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REGULAR OPERATOR APPROXIMATION THEORY

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Regular operator approximation theory applies to finite difference approximations for differential equations and numerical integration approximations for integral equations. New relationships and efficient derivations of known results are presented. The analysis is based on the systematic use of convergence and compactness properties of sequences of sets. Since the purpose is theoretical, applications are merely indicated and references are cited.

1. Introduction and Summary. Regular operator approximation theory applies to numerical solutions of differential and integral equations. Some pertinent references are Anselone and Ansorge [**3, 4**], Chatelin [**6**], Grigorieff [**7, 8**], Stummel [**10**], and Vainikko [**12**].

We focus here on linear equations in a Banach space setting. Approximations are defined in the same setting. This case serves to motivate ideas and results for the more complicated situation with approximations defined in different spaces related by connection maps, such as restriction and interpolation. We present particularly efficient and revealing derivations of basic convergence results and we extend the theory in several significant respects. The analysis is based on systematic use of convergence and compactness concepts for sequences of sets. Some of these ideas were exploited in nonlinear operator approximation theory in [**3**]. These concepts should be useful also in general approximation theory.

Let X and Y be Banach spaces. A sequence of elements in X or Y is discretely compact (d-compact for short) if every subsequence has a convergent subsequence.

Let $A, A_n \in L(X, Y)$ the space of bounded linear operators from X to Y. We shall compare equations

$$Ax = y, \qquad A_n x_n = y,$$

where $A_n \to A$ pointwise and $\{A_n\}$ is asymptotically regular, i.e., if $\{x_n\}$ is bounded and $\{A_n x_n\}$ is d-compact, then $\{x_n\}$ is d-compact. Results concern inverse operators, null spaces, and ranges. There are implications for eigenvalues and eigenvectors.

Sharper results can be given for equations

$$(I - K)x = y, \qquad (I - K_n)x_n = y,$$

where K, $K_n \in L(X)$ with $K_n \to K$ and either $\{K_n\}$ is collectively compact, i.e., $\overline{\bigcup K_n S}$ is compact for any bounded set $S \subset X$, or $\{K_n\}$ is asymptotically compact, i.e., $\{K_n x_n\}$ is d-compact for any bounded sequence $\{x_n\}$ in X. Results for the collectively compact case presented in [**1**] will be extended to the more general asymptotically compact case in a forthcoming sequel.

For expository continuity, the paper is largely self-contained.

2. Convergence and compactness concepts. This material is valid also in a metric space. For convenience, we stay in a Banach space X. Notation:

$$x, x_n \in X, \qquad S, S_n \subset X, \qquad n \in N = \{1, 2, \dots\}.$$

We introduce ε-neighborhoods of sets,

$$\Omega_\varepsilon(S) = \bigcup_{x \in S} \{y \in X : \|x - y\| < \varepsilon\}, \qquad \varepsilon > 0.$$

By analogy with point convergence, $x_n \to x$ as $n \to \infty$, set convergence is defined by

$$S_n \to S \quad \text{if } S_n \subset \Omega_\varepsilon(S) \ \forall\, n \text{ large } \forall\, \varepsilon > 0,$$

i.e., for $n \geq n_\varepsilon$ with some n_ε. Such set limits are not unique:

$$S_n \to S \subset S' \Rightarrow S_n \to S'. \tag{2.1}$$

The void set $\varnothing$ plays a special role: $\Omega_\varepsilon(\varnothing) = \varnothing$ and

$$S_n \to \varnothing \Rightarrow S_n = \varnothing \quad \forall\, n \text{ large}, \tag{2.2}$$

$$S_n = \varnothing \ \forall\, n \text{ large} \Rightarrow S_n \to S \ \forall\, S \subset X. \tag{2.3}$$

For S compact,

$$S_n \to S \Leftrightarrow \forall \text{ open } \Omega \supset S, \Omega \supset S_n \quad \forall\, n \text{ large}, \tag{2.4}$$

i.e., for $n \geq n_\Omega$ with some n_Ω.

Denote infinite subsequences of N by N', N'', etc. Sets of cluster points (limits of subsequences) of $\{x_n\}$ and $\{S_n\}$ are:

$$\{x_n\}^* = \{x \in X : x_n \to x, n \in N'\},$$

$$\{S_n\}^* = \{x \in X : x_n \to x, x_n \in S_n, n \in N'\}.$$

All such cluster point sets are closed. A sequence $\{x_n\}$ is *discretely compact* (d-compact) if every subsequence has a convergent subsequence, hence a cluster point. Similarly, $\{S_n\}$ is d-compact if every sequence $\{x_n : n \in N'\}$ with $x_n \in S_n$ has a convergent subsequence.

Elementary properties include
(2.5) $\{S_n\}$ d-compact, $\{S_n\}^* = \varnothing \Rightarrow S_n = \varnothing\ \forall\, n$ large,
(2.6) $\{\overline{S_n}\}^* = \{S_n\}^*$,
(2.7) $\{\overline{S_n}\}$ d-compact $\Leftrightarrow$ $\{S_n\}$ d-compact,
(2.8) $\bar{S}_n \to S \Leftrightarrow S_n \to S$,
(2.9) $\overline{\bigcup S_n} = (\bigcup \bar{S}_n) \cup \{S_n\}^*$,
(2.10) $\overline{\bigcup S_n}$ compact $\Leftrightarrow$ $\bar{S}_n$ compact $\forall\, n$, $\{S_n\}$ d-compact.

THEOREM 2.1.
(a) $\{S_n\}$ *d-compact* $\Leftrightarrow$ $\{S_n\}^*$ *compact*, $S_n \to \{S_n\}^*$.
(b) $\{S_n\}$ *d-compact*, $\{S_n\}^* \subset S \Rightarrow S_n \to S$.

Proof. (a) $\Rightarrow$ (b). Prove (a).

1. Assume $\{S_n\}$ d-compact. Let $x_i \in \{S_n\}^*$ for $i \in N$. Then $\exists\, n_i < n_{i+1}$ and $x_{n_i} \in S_{n_i}$ such that $\|x_{n_i} - x_i\| \to 0$. Since $\{S_n\}$ is d-compact, $\exists\, N'$ and $x \in X$ such that $x_{n_i} \to x$ with $i \in N'$. Hence, $x_i \to x$ with $i \in N'$ and $\{S_n\}^*$ is compact.

2. Assume $S_n \to \{S_n\}^*$ fails. Then $\exists\, \varepsilon > 0$, N' and $x_n \in S_n$ for $n \in N'$ such that $\|x_n - x\| > \varepsilon$ for all $n \in N'$ and $x \in \{S_n\}^*$. Hence, $\{x_n\colon n \in N'\}^* = \varnothing$ and $\{S_n\}$ is not d-compact. Thus, $\{S_n\}$ d-compact $\Rightarrow S_n \to \{S_n\}^*$.

3. Assume $\{S_n\}^*$ compact and $S_n \to \{S_n\}^*$. Let $x_n \in S_n$ for $n \in N'$. Since $S_n \to \{S_n\}^*$, $\exists\, N'' \subset N'$ and $x^n \in \{S_n\}^*$ for $n \in N''$ such that $\|x^n - x_n\| \to 0$ with $n \in N''$. Since $\{S_n\}^*$ is compact, $\exists\, N''' \subset N''$ and $x \in X$ such that $x^n \to x$ with $n \in N'''$. Then $x_n \to x$ with $n \in N'''$ and $\{S_n\}$ is d-compact.

For other relations involving convergence and d-compactness of sequences of sets, see [3].

Next we introduce analogues of "cover compactness" and total boundedness for sequences of sets. Let $\{S_n\}$ be *asymptotically compact* if for any open cover of $\overline{\bigcup S_n}$ $\exists\, n_0$ and a finite subcover of $\overline{\bigcup_{n \geq n_0} S_n}$. Let $\{S_n\}$ be *asymptotically totally bounded* if $\forall\, \varepsilon > 0\ \exists\, n_\varepsilon$ such that $\bigcup_{n \geq n_\varepsilon} S_n$ has a finite ε-net. The following theorem generalizes the equivalence of relative compactness, sequential compactness and total boundedness for sets in a complete metric space.

THEOREM 2.2.
(a) $\{S_n\}$ *d-compact* $\Leftrightarrow$ $\{S_n\}$ *asymptotically compact.*
(b) $\{S_n\}$ *d-compact* $\Leftrightarrow$ $\{S_n\}$ *asymptotically totally bounded.*

Proof.

1. Assume $\{S_n\}$ d-compact. Let $\overline{\bigcup S_n} \subset \bigcup_{\alpha \in A} O_\alpha$, O_α open. By (2.9) and Theorem 2.1, $\{S_n\}^* \subset \bigcup_{\alpha \in A} O_\alpha$ and $\{S_n\}^*$ is compact. So $\{S_n\}^* \subset \bigcup_{\alpha \in B} O_\alpha$ for some finite $B \subset A$. By (2.8) and Theorem 2.1, $\bar{S}_n \to \{S_n\}^*$. By (2.4) and (2.9), $\overline{\bigcup_{n \geq n_0} S_n} \subset \bigcup_{\alpha \in B} O_\alpha$ for some n_0. Thus, $\{S_n\}$ is asymptotically compact.

2. Assume $\{S_n\}$ is not d-compact. Then $\exists\, x_n \in S_n$, $n \in N'$, such that $\{x_n\}^* = \varnothing$ and, moreover, $x_m \neq x_n$ for $m \neq n$. Then $\bigcup\{x_n\}$ is closed and $\Omega = [\bigcup\{x_n\}]^c$ is open. For each $n \in N'$ $\exists\, \Omega_n$ open with $x_n \in \Omega_n$ and $x_m \notin \Omega_n$ for $m \neq n$. Now $\overline{\bigcup S_n} \subset (\bigcup \Omega_n) \cup \Omega = X$. There is no finite subcover of $\overline{\bigcup_{n \geq n_0} S_n}$ for any n_0. So $\{S_n\}$ is not asymptotically compact. This proves part (a) of the theorem.

3. Assume $\{S_n\}$ d-compact. By Theorem 2.1, $\{S_n\}^*$ is compact, hence totally bounded: $\forall\, \varepsilon > 0$ $\exists$ a finite set $C_\varepsilon \subset X$ such that $\{S_n\}^* \subset \Omega_\varepsilon(C_\varepsilon)$. By Theorem 2.1, $S_n \to \{S_n\}^*$ and $S_n \subset \Omega_\varepsilon(\{S_n\}^*) \subset \Omega_{2\varepsilon}(C_\varepsilon)$ for n large. So $\{S_n\}$ is asymptotically totally bounded.

4. Assume $\{S_n\}$ asymptotically totally bounded. For each $m = 1, 2, \ldots$, $\exists$ a finite set $C_m \subset X$ and $n_m \in N$ such that $\bigcup_{n \geq n_m} S_n \subset \Omega_{2^{-m}}(C_m)$. Let $x_n \in S_n$ for $n \in N'$. Then

$$\exists\, N_1 \subset N' \text{ and } y_1 \in C_1 \text{ such that } x_n \in \Omega_{2^{-1}}(y_1) \quad \forall\, n \in N_1,$$
$$\exists\, N_2 \subset N_1 \text{ and } y_2 \in C_2 \text{ such that } x_n \in \Omega_{2^{-2}}(y_2) \quad \forall\, n \in N_2,$$

and so on. Choose $n_1 \in N_1$ and $n_1 < n_2 \in N_2$, etc. Then $x_{n_i} \in \Omega_{2^{-i}}(y_i)$ with $y_i \in C_i$. Hence, $\|x_{n_i} - x_{n_j}\| < 1/2^{i-1}$ for $i < j$, so $\{x_{n_i}\}$ is Cauchy. Since X is complete, $x_{n_i} \to x$ for some $x \in X$. Therefore, $\{S_n\}$ is d-compact. This proves part (b) of the theorem.

3. Dimension and codimension. Let E, E_n and F be closed subspaces of X. Let

$$U = \{x \in X : \|x\| = 1\}, \tag{3.1}$$

the unit sphere in X. Recall that

$$\dim F < \infty \Leftrightarrow F \cap U \text{ compact}. \tag{3.2}$$

LEMMA 3.1. *Assume $\{E_n\}^* \subset E$ and either $\{E_n \cap U\}$ is d-compact or* $\dim F < \infty$. *Then*

$$E \cap F = \{0\} \Rightarrow E_n \cap F = \{0\} \quad \forall \textit{ large}.$$

Proof. In either case, $\{E_n \cap F \cap U\}$ is d-compact and

$$\{E_n \cap F \cap U\}^* \subset E \cap F \cap U = \varnothing.$$

By (2.5),

$$E_n \cap F \cap U = \varnothing \quad \forall\, n \text{ large.}$$

Hence,

$$E_n \cap F = \{0\} \quad \forall\, n \text{ large.}$$

THEOREM 3.2. *Assume* $\{E_n \cap U\}$ *d-compact*, $\{E_n\}^* \subset E$, *and* $\dim E < \infty$. *Then* $\dim E_n \leq \dim E\ \forall\, n$ *large*.

Proof. Since $\dim E < \infty$, there is a closed subspace F such that $X = E \oplus F$. Thus, $E \cap F = \{0\}$. By Lemma 3.1, $E_n \cap F = \{0\}\ \forall\, n$ large. Hence, $\dim E_n \leq \operatorname{codim} F = \dim E\ \forall\, n$ large.

THEOREM 3.3. *Let* $\{E_n\}^* \subset E$.

(a) $\operatorname{codim} E < \infty \Rightarrow \operatorname{codim} E_n \geq \operatorname{codim} E \quad \forall\, n$ *large*.

(b) $\operatorname{codim} E = \infty \Rightarrow \operatorname{codim} E_n \to \infty \quad$ *as* $n \to \infty$.

Proof. Let $\dim F < \infty$ and $E \cap F = \{0\}$. By Lemma 3.1, $E_n \cap F = \{0\}\ \forall\, n$ large. Therefore,

$$\operatorname{codim} E \geq m \Rightarrow \operatorname{codim} E_n \geq m \quad \forall\, n \text{ large.}$$

This implies both (a) and (b).

4. Operator fundamentals. We record for later use some basic facts about bounded linear operators. Let $N(A)$ and $R(A)$ denote the null space and range of $A \in L(X, Y)$. Recall $\|A\| = \sup\{\|Ax\|: x \in U\}$.

(4.1) $\exists A^{-1} \Leftrightarrow N(A) = \{0\} \Leftrightarrow 0 \notin AU$.

(4.2) $\exists A^{-1}$ bounded $\Leftrightarrow 0 \notin \overline{AU}$.

(4.3) AU closed, $\exists A^{-1} \Rightarrow A^{-1}$ bounded.

(4.4) Let $\exists A^{-1}$. Then A^{-1} bounded $\Leftrightarrow R(A)$ closed.

LEMMA 4.1. *Assume* $\dim N(A) < \infty$. *Then there is a closed subspace* $F \subset X$ *such that* $X = N(A) \oplus F$. *Let* $A_F = A|_F$. *Then* $\exists A_F^{-1}$, $R(A_F) = R(A)$, *and*

$$R(A) \text{ is closed} \Leftrightarrow A_F^{-1} \text{ is bounded.}$$

Let P *and* Q *be the complementary projections in* $L(X)$ *with ranges* $R(P) = N(A)$ *and* $R(Q) = F$. *Then* $P + Q = I$, $AP = 0$, *and* $A = AQ = A_F Q$.

The *index* of A is given by

$$\operatorname{ind}(A) = \dim N(A) - \operatorname{codim} R(A)$$

when the right member is well defined. Thus, $-\infty \leq \operatorname{ind}(A) \leq +\infty$.

LEMMA 4.2 [**9, 11**]. *Let* $A = A_1 + A_2$ *with* $R(A_1)$ *closed*, $\exists \operatorname{ind}(A_1)$, *and* $\dim R(A_2) < \infty$. *Then* $R(A)$ *is closed and* $\operatorname{ind}(A) = \operatorname{ind}(A_1)$.

5. Compact operators. Two equivalent definitions of a compact operator $K \in L(X)$ are given.

K compact:
S bounded $\Rightarrow \overline{KS}$ compact,
$\{x_n\}$ bounded $\Rightarrow \{Kx_n\}$ d-compact.

For example, if $\dim R(K) < \infty$ then K is compact. Some well-known properties of compact operators are

(5.1) $\dim N(I-K) < \infty$, $R(I-K)$ closed,

and the extended Fredholm alternative:

(5.2) $R(I-K) = X \Leftrightarrow \exists (I-K)^{-1} \Rightarrow (I-K)^{-1} \in L(X)$,

(5.3) $\operatorname{ind}(I-K) = 0$.

6. Regular operators. Three equivalent definitions of a regular operator $A \in L(X, Y)$ are given.

A regular:
S bounded, $\overline{AS}$ compact $\Rightarrow \bar{S}$ compact,
$\{x_n\}$ bounded, $\{Ax_n\}$ d-compact $\Rightarrow \{x_n\}$ d-compact,
$\{x_n\}$ bounded, $Ax_n \to y \Rightarrow \{x_n\}^* \neq \varnothing$.

Examples of regular operators are

(6.1) $\exists A^{-1}$ bounded $\Rightarrow A$ regular,

(6.2) $K \in L(X)$, K compact $\Rightarrow I - K$ regular.

From the definition of a regular operator, restrictions and products of regular operators are regular, and

(6.3) A regular, $\{x_n\}$ bounded, $Ax_n \to y \Rightarrow Ax = y \ \forall x \in \{x_n\}^*$,

(6.4) A regular, S closed bounded $\Rightarrow AS$ closed.

Thus, A regular $\Rightarrow AU$ closed. By (4.3), (4.4), and (6.1),

(6.5) whenever $\exists A^{-1}$,

$$A \text{ regular} \Leftrightarrow A^{-1} \text{ bounded} \Leftrightarrow R(A) \text{ closed}.$$

THEOREM 6.1 (Wolf [**13**]).

$$A \textit{ regular} \Leftrightarrow \dim N(A) < \infty,\ R(A) \textit{ closed}. \tag{6.6}$$

Proof. Let $S = U \cap N(A)$ in the definition of A regular. By (3.2), A regular $\Rightarrow$ dim $N(A) < \infty$. Now assume dim $N(A) < \infty$. Refer to Lemma 4.1. Since dim $R(P) < \infty$, P is compact. By (6.2), $Q = I - P$ is regular. Since $A = A_F Q$, and restrictions and products of regular operators are regular, A regular $\Leftrightarrow A_F$ regular. By (6.5), A_F regular $\Leftrightarrow R(A_F)$ closed. Since $R(A_F) = R(A)$,

$$A \text{ regular} \Leftrightarrow A_F \text{ regular} \Leftrightarrow R(A_F) \text{ closed} \Leftrightarrow R(A) \text{ closed},$$

when dim $N(A) < \infty$. The theorem follows.

In other terminology, A is regular iff A is a semi-Fredholm operator with $-\infty \leq \text{ind}(A) < \infty$.

REMARK. Regular operators are related to the notion of a proper mapping. Recall that a continuous map f between topological spaces is *proper* if $f^{-1}(S)$ is compact whenever S is compact. It is not difficult to show that a linear operator A is regular iff the restriction of A to each closed, bounded set is proper. It is from this viewpoint that the previous theorem can also be attributed to Yood [**14**].

7. Stable convergence. We shall need a few basic facts about operator convergence. Let $A, A_n \in L(X, Y)$. Denote pointwise convergence on X by $A_n \to A$. If $A_n \to A$ then $\{A_n\}$ is bounded uniformly and there is continuous convergence: $x_n \to x \Rightarrow A_n x_n \to Ax$. Moreover,

$$A_n \to A \Rightarrow \{N(A_n)\}^* \subset N(A), \qquad \{R(A_n)\}^* \supset R(A). \tag{7.1}$$

The special cases with $N(A) = \{0\}$ and $R(A) = Y$ are worth noting.

Define

$\{A_n\}$ *stable*:

$\exists A_n^{-1}$ bounded uniformly $\forall\, n$ large,

stable convergence $A_n \overset{s}{\to} A$:

$A_n \to A$, $\{A_n\}$ stable, $R(A_n) = Y \;\forall\, n$ large.

From (4.4)

$$\{A_n\} \text{ stable} \Rightarrow R(A_n) \text{ closed} \quad \forall\, n \text{ large}. \tag{7.2}$$

By analogy with (4.2),

$$\{A_n\} \text{ stable} \Leftrightarrow 0 \notin \{A_n U\}^*. \tag{7.3}$$

Since $A_n \to A$ implies $\overline{AU} \subset \{A_n U\}^*$,

$$A_n \overset{s}{\to} A \Rightarrow \exists A^{-1} \text{ bounded}, R(A) \text{ closed}. \tag{7.4}$$

Since $A_n^{-1} - A^{-1} = A_n^{-1}(A - A_n)A^{-1}$ when $A^{-1}, A_n^{-1} \in L(Y, X)$,

$$(7.5) \qquad \begin{aligned} &A_n \xrightarrow{s} A, R(A) = Y, Ax = y, A_n x_n = y \\ &\Rightarrow A_n^{-1} \to A^{-1}, \|x_n - x\| \leq \|A_n^{-1}\| \, \|A_n x - Ax\| \to 0. \end{aligned}$$

For this error bound to be practical, estimates of $\|A_n^{-1}\|$ and $\|A_n x - Ax\|$ are needed. Another limitation of (7.5) is the need to assume $R(A) = Y$. We have no Fredholm alternative in this generality.

8. Regular operator approximation. Let $A, A_n \in L(X, Y)$. Basic properties, the first in two equivalent forms, are

$\{A_n\}$ *asymptotically regular*:
$\{x_n\}$ bounded, $\{A_n x_n\}$ d-compact $\Rightarrow$ $\{x_n\}$ d-compact,
$\{x_n\}$ bounded, $A_n x_n \to y \Rightarrow \{x_n\}^* \neq \varnothing$,

regular convergence $A_n \xrightarrow{r} A$:

$A_n \to A$, $\{A_n\}$ asymptotically regular.

Examples of regular convergence will be given later. Straightforward reasoning yields

$$(8.1) \qquad A_n \xrightarrow{r} A \Rightarrow A \text{ regular},$$

$$(8.2) \qquad \begin{aligned} &A_n \xrightarrow{r} A, \{x_n\} \text{ bounded}, A_n x_n \to y \\ &\Rightarrow y \in R(A), Ax = y \quad \forall x \in \{x_n\}^*. \end{aligned}$$

$$(8.3) \qquad A_n \xrightarrow{r} A, S \text{ closed bounded} \Rightarrow \{A_n S\}^* = AS.$$

Thus, $A_n \xrightarrow{r} A \Rightarrow \{A_n U\}^* = AU$. By (4.1) and (7.3),

$$(8.4) \qquad \text{when } A_n \xrightarrow{r} A: \exists A^{-1} \Leftrightarrow \{A_n\} \text{ stable}.$$

THEOREM 8.1 (cf. Vainikko [**12**]).

$$\begin{aligned} &A_n \xrightarrow{r} A, \exists A^{-1}, R(A_n) = Y \quad \forall n \text{ large} \\ &\Rightarrow A_n \xrightarrow{s} A, R(A) = Y, \quad \text{and} \quad A_n^{-1} \to A^{-1}. \end{aligned}$$

Proof. By (8.4), $A_n \xrightarrow{s} A$. To show $R(A) = Y$, let $y \in Y$. Let $x_n = A_n^{-1} y$ $\forall n$ large. Since $\{A_n\}$ is stable, $\{x_n\}$ is bounded. By (8.2), $y \in R(A)$ and $R(A) = Y$. By (7.5), $A_n^{-1} \to A^{-1}$.

The error bound from (7.5) is theoretical, since regular convergence does not yield an estimate for $\|A_n^{-1}\|$. Another limitation of Theorem 8.1 is the hypothesis that $R(A_n) = Y \; \forall \; n$ large.

Next, null spaces and ranges of A_n and A are compared with the aid of the set convergence introduced in §2. For other derivations of some of the results, see Grigorieff [7, **8**] and Vainikko [**12**].

THEOREM 8.2. *Let* $A_n \xrightarrow{\mathrm{r}} A$.

(a) $N(A_n) \cap U \to N(A) \cap U$.

(b) $\dim N(A_n) \leq \dim N(A) \; \forall \; n$ *large*.

Proof. By (7.1), $\{N(A_n) \cap U\}^* \subset N(A) \cap U$. Since $\{A_n\}$ is asymptotically regular, $\{N(A_n) \cap U\}$ is d-compact. Theorem 2.1 yields (a). Theorems 3.2 and 6.1 imply (b).

THEOREM 8.3. *Let* $A_n \xrightarrow{\mathrm{r}} A$ *and* $\exists \, A^{-1}$.

(a) $\{R(A_n)\}^* = R(A)$.

(b) $\dim F < \infty$, $R(A) \cap F = \{0\} \Rightarrow R(A_n) \cap F = \{0\} \; \forall \; n$ *large*.

(c) codim $R(A_n) \geq$ codim $R(A) \; \forall \; n$ *large if* codim $R(A) < \infty$.

(d) codim $R(A_n) \to \infty$ *if* codim $R(A) = \infty$.

Proof. By (7.1), $R(A) \subset \{R(A_n)\}^*$. From (8.4) and (8.2), $\{R(A_n)\}^* \subset R(A)$. Thus, (a) holds. Lemma 3.1 gives (b).

THEOREM 8.4. *Let* $A_n \xrightarrow{\mathrm{r}} A$.

(a) $R(A_n)$ *is closed* $\forall \; n$ *large*.

(b) $\operatorname{ind}(A_n) \leq \operatorname{ind}(A) \; \forall \; n$ *large if* $\operatorname{ind}(A) \geq -\infty$,

(c) $\operatorname{ind}(A_n) \to -\infty$ *if* $\operatorname{ind}(A) = -\infty$.

Proof. By (8.1) and (6.6), A is regular and $\dim N(A) < \infty$. Refer to Lemma 4.1. Now

$$A_n = A_n P + A_n Q,$$
$$A_n P \to AP = 0, \quad \dim R(P) < \infty \Rightarrow \|A_n P\| \to 0.$$

It follows easily that $A_n Q \xrightarrow{\mathrm{r}} A$. By Theorem 8.2,

$$\dim N(A_n Q) \leq \dim N(A) \quad \forall \; n \text{ large}.$$

The restrictions of A_n and A to F satisfy

$$A_{nF} \xrightarrow{\mathrm{r}} A_F, \quad \exists \, A_F^{-1}.$$

By (8.4), (7.2) and $R(A_{nF}) = R(A_n Q)$,

$$R(A_n Q) \text{ is closed} \quad \forall \; n \text{ large}.$$

Now Theorem 8.3, $R(A_{nF}) = R(A_nQ)$ and $R(A_F) = R(A)$ imply

$$\operatorname{codim} R(A_nQ) \geq \operatorname{codim} R(A) \ \forall\, n \text{ large} \quad \text{if } \operatorname{codim} R(A) < \infty,$$
$$\operatorname{codim} R(A_nQ) \to \infty \quad \text{if } \operatorname{codim} R(A) = \infty.$$

So Theorem 8.4 holds for $A_nQ \xrightarrow{r} A$. Lemma 4.2 with $A_n = A_nP + A_nQ$ implies $R(A_n)$ is closed and $\operatorname{ind}(A_n) = \operatorname{ind}(A_nQ)$. The theorem follows. By Theorems 6.1, 8.2, and 8.4:

THEOREM 8.5. *$A_n \xrightarrow{r} A \Rightarrow A_n$ regular $\forall$ n large.*

Regular convergence applies to finite difference approximations for differential equations. See [**3**], [**12**]. It also applies to numerical integration approximations of integral equations. However, in the latter situation, sharper results can be obtained from special cases of regular convergence involving compact operators. This is the next topic.

9. Compact operator approximation. Let $K, K_n \in L(X)$ for $n \in N$. Key definitions, each expressed in two equivalent forms, are

$\{K_n\}$ *collectively compact*:
S bounded $\Rightarrow \overline{\bigcup K_nS}$ compact,
$\{x_n\}$ bounded $\Rightarrow \{K_{m_n}x_n\}$ d-compact $\forall \{m_n\}$,

$\{K_n\}$ *asymptotically compact*:
S bounded $\Rightarrow \{K_nS\}$ d-compact,
$\{x_n\}$ bounded $\Rightarrow \{K_nx_n\}$ d-compact.

It suffices if S is the unit ball in these definitions. Define

collectively compact convergence $K_n \xrightarrow{cc} K$:
$K_n \to K$, $\{K_n\}$ collectively compact,

asymptotically compact convergence $K_n \xrightarrow{ac} K$:
$K_n \to K$, $\{K_n\}$ asymptotically compact.

By (2.9) and (2.10),

(9.1) $\{K_n\}$ collectively compact

$$\Leftrightarrow \{K_n\} \text{ asymptotically compact, each } K_n \text{ compact},$$

(9.2) $$K_n \xrightarrow{cc} K \Leftrightarrow K_n \xrightarrow{ac} K, \quad \text{each } K_n \text{ compact},$$

(9.3) $$K_n \overset{\mathrm{cc}}{\to} K \Rightarrow K_n \overset{\mathrm{ac}}{\to} K \Rightarrow K \text{ compact}.$$

The difference between $K_n \overset{\mathrm{cc}}{\to} K$ and $K_n \overset{\mathrm{ac}}{\to} K$ is illustrated by

(9.4) $$\|L_n\| \to 0 \Rightarrow L_n \overset{\mathrm{ac}}{\to} 0,$$

(9.5) $$K_n \overset{\mathrm{cc}}{\to} K, \quad \|L_n\| \to 0 \Rightarrow K_n + L_n \overset{\mathrm{ac}}{\to} K,$$

but $L_n \overset{\mathrm{cc}}{\to} 0$ and $K_n + L_n \overset{\mathrm{cc}}{\to} K$ only when every L_n is compact. They are not compact if $L_n = c_n I$, $c_n \downarrow 0$, and $\dim X = \infty$.

Consider equations

(9.6) $$(I - K)x = y, \qquad (I - K_n)x_n = y.$$

Examples with $K_n \overset{\mathrm{cc}}{\to} K$ are integral equations and numerical integration approximations. See [**1**], [**5**]. An example with $K_n \overset{\mathrm{ac}}{\to} K$ is a weakly singular integral equation and numerical approximations based on the singularity subtraction technique of Kantorovich and Krylov. The situation in (9.5) arises. See [**2**].

By an easy argument,

(9.7) $$K_n \overset{\mathrm{cc}}{\to} K \Rightarrow K_n \overset{\mathrm{ac}}{\to} K \Rightarrow I - K_n \overset{\mathrm{r}}{\to} I - K.$$

So Theorem 8.1 applies to solutions of $(I - K)x = y$ and $(I - K_n)x_n = y$.

However, in the present situation, sharper results are available.

THEOREM 9.1. *Let* $K_n \overset{\mathrm{cc}}{\to} K$ *or* $K_n \overset{\mathrm{ac}}{\to} K$.

(a) (*Fredholm alternative*) $\forall$ *n large*,

$$R(I - K_n) = X \Leftrightarrow \exists (I - K_n)^{-1}$$
$$\Rightarrow (I - K_n)^{-1} \in L(X), \dim N(I - K_n) = \operatorname{codim} R(I - K_n) < \infty.$$

(b) $\|(K_n - K)K\| \to 0$, $\|(K_n - K)K_n\| \to 0$ *as* $n \to \infty$.

(c) $\exists (I - K)^{-1} \Leftrightarrow \exists (I - K_n)^{-1}$ *uniformly bounded* $\forall$ *n large*, *in which case* $(I - K_n)^{-1} \to (I - K)^{-1}$ *on X and there are practical error bounds.*

Theorem 9.1 is presented under the collectively compact hypothesis in [**1**]. While many of the proofs from the classical compact operator theory [**11**] apply under the weaker hypothesis, a different approach will be taken in the sequel which will handle both the collectively and asymptotically compact cases with equal efficiency. The regular convergence in (9.7) will prove useful in this regard.

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