

ON WEAK EPIMORPHISMS IN HOMOTOPY THEORY

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Weakened versions of the categorical notions of epimorphism and monomorphism have proved to be of some interest in pointed homotopy theory. A weak epimorphism, for instance, is a morphism e (in any category with 0 objects) such that $g \circ e = 0$ implies $g = 0$.

In 1967, Ganea utilized extensive homotopy-theoretic calculations to exhibit examples, in the pointed homotopy category, of weak monomorphisms which are not monomorphisms. In this note, we exploit the properties of a remarkable group discovered by Higman in 1951 to exhibit examples, again in the pointed homotopy category, of weak epimorphisms which are not epimorphisms, thereby confirming a suspicion enunciated by Hilton in the early 1960's.

1. A *weak epimorphism* in a category with 0 objects is a morphism $X \xrightarrow{e} Y$ satisfying *weak right cancellation*: if $Y \xrightarrow{g} Z$ is a morphism such that $g \circ e = 0$, then $g = 0$. The notion of weak epimorphism, as well as the dual notion of weak monomorphism (a morphism satisfying weak left cancellation), arises rather naturally in the pointed homotopy category \mathcal{H} of topological spaces (compare [R]; in [R], the objects in \mathcal{H} are taken to be path-connected CW-spaces, but this restriction is unnecessary here) and our purpose here is to compare this notion with the more traditional notion of homotopy-epimorphism (= epimorphism in \mathcal{H}).

A study of the comparison between weak monomorphisms and monomorphisms in homotopy theory was carried out by Ganea [G] who, in particular, answered a number of questions raised in Hilton's notes [H2]. An additional problem hinted at in [H2; p. 180] is settled here; we show that weak epimorphisms in \mathcal{H} are, in general, genuinely weaker than epimorphisms in \mathcal{H} .

Before proceeding with the details, a few remarks linking the notions of (weak) epimorphism and (weak) monomorphism in \mathcal{H} may be in order. By definition, a (weak) monomorphism $X \rightarrow Y$ induces a (weak) monomorphism of pointed morphism sets $[W, X] \rightarrow [W, Y]$ for each W in \mathcal{H} . Similarly, a (weak) epimorphism $X \rightarrow Y$ induces a (weak) monomorphism of pointed morphism sets $[Y, Z] \rightarrow [X, Z]$ for each Z in \mathcal{H} . Sharper statements are provided by the following.

PROPOSITION 1.1. (i) *A (weak) monomorphism $X \rightarrow Y$ induces a (weak) monomorphism $X^W \rightarrow Y^W$ of pointed mapping spaces for each W in \mathcal{H} .*

(ii) *A (weak) epimorphism $X \rightarrow Y$ induces a (weak) monomorphism $Z^Y \rightarrow Z^X$ of pointed mapping spaces for each Z in \mathcal{H} .*

Proposition 1.1 (ii) provides a method for creating (weak) monomorphisms out of (weak) epimorphisms. However, this half of Proposition 1.1 does not dualize and we merely have

PROPOSITION 1.1'. *A (weak) epimorphism $X \rightarrow Y$ induces a (weak) epimorphism $X \wedge W \rightarrow Y \wedge W$ of smash products.*

It thus appears unlikely that Ganea's examples of weak monomorphisms which are not monomorphisms can be somehow transformed into examples of weak epimorphisms which are not epimorphisms (all in \mathcal{H}). In any event, Ganea's examples and the examples presented in §2 and §3 below utilize radically different techniques. Ganea requires very detailed knowledge of the homotopy groups of spheres to establish that his examples, though clearly weak monomorphisms, are not monomorphisms. On the other hand, our examples are based on the properties of a certain group introduced by Higman [H1] and further studied by Dyer and Vasquez [DV].

2. In most categories with 0 objects (for example, the category of pointed sets, or the category $\mathcal{H}[\mathbf{G}]$) the notions of weak monomorphism and monomorphism differ, but in the category \mathcal{G} of groups, they clearly coincide. However, the notions of weak epimorphism and epimorphism do *not* coincide in \mathcal{G} . In fact, we have the following intrinsic characterization of weak epimorphisms in \mathcal{G} ([R; §3]).

PROPOSITION 2.1. *A group homomorphism $\phi: G_1 \rightarrow G_2$ is a weak epimorphism if and only if the normal closure of ϕG_1 in G_2 is all of G_2 .*

It is also possible to intrinsically characterize weak epimorphisms and weak monomorphisms in \mathcal{H} .

PROPOSITION 2.2. (i) *$X \xrightarrow{f} Y$ is a weak epimorphism in \mathcal{H} if and only if Y is contractible in the homotopy-cofiber of f .*

(ii) *$X \xrightarrow{f} Y$ is a weak monomorphism in \mathcal{H} if and only if the homotopy-fiber of f is contractible in X .*

Now any group homomorphism $\phi: G_1 \rightarrow G_2$ gives rise to a unique morphism $f: K(G_1, 1) \rightarrow K(G_2, 1)$ in \mathcal{H} , where $K(G_i, 1)$ is an Eilenberg-MacLane complex and where $f_*: \pi_1 K(G_1, 1) \rightarrow \pi_1 K(G_2, 1)$ coincides with ϕ . But knowing that ϕ is a (weak) epimorphism in \mathcal{G} does not allow us to infer that f is a (weak) epimorphism in \mathcal{H} . Nevertheless, if G_2 is chosen carefully, we may in fact make this latter inference and thus arrive at a suitable example.

To this end, let H be the Higman group [H1] with presentation.

$$H = \langle a, b, c, d; bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2 \rangle$$

According to [H1], if N is a maximal normal subgroup of H , $N \neq H$, the quotient group H/N is an infinite simple group. Now let x be an element of H not in N and take H_0 to be the subgroup of H generated by N and x . Further, let $\phi: H_0 \hookrightarrow H$ be the inclusion homomorphism and $f: K(H_0, 1) \rightarrow K(H, 1)$ the induced morphism of Eilenberg-MacLane complexes.

THEOREM 2.1. *$f: K(H_0, 1) \rightarrow K(H, 1)$ is a weak epimorphism but not an epimorphism in \mathcal{H} .*

Proof. As H_0/N is a cyclic subgroup of H/N (generated by the coset xN) and as H/N is infinite simple, it follows that H_0/N is a proper subgroup of H/N . Thus $\phi: H_0 \hookrightarrow H$ is not an epimorphism in \mathcal{G} and so ([HR; Prop. 1]) $f: K(H_0, 1) \rightarrow K(H, 1)$ is not an epimorphism in \mathcal{H} . By the maximality of N and Proposition 2.1, ϕ is a weak epimorphism in \mathcal{G} . To see that f is a weak epimorphism in \mathcal{H} , let $g: K(H, 1) \rightarrow Z$ be a morphism with $g \circ f = 0$. As $g_* \circ f_* = 0: \pi_1 K(H_0, 1) \rightarrow \pi_1 Z$ and as $f_* = \phi$ is a weak epimorphism in \mathcal{G} , we see that $g_* = 0$, so that g lifts (uniquely) to $\tilde{g}: K(H, 1) \rightarrow \tilde{Z}$, \tilde{Z} the universal cover of Z . (Since $K(H, 1)$ is a CW-space, Z may be assumed, without loss of generality, to admit a universal cover. Alternatively, using Proposition 2.2 (i), g may be taken to be the canonical map from $K(H, 1)$ to the homotopy-cofiber of f and the latter is 1-connected by van Kampen's theorem.) But the k -dimensional obstruction to $\tilde{g} = 0$ lies in the (constant coefficient) cohomology group $H^k(K(H, 1); \pi_k \tilde{Z})$, $k \geq 2$. As H is acyclic and geometrically finite ([DV; §4]), these obstructions vanish, $\tilde{g} = 0$, and hence $g = 0$.

REMARK. While the model of $K(H, 1)$ constructed in [DV] is a finite 2-dimensional complex, we do not know whether $K(H_0, 1)$ has the homotopy type of a finite complex. Of course, since it may be viewed as a covering space of $K(H, 1)$, $K(H_0, 1)$ certainly has the homotopy type of a 2-dimensional complex.

3. The example given in this section requires a more elaborate argument than that of §2 but it yields a somewhat stronger result. To explain, we recall that a morphism $X \xrightarrow{f} Y$ in \mathcal{H} gives rise to a ‘cofibration sequence’

$$X \xrightarrow{f} Y \xrightarrow{q} C \xrightarrow{s} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \rightarrow \dots$$

Here, q is the canonical morphism from Y to the homotopy-cofiber of f , s is the ‘connecting’ morphism from C to the reduced suspension of X , and so on.

Theorem 2.1 shows that f need not be an epimorphism even if $q = 0$. Theorem 3.1 below will show that q need not be an epimorphism even if $s = 0$; in other words, even if a morphism, when viewed as a cofibration, is *co-induced*, it may fail to be an epimorphism while being a weak epimorphism. Since Hilton [H2; p. 180] has shown that s is an epimorphism if and only if $\Sigma f = 0$, it follows that Theorem 3.1 provides an example which is, in some sense, optimal.

We return now to the space $K(H, 1)$ studied in §2. If W is the 4-fold wedge of circles $S^1 \vee S^1 \vee S^1 \vee S^1$, there is a morphism $W \xrightarrow{f} W$ derived from the relators in the presentation of H given in §2 and according to [DV; §4] the homotopy-cofiber of f is a $K(H, 1)$. Thus we have a cofibration sequence

$$W \xrightarrow{f} W \xrightarrow{q} K(H, 1) \xrightarrow{s} \Sigma W \xrightarrow{\Sigma f} \Sigma W \rightarrow \dots$$

THEOREM 3.1. $q: W \rightarrow K(H, 1)$ is a weak epimorphism but not an epimorphism in \mathcal{H} .

Proof. The proof that q is a weak epimorphism is precisely the same as in Theorem 2.1. However, as $q_*: \pi_1 W \rightarrow \pi_1 K(H, 1)$, it is not so simple to deduce that q is not an epimorphism.

For that purpose, recall ([DV; §4]) that there exists an H -module M such that the (twisted) cohomology group $H^2(K(H, 1); M)$ is non-0. The homomorphism $H \rightarrow \text{Aut } M$ describing this H -module structure gives rise to a sectioned fibration $E \rightarrow K(H, 1)$ with fiber the Eilenberg-MacLane space $K(M, 2)$ and a twisted version of the usual representation theorem ([S]) identifies $H^2(K(H, 1); M)$ with the homotopy classes (= vertical homotopy classes) of sections of this fibration. Let then $u, v \in H^2(K(H, 1); M)$, $u \neq v$ and view u, v as morphisms from $K(H, 1)$ to

E. Pulling back via q yields a morphism of fibrations

$$\begin{array}{ccc} E^* & \rightarrow & E \\ \downarrow & & \downarrow \\ W & \xrightarrow{q} & K(H, 1) \end{array}$$

and u, v pull back to sections u^*, v^* of the induced fibration $E^* \rightarrow W$. But $u^* = v^*$ since each may be viewed as an element of $H^2(W; q^*M) = 0$; here q^*M denotes the module over the free group on 4 generators associated to the homomorphism

$$\pi_1 W \xrightarrow{q^*} \pi_1 K(H, 1) = H \xrightarrow{\alpha} \text{Aut } M.$$

It follows then that $u \circ q = v \circ q$, hence that q is not an epimorphism.

REMARK. Since $H^2(K(H, 1); M) = 0$ for any H -module M which is *finitely generated* as an abelian group ([DV; §4]), obstruction theory shows that q is an epimorphism in the full subcategory of \mathcal{H} whose objects are pointed spaces with finitely generated second homotopy groups. In contrast, the morphism $f: K(H_0, 1) \rightarrow K(H, 1)$ of §2 is plainly not an epimorphism in this subcategory nor, indeed, in any full subcategory of \mathcal{H} containing the Eilenberg-MacLane spaces $K(G, 1)$, G a finitely generated group.

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