \to \mathscr{Y}$ is a semi-open surjection.*

Proof. Let $\mathscr{C}$ be the collection of all closed invariant subsets of X that are mapped onto Y by ϕ. Then $\mathscr{C}$ is nonempty and ordered by inclusion. It is easily seen that $\mathscr{C}$ is inductively ordered, so by Zorn's lemma there is an $A = \overline{TA} \subseteq X$ with $\phi' = \phi|_A: A \to Y$ is surjective and that is minimal under that condition. We shall show that A is ergodic and that ϕ' is semi-open.

Let $F \subseteq A$ be a closed subset of A such that F is mapped irreducibly onto Y by $\phi'|_F$ (e.g.: **[Wi 70]** 14.2.1). As every $t \in T$ acts as a homeomorphism on A, also $\phi'|_{tF}: tF \to Y$ is an irreducible surjection for every $t \in T$. Hence, for every $t \in T$, $\phi'|_{tF}: tF \to Y$ is semi-open (compare 2.4(a) where equivariance is not needed).

As $\phi'|_{\overline{TF}}: \overline{TF} \to Y$ is a surjection and $\overline{TF} \subseteq A$, it follows that $\overline{TF} = A$. Suppose $U \subseteq A$ is open in A. Then for some $t \in T$, $U \cap tF \neq \varnothing$; so $U \cap tF$ is open in tF. By irreducibility of $\phi'|_{tF}$ we know that $\phi'|_{tF}[U \cap tF]$ has a nonempty interior in Y. Consequently $\phi'|_{\overline{TF}}[U]$ has a nonempty interior in Y, which proves semi-openness for $\phi'|_{\overline{TF}} = \phi'|_A$. Moreover, ergodicity of $\mathscr{Y}$ implies that $\phi'|_A[\overline{TU}] = \overline{T \cdot \phi'|_A[U]} = Y$. But then by the construction of A it follows that $A = \overline{TU}$. Hence A is ergodic. □

III. A characterization for PI extensions. In [B 77] it was proven that a homomorphism $\phi\colon \mathscr{X} \to \mathscr{Y}$ of metric minimal ttgs is a PI extension iff every orbit closure in R_ϕ with a dense set of almost periodic points is minimal.

Before we state the non-metric version we need the following definition: Let $n \in \mathbf{N}$, $n \geq 2$; a homomorphism $\phi\colon \mathscr{X} \to \mathscr{Y}$ is said to satisfy *the n-C′-condition* (or ϕ is an *n-C′-extension*) iff every ergodic subset $A = \overline{TA}$ of R^n_ϕ with a dense set of almost periodic points is minimal. We say that ϕ is a *C′-extension* if ϕ is 2-C′. We shall prove:

3.1. THEOREM. *Let $\phi\colon \mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs. Then the following statements are equivalent*:

(a) *ϕ is a* PI *extension*;

(b) *ϕ is a* C′*-extension*;

(c) *ϕ is an n-*C′*-extension for some $n \in \mathbf{N}$, $n \geq 2$.*

(d) *ϕ is an n-*C′*-extension for every $n \in \mathbf{N}$, $n \geq 2$.*

The proof of 3.1 requires several steps that will be formulated in separate lemmas. The straightforward proofs of the following remark will be omitted.

3.2. REMARK: (a) A weakly mixing Bc extension that satisfies the C′ condition is an isomorphism.

(b) Let ϕ and ψ be extensions of minimal ttgs such that $\psi \circ \phi$ is an n-C′-extension. Then ϕ is an n-C′-extension.

(c) Let $\{\phi_\alpha\colon \mathscr{X}_\alpha \to \mathscr{Y} \mid \alpha < \nu\}$ be an inverse system of n-C′-extensions. Then $\phi := \operatorname{inv\,lim} \phi_\alpha$ is an n-C′-extension. □

3.3. LEMMA. *Let ϕ be an n-*C′*-extension for some $n \geq 2$. Then ϕ is an m-*C′*-extension for every $m \in \mathbf{N}$ with $2 \leq m \leq n$.*

Proof. Suppose for $n \geq 3$, ϕ is an n-C′-extension. We shall prove that ϕ is an n-1-C′-extension.

Let $A = \overline{TA}$ be an ergodic subset of R^{n-1}_ϕ with a dense set of almost periodic points. Let $\pi\colon R^n_\phi \to R^{n-1}_\phi$ be the projection. By the proof of 2.5 there exists an $A' = \overline{TA'} \subseteq R^n_\phi$ which is ergodic such that $\pi|_{A'}\colon A' \to A$ is a semi-open surjection and, moreover, such that A' does not contain a proper closed invariant subset that is mapped onto A. Let $U \subseteq A'$ be open. Suppose that U contains no almost periodic points. Then TU contains no almost periodic points, which means that $A' \setminus TU$ contains all almost periodic points that are in the preimages of the almost periodic points in A. This shows that $A' \setminus TU$ is a closed invariant subset which is mapped onto A. Hence $A' = A' \setminus TU$, so $U = \emptyset$. Consequently, A' has a dense set of almost periodic points. As ϕ is an n-C$'$-extension, it follows that A' is minimal, so A is minimal. □

Using the same reasoning as in the proof of 3.3 one shows:

3.4. LEMMA. *A factor of an n-C$'$-extension is an n-C$'$-extension; i.e.: If $\psi \circ \phi$ is an n-C$'$-extension then ψ is an n-C$'$-extension too.* □

3.5. LEMMA. *Let ϕ: $\mathscr{X} \to \mathscr{Y}$ and ψ: $\mathscr{Y} \to \mathscr{Z}$ be homomorphisms of minimal ttgs and let $n \in \mathbf{N}$ with $n \geq 2$.*

(a) *If ψ is an n-C$'$-extension and if ϕ is almost periodic then $\psi \circ \phi$ is an n-C$'$-extension.*

(b) *If ψ is an n-C$'$-extension and if ϕ is proximal then $\psi \circ \phi$ is an n-C$'$-extension.*

Proof. (a) Let $A \subseteq R^n_{\psi \circ \phi}$ be ergodic with a dense set of almost periodic points. Then $\phi^n[A] \subseteq R^n_\psi$ is ergodic with a dense set of almost periodic points. As ψ is an n-C$'$-extension, $\phi^n[A]$ is minimal. As ϕ^n: $A \to \phi^n[A]$ is an almost periodic extension of a minimal set and as A is ergodic, it follows from 2.1 that A is minimal.

(b) Suppose ψ is an n-C$'$-extension and let $A = \overline{TA} \subseteq R^n_{\psi \circ \phi}$ be ergodic with a dense set of almost periodic points. Then $\phi^n[A] \subseteq R^n_\psi$ is ergodic with a dense set of almost periodic points. As ψ was n-C$'$, $\phi^n[A]$ is minimal. By proximality of the map ϕ^n, A has a unique minimal subset; hence A is minimal. □

By now we are able to prove Theorem 3.1:

Proof (of 3.1). The implication (d) $\Rightarrow$ (c) is trivial and (c) $\Rightarrow$ (b) is just 3.3.

(b) $\Rightarrow$ (a). Suppose ϕ is a C'-extension and consider the canonical PI shadow diagram for ϕ (**[EGS 75]**, **[V 77]**):

$$\begin{array}{ccc} \mathscr{X}_\infty & \xrightarrow{\sigma} & \mathscr{X} \\ \downarrow \phi_\infty & & \downarrow \phi \\ \mathscr{Y}_\infty & \xrightarrow[\tau]{} & \mathscr{Y} \end{array}$$

where σ is proximal, τ is strictly PI and ϕ_∞ is a weakly mixing Bc extension (**[V 77]** 2.1.3).

By 3.5(b), $\phi \circ \sigma$ is a C′-extension; and, by 3.2(b), ϕ_∞ turns out to be a C′-extension. Hence, by 3.2(a), ϕ_∞ is an isomorphism; which shows that ϕ is a PI extension.

(a) $\Rightarrow$ (d). Let ϕ be a PI extension and let θ: $\mathscr{W} \to \mathscr{Y}$ be a strictly PI-extension such that $\theta = \phi \circ \kappa$ for some κ: $\mathscr{W} \to \mathscr{X}$. Let $n \in \mathbf{N}$, $n \geq 2$. Then, by 3.5(a) and (b) and by 3.2(c), it follows that θ is an n-C′-extension. Hence, by 3.4, ϕ is an n-C′-extension. As n was chosen arbitrarily, this proves the theorem. □

One might wonder how this characterization for PI extensions relates to other known ones. The relation to the PI characterization in **[EGS 75]** is obvious and it is made through **[V 77]** 2.6.3. In order to relate our characterization to the one in **[MN 80]** we need the following definition.

Let ϕ: $\mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs, let $y \in Y$ and let $\{t_i\}_i$ be a net in T. For a closed subset $K \subseteq \phi^{\leftarrow}(y)$ and for x and z elements of K, we say that x is *strongly regionally proximal to y in K with respect to the net $\{t_i\}_i$* if there are $z_i \in K$ such that $t_i x \to x$, $z_i \to z$ and $t_i z_i \to x$. In other words: for x and z we can find a regionally proximal making net in $K \times K$ such that the coordinate tending to x is fixed (x) and such that the net in T chasing it to the diagonal is the prescribed one. Notation: $x \in \mathrm{SRP}(\phi, K, z, \{t_i\}_i)$.

This definition is slightly different from the one in **[MN 80]**. First, because it is a generalization to the relativized case. (Note that the results in the first section of **[MN 80]** can easily be transferred to the relativized case.) Second, because it takes a net into account, which is necessary to state the characterization in **[MN 80]** as strong as it was proven.

3.6. THEOREM (**[MN 80]** 1.1). *Let ϕ: $\mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs. Then ϕ is not a* PI *extension iff for some $w \in J$ there is a nontrivial set $K = \overline{wK} \subseteq \phi^{\leftarrow}(y)$, for some $y = wy$, such that for some (every) net $\{t_i\}_i \to w$ in T and for every subnet $\{s_i\}_i$ of $\{t_i\}_i$ there is a subnet $\{t_i^*\}_i$ of $\{s_i\}_i$ such that $x \in$* SRP*$(\phi, K, z, \{t_i^*\}_i)$ for every $x \in wK$ and $z \in K$. (Nontriviality for K means $|K| \geq 2$.)* □

We shall show that such a K (as in the theorem) defines an ergodic subset $\overline{T(K \times K)}$ in R_ϕ with a dense set of almost periodic points. It then follows that if ϕ is PI, $\overline{T(K \times K)}$ has to be minimal, hence K has to be trivial. This illustrates how our characterization relates to the one in **[MN 80]**.

First note that $wK \times wK$ is dense in $K \times K$, so $T(wK \times wK)$ is a dense set of almost periodic points in $\overline{T(K \times K)} \subseteq R_\phi$. Let $U \subseteq \overline{T(K \times K)}$ be open, then for some $t_1 \in T$, $t_1 U \cap K \times K \neq \varnothing$ and open in $K \times K$. Then there is a point $(x_1, x_2) = (wx_1, wx_2) \in K \times K \cap t_1 U$. Let $V \subseteq \overline{T(K \times K)}$ be open, then for some $t_2 \in T$, $t_2 V \cap K \times K \neq \varnothing$. Choose $(z_1, z_2) \in t_2 V \cap K \times K$. Let $\{s_i\}_i$ be a subnet of $\{t_i\}_i \to w$ such that for certain $z_1^i \in K$ we have $z_1^i \to z_1$, $s_i z_1^i \to x_1$. Let $\{t_i^*\}_i$ be a subnet of $\{s_i\}_i \to w$ such that for certain $z_2^i \in K$ we have $z_2^i \to z_2$ and $t_i^* z_2^i \to x_2$. Then $(z_1^i, z_2^i) \to (z_1, z_2)$ and $t_i^*(z_1^i, z_2^i) \to (x_1, x_2)$. So for certain i_0, $t_{i_0}^*(z_1^{i_0}, z_2^{i_0}) \in t_1 U$ and $(z_1^{i_0}, z_2^{i_0}) \in t_2 V$. Hence $t_{i_0}^* t_2 V \cap t_1 U \neq \varnothing$, which shows ergodicity of $\overline{T(K \times K)}$.

Note that for non-PI extensions it is easy to find a nontrivial $K = \overline{wK}$ as in 3.6, namely $K = \overline{wF_\infty x_0}$ where $F = \mathfrak{G}(\mathscr{Y}, \phi(x_0))$ and $w \in J$ such that $\langle \overline{wF_\infty}, F_\infty \rangle$ is a minimal right ttg for F_∞.

IV. A characterization of HPI extensions. Although in the metric case HPI extensions are known as the pointdistal extensions, there does not exist a satisfying intrinsic characterization for HPI extensions of nonmetric minimal ttgs. (The most interesting up to now is the one in [**AG 77**] Thm. III.2).

Note that universal PI extensions may be characterized in terms of quasifactors of $\mathscr{M}$. The known characterization of the universal HPI extensions in terms of quasifactors of $\mathscr{M}$, however, is rather artificial ([**Wo 82**] V.4).

In this section we shall give a characterization in the same spirit as the PI characterization in 3.1. The next definition is the HPI counterpart of the one for n-C′-extensions in the previous section: Let $n \in \mathbf{N}$, $n \geq 2$ and let ϕ: $\mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs. Then ϕ is said to satisfy the n-ESOM *property* iff every ergodic sub ttg $A = \overline{TA} \subseteq R_\phi^n$ such that every projection π_i: $\mathscr{A} \to \mathscr{X}$ is semi-open is minimal. If ϕ is a 2-ESOM extension, we say ϕ is an ESOM *extension*.

4.1. Remark. Let ϕ: $\mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs.

(a) If ϕ is an ESOM extension, then ϕ is a PI extension.

(b) If ϕ is a pointdistal PI extension then ϕ is an ESOM extension.

Proof. (a) Let $A \subseteq R_\phi$ be ergodic with a dense set of almost periodic points. Then, by 2.4(b), we know that the projections π_i: $A \to \mathscr{X}$ are semi-open. So by the definition of ESOM, A is minimal. Hence, by 3.1, ϕ is a PI extension.

(b) Let $A = \overline{TA} \subseteq R_\phi$ be ergodic with semi-open projections to $\mathscr{X}$. Let $U \subseteq A$ be open and let $V = (\pi_1[U])^\circ \subseteq X$ then V contains a ϕ-distal point x. So there is a point $(x, z) \in U$ and as x is ϕ-distal, (x, z) is an almost periodic point. This shows that A has a dense set of almost periodic points. As ϕ is PI, it follows from 3.1 that A is minimal. Hence ϕ is an ESOM map. □

After our main result (4.8) it follows that a pointdistal PI extension is n-ESOM for every $n \in \mathbf{N}$, $n \geq 2$ (which, by then, is obvious).

The straightforward proofs of the following remark are omitted.

4.2. Remark. (a) A weakly mixing ESOM extension ϕ of minimal ttgs which is either an open- or a BC extension is an isomorphism (use 2.4(b), (c)).

(b) Let ϕ and ψ be extensions of minimal ttgs such that $\psi \circ \phi$ is an n-ESOM-extension. Then ϕ is an n-ESOM-extension.

(c) Let $\{\phi_\alpha\colon \mathscr{X}_\alpha \to \mathscr{Y} | \alpha < \nu\}$ be an inverse system of n-ESOM-extensions. Then $\phi = \operatorname{inv\,lim} \phi_\alpha$ is an n-ESOM-extension. □

4.3. Lemma. *Let $\phi\colon \mathscr{X} \to \mathscr{Y}$ and $\psi\colon \mathscr{Y} \to \mathscr{Z}$ be homomorphisms of minimal ttgs and let $n \in \mathbf{N}$ with $n \geq 2$.*

(a) *If ψ is an* n-ESOM-*extension and ϕ is almost periodic, then $\psi \circ \phi$ is an* n-ESOM-*extension.*

(b) *If ψ is an* n-ESOM-*extension and ϕ is highly proximal, then $\psi \circ \phi$ is an* n-ESOM-*extension.*

Proof. (a) Let $A = \overline{TA}$ be an ergodic subset of $R^n_{\psi \circ \phi}$ with semi-open projections to $\mathscr{X}$. Then $\phi^n[A]$ is an ergodic subset of R^n_ψ. That the projections to $\mathscr{Y}$ are semi-open follows from the fact that $\phi\colon \mathscr{X} \to \mathscr{Y}$ as a homomorphism of minimal ttgs is semi-open (2.4(b)). As ψ is n-ESOM, $\phi^n[A]$ is minimal. Hence, by 2.1, A is minimal; so $\psi \circ \phi$ is an n-ESOM-extension.

(b) Let $A = \overline{TA}$ be an ergodic subset of $R^n_{\psi \circ \phi}$ with semi-open projections to $\mathscr{X}$. Then $\phi^n[A]$ is ergodic with semi-open projections to $\mathscr{Y}$. Hence, as ψ is n-ESOM, $\phi^n[A]$ is minimal in R^n_ψ.

Let $U_i \subseteq X$ be open with $U_1 \times \cdots U_n \cap A \neq \varnothing$. By semi-openness of π_1, there is a nonempty open $V_1 \subseteq (\pi_1[U_1 \times \cdots \times U_n \cap A])^\circ$. As ϕ is irreducible, there is a nonempty open $W_1 = \phi^{\leftarrow} \phi[W_1] \subseteq V_1$ and $\phi[W_1]$ is nonempty and open in Y. Note that $W_1 \times U_2 \times \cdots \times U_n \cap A \neq \varnothing$ and open. By semi-openness of π_2 and irreducibility of ϕ there is an open W_2

in X with $W_2 = \phi^{\leftarrow}\phi[W_2] \subseteq U_2$ and $W_1 \times W_2 \times U_3 \times \cdots \times U_n \cap A \neq \varnothing$ and open. Proceeding this way, there are open $W_i \subseteq U_i$ with $W_i = \phi^{\leftarrow}\phi[W_i]$ and $W_1 \times \cdots \times W_n \cap A \neq \varnothing$. From the choice of the W_i it follows that $W_1 \times \cdots \times W_n \cap A$ contains a full preimage of some $b \in \phi^n[A]$ under the map $\phi^n|_A\colon A \to \phi^n[A]$.

As this is true for every open $U = U_1 \times \cdots \times U_n \cap A$ in A, it follows that $\phi^n|_A\colon A \to \phi^n[A]$ is irreducible. So, by 2.3, A is minimal; which proves that $\psi \circ \phi$ is an n-ESOM extension. □

In order to show that every HPI extension satisfies the n-ESOM property we just have to prove that a factor of an n-ESOM-extension under a highly proximal map is an n-ESOM-extension.

4.4. LEMMA. *Let $\phi\colon \mathscr{X} \to \mathscr{Y}$ and $\psi\colon \mathscr{Y} \to \mathscr{Z}$ be homomorphisms of minimal ttgs with ϕ highly proximal. If $\psi \circ \phi$ is n-ESOM then ψ is an n-ESOM extension.*

Proof. Let $A = \overline{TA} \subseteq R^n_\psi$ be ergodic with semi-open projections to $\mathscr{Y}$. By 2.5, there is an ergodic $A' = \overline{TA'} \subseteq R^n_{\psi\circ\phi}$ such that $\phi^n|_{A'}\colon A' \to A$ is a semi-open surjection. We shall show that the projections from A' onto $\mathscr{X}$ are semi-open. It then follows from the n-ESOM property for $\psi \circ \phi$ that A' is minimal; hence A as a factor of A' is minimal. So ψ turns out to satisfy the n-ESOM property.

Let $U \subseteq A'$ be open and choose an open $\varnothing \neq V \subseteq \overline{V} \subseteq U$. By semi-openness of $\phi^n|_{A'}$, $\phi^n[V]$ has a nonempty interior in A. So, by semi-openness of the ith projection $\pi_i^Y\colon A \to \mathscr{Y}$, it follows that $(\pi_i^Y \circ \phi^n[V])^\circ = W$ is nonempty in Y. But then $\pi_i^Y \circ \phi^n[\overline{V}] = \phi \circ \pi_i^X[\overline{V}]$ has a nonempty interior in Y; and so for some finite $F \subseteq T$ we have

$$Y = F \cdot \left(\phi \circ \pi_i^X[\overline{V}]\right) = \phi \circ \pi_i^X[F\overline{V}].$$

(Here π_i^X is the ith projection from A' onto $\mathscr{X}$). As ϕ is irreducible, $\pi_i^X[F\overline{V}] = X = F \cdot \pi_i^X[\overline{V}]$. But then $\pi_i^X[\overline{V}]$ has a nonempty interior (F is finite); and so $\pi_i^X[U]$ has a nonempty interior in X. As this holds for every $i \leq n$, it follows that all projections from A' onto $\mathscr{X}$ are semi-open. □

4.5. THEOREM. *If $\phi\colon \mathscr{X} \to \mathscr{Y}$ is an HPI extension of minimal ttgs, then ϕ is an n-ESOM-extension for every $n \in \mathbf{N}$, $n \geq 2$.*

Proof. Let $\theta\colon \mathscr{W} \to \mathscr{Y}$ be a strictly HPI extension and let $\xi\colon \mathscr{W} \to \mathscr{X}$ be highly proximal. Then, by 4.3(a) and (b) and 4.2(c), θ is an n-ESOM-extension for every $n \in \mathbf{N}$, $n \geq 2$. As ξ is highly proximal, the theorem follows from 4.4. □

For the following (the converse of 4.5) we need some details about the EGS(ϕ) diagram for $\phi: \mathscr{X} \to \mathscr{Y}$ (see **[EGS 75]**, **[G 76]**, **[V 77]** and, mainly for reasons of notation and for useful details, **[Wo 82]**).

Let $\phi: \mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs. Let M be a fixed minimal left ideal in the universal ambit S_T considered as the universal minimal ttg $\mathscr{M}$ for T, and let J denote the collection of idempotents in M.

Define the shadow diagram EGS(ϕ) for ϕ as follows:

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{\sigma} & \mathscr{X} \\ \phi' \downarrow & \mathrm{EGS}(\phi) & \downarrow \phi \\ \mathscr{Y}' & \xrightarrow[\tau]{} & \mathscr{Y} \end{array}$$

$Y' = \{ v \circ u\phi^{\leftarrow}(y) | y \in Y, v \in J_y \}$, where $J_y = \{ w \in J | wy = y \}$ and $u \in J$. (The circle operation refers to the action of M on the hyperspace 2^X of X.) The map τ is defined by $\tau(v \circ u\phi^{\leftarrow}(y)) = y$. Then define $X' = \{(x, y') | x \in y' \in Y'\} \subseteq X \times Y'$, and let τ and ϕ' be the projections. Then $\mathscr{X}'$ and $\mathscr{Y}'$ are minimal ttgs, σ and τ are proximal extensions and ϕ' is a RIC extension.

Note that there are several equivalent descriptions for the elements of Y': $Y' = \{ v \circ v\phi^{\leftarrow}(y) | v \in J,\ y \in Y \}$; $Y' = \{ p \circ u\phi^{\leftarrow} \phi(x_0) | p \in M \}$, here $x_0 \in X$ is fixed and $u \in J$; $Y' = \{ p \circ Fx_0 | p \in M \}$, where (again) $x_0 \in X$ is fixed and $F = \mathfrak{G}(\mathscr{Y}, \phi(x_0))$ is the Ellis group of $\mathscr{Y}$ with respect to $\phi(x_0)$.

If $\phi: \mathscr{X}^* \to \mathscr{Y}^*$, then σ and τ are open and with respect to the open sets in X' we have the following useful remark (in this situation):

As $X' \subseteq X \times Y'$ and as the projections are open ($\mathscr{X} = \mathscr{X}^*$!), X' has a base of open sets of the form $U \times V \cap X' \neq \varnothing$, where U and V are open in X and Y' and such that for every $u \in U$ there is a $v \in V$ with $(u, v) \in X'$. In addition, note that for such basic open sets we have $\sigma[U \times V \cap X'] = U$.

4.6. LEMMA. *Let* $\phi: \mathscr{X}^* \to \mathscr{Y}^*$ *be a homomorphism of minimal ttgs and consider* EGS(ϕ). *If* $A \subseteq R_{\tau \circ \phi'}$ *is such that* $\phi' \times \phi'|_A: A \to R_\tau$ *is a semi-open surjection, then* $\sigma \times \sigma[A]$ *has semi-open projections onto* $\mathscr{X}^*$.

Proof. As τ is open, the projections $\pi_i^{Y'}: R_\tau \to Y'$ are semi-open, so the maps $\pi_i^{Y'} \circ \phi' \times \phi': A \to Y'$ are semi-open.

Let $O \subseteq \sigma \times \sigma[A]$ be a nonempty open set in $\sigma \times \sigma[A]$, and let $W := (\sigma \times \sigma)^{\leftarrow}[O] \cap A$ which is open in A. Choose basic open sets W_1 and W_2 in X' with $\varnothing \neq W_1 \times W_2 \cap A \subseteq W$. By semi-openness of

$\pi_1^{Y'} \circ \phi' \times \phi'$: $A \to Y'$, we may find an open $\tilde{W}_1 \subseteq W_1$ such that for every $w_1 \in \tilde{W}_1$ there is a $w_2 \in W_2$ with $(w_1, w_2) \in A$. Without loss of generality $\tilde{W}_1$ may be chosen to be basic open in X'. Say $\tilde{W}_1 = U_1 \times V_1 \cap X'$ and $W_2 = U_2 \times V_2 \cap X'$. We show that $U_1 \subseteq \pi_1^X[O]$; hence it follows that π_1^X: $\sigma \times \sigma[A] \to \mathscr{X}^*$ is semi-open:

Let $u \in U_1$; then, by the discussion above, there is a $v \in V$ with $w = (u, v) \in X'$. As $w \in \tilde{W}_1$, there is a $w_2 \in W_2$ with $(w, w_2) \in W \subseteq A$, say $w_2 = (u_2, v_2)$. Clearly $\sigma \times \sigma(w, w_2) = (u, u_2) \in \sigma \times \sigma[W] \subseteq O$, which shows that $U_1 \subseteq \pi_1^X[O]$, hence that π_1^X is semi-open.

Similarly, using semi-openness of $\pi_2^{Y'} \circ \phi' \times \phi'$: $A \to Y'$, one shows that π_2^X is semi-open. □

4.7. Lemma. *Let* ϕ: $\mathscr{X}^* \to \mathscr{Y}^*$ *be an* ESOM *extension, then* ϕ *is a* RIC *extension.*

Proof. Consider the shadow diagram EGS(ϕ) for ϕ.

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{\sigma} & \mathscr{X}^* \\ \phi' \downarrow & \searrow \psi & \downarrow \phi \\ \mathscr{Y}' & \xrightarrow[\tau]{} & \mathscr{Y}^* \end{array}$$

As $\phi \circ \sigma = \psi = \tau \circ \phi'$ is open and τ is proximal, $R_{\psi\tau}$ is ergodic ([Wo 82] VII.2.14). By 2.5, there is an ergodic subset $A \subseteq R_\psi$ such that $1 \times \phi'$: $A \to R_{\psi\tau}$ is a semi-open surjection. As $\phi' \times 1$: $R_{\psi\tau} \to R_\tau$ is an open map, $\phi' \times \phi'$: $A \to R_\tau$ is semi-open. So, by 4.6, $\sigma \times \sigma[A]$ has semi-open projections onto $\mathscr{X}^*$. Since $\sigma \times \sigma[A]$ is ergodic and ϕ is an ESOM extension, it follows that $\sigma \times \sigma[A]$ is minimal, say $\sigma \times \sigma[A] = \overline{T(x_0, \alpha x_0)}$ for some $x_0 \in u\mathscr{X}^*$ and $\alpha \in F$. Here $F = \mathfrak{G}(\mathscr{Y}^*, \phi(x_0))$ is the Ellis group of $\mathscr{Y}^*$ with respect to $\phi(x_0)$. Let $H = \mathfrak{G}(\mathscr{X}^*, x_0)$ and let $K = \bigcap\{fHf^{-1} \mid f \in F\}$ be the largest normal subgroup of F which is contained in H.

We shall now construct an ergodic subset $\tilde{A}$ of R_ψ such that $1 \times \phi'$: $\tilde{A} \to R_{\psi\tau}$ again is semi-open and, what is even more important, such that $\Delta_X \subseteq \sigma \times \sigma[\tilde{A}]$. As, by 4.6, the projections of $\sigma \times \sigma[\tilde{A}]$ are semi-open, it follows by the ESOM property for ϕ that $\Delta_X = \sigma \times \sigma[\tilde{A}]$.

Consider $\mathfrak{A}(K)$, the universal proximal extension of ttgs with Ellis group K and define κ: $\mathfrak{A}(K) \to X'$ by $\kappa(p \circ K) = (px_0, p \circ u\phi^{\leftarrow}\phi(x_0))$. Then, by 2.5, there is an ergodic subset B of $R_{\psi \circ \kappa}$ such that $\kappa \times \kappa$: $B \to A$ is a semi-open surjection. Define the endomorphism $\rho_{\alpha^{-1}}$: $\mathfrak{A}(K) \to \mathfrak{A}(K)$ by $p \circ K \mapsto p\alpha^{-1} \circ K$. As $K \triangleleft F$ and $\alpha^{-1} \in F$ it follows that $\rho_{\alpha^{-1}}$ is a well defined isomorphism. Then define $\tilde{B} \subseteq R_{\psi \circ \kappa}$ by $\tilde{B} = 1 \times \rho_{\alpha^{-1}}[B]$. As

$\tilde{B}$ and B are isomorphic, $\tilde{B}$ is ergodic. Let $\tilde{A} \subseteq R_{\psi}$ be defined as $\tilde{A} = \kappa \times \kappa[\tilde{B}]$. Then $\tilde{A}$ is ergodic. As $\alpha \in F$, $p\alpha^{-1} \circ u\phi^{\leftarrow}\phi(x_0) = p \circ u\alpha^{-1}\phi^{\leftarrow}\phi(x_0) = p \circ u\phi^{\leftarrow}\phi(x_0)$, so the difference between A and $\tilde{A}$ is only in the "third" coordinate which is filtered out after the "second" projection ϕ'. Hence $1 \times \phi'[\tilde{A}] = R_{\psi\tau}$.

Let $O \subseteq \tilde{A}$ be relatively open. Then by continuity of $\kappa \times \kappa|_B$, by isomorphy of $1 \times \rho_{\alpha^{-1}}$ and by semi-openness of $\kappa \times \kappa|_B$ it follows that there exists an open $O' \subseteq A$ with $1 \times \phi'[O'] \subseteq 1 \times \phi'[O]$. As $1 \times \phi'|_A$ is semi-open, also $1 \times \phi'|_{\tilde{A}}$ is semi-open. Then, by 4.6, $\sigma \times \sigma[\tilde{A}]$ has semi-open projections, so $\sigma \times \sigma[\tilde{A}]$ is minimal. Since $(x_0, \alpha x_0) \in \sigma \times \sigma[A]$, $(x_0, x_0) = (x_0, \alpha^{-1}\alpha x_0) \in \sigma \times \sigma[\tilde{A}]$. So $\sigma \times \sigma[\tilde{A}] = \Delta_X$. Note that this implies that an element $a \in \tilde{A}$ has the form

$$a = ((x, v \circ u\phi^{\leftarrow}\phi(x)), (x, w \circ u\phi^{\leftarrow}\phi(x))) \quad \text{with } v, w \in J_{\phi(x)}.$$

This enables us to show that $R_{\phi} = \overline{JR_{\phi}}$ and better, that ϕ is RIC.

Let $(x_1, x_2) \in R_{\phi}$ then there are $v, w \in J_{\phi(x_1)}$ such that $x_1 \in v \circ u\phi^{\leftarrow}\phi(x_1)$ and $x_2 \in w \circ u\phi^{\leftarrow}\phi(x_1)$. As

$$z = ((x_1, v \circ u\phi^{\leftarrow}\phi(x_1)), w \circ u\phi^{\leftarrow}\phi(x_1)) \in R_{\psi\tau},$$

there is an $a \in A$ with $1 \times \phi'(a) = z$. But $\sigma \times \sigma(a) = (x_1, x_1)$ so $x_1 \in w \circ u\phi^{\leftarrow}\phi(x_1)$ for

$$a = ((x_1, v \circ u\phi^{\leftarrow}\phi(x_1)), (x_1, w \circ u\phi^{\leftarrow}\phi(x_1)))$$

and $(x_1, w \circ u\phi^{\leftarrow}\phi(x_1)) \in X'$. Hence x_1 and x_2 are elements of $w \circ u\phi^{\leftarrow}\phi(x_1)$ and so $(x_1, x_2) \in \overline{JR_{\phi}}$.

Moreover, as x_1 and x_2 were chosen arbitrarily it follows that for every $x_1 \in X^*$, $x_1 \in \bigcap\{w \circ u\phi^{\leftarrow}\phi(x_1) | w \in J_{\phi(x_1)}\}$. But then, as is easily seen, $\phi^{\leftarrow}\phi(x_1) = w \circ u\phi^{\leftarrow}\phi(x_1)$ for every $w \in J_{\phi(x_1)}$ and for every $x_1 \in X^*$, which proves that ϕ is a RIC extension. □

Note how the lemma above resembles the characterization of HPI extensions in ([**AG 77**]) Thm. III.2).

4.8. THEOREM. *Let ϕ: $\mathscr{X} \to \mathscr{Y}$ be a homomorphism of minimal ttgs. Then the following statements are equivalent*:

(a) ϕ *is an* HPI *extension*;

(b) ϕ *is an* ESOM *extension*;

(c) ϕ *is an n*-ESOM *extension for every* $n \in \mathbf{N}$, $n \geq 2$.

Proof. (a) $\Rightarrow$ (c). Is just Theorem 4.5.

(c) $\Rightarrow$ (b). Is trivial.

(b) $\Rightarrow$ (a). Consider ϕ^* and note that ϕ^* is an HPI extension iff ϕ is an HPI extension. Let ψ be the maximal HPI factor under ϕ^*, say $\phi^* = \theta \circ \psi$ and remark that $\theta = \theta^*$, $\psi = \psi^*$ and θ^* does not admit nontrivial almost periodic factors (maximality of ψ^*). By 4.2(b), θ^* is an ESOM extension and, by 4.7, θ^* is a RIC extension. So, by [V 77] 2.6.3, θ^* is weakly mixing, i.e., R_{θ^*} is ergodic. As θ^* is open, the projections from R_{θ^*} onto $\mathscr{X}^*$ are semi-open. Hence R_{θ^*} is minimal and θ^* is an isomorphism, which proves that ϕ^* is an HPI extension and so ϕ is an HPI extension. $\square$

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