PACIFIC JOURNAL OF MATHEMATICS
Vol. 122, No. 1, 1986

ESTIMATES FOR PARTIAL SUMS OF CONTINUED FRACTION PARTIAL QUOTIENTS

HAROLD G. DIAMOND AND JEFFREY D. VAALER

Metric-type estimates are given for a class of partial sums involving continued fraction partial quotients. These results extend a well known theorem of Khinchin and yield an almost-everywhere estimate for the quantity in the title.

1. Introduction. For α an irrational number in $(0,1)$ let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = \langle 0, a_1, a_2, \dots \rangle$$

be the representation of α as a regular continued fraction ([**4**, Ch. X], [**5**]). The numbers $a_n = a_n(\alpha)$ are called the *partial quotients* of α.

A well-known theorem of Khinchin [**5**], [**6**] in the metric theory of continued fractions asserts that if F is an arithmetic function satisfying $F(r) \ll r^{1/2-\delta}$ for some $\delta > 0$ and if $S_N(F, \alpha) := F(a_1(\alpha)) + \cdots + F(a_N(\alpha))$ for each positive integer N, then

$$(1) \qquad \lim_{N\to\infty} \frac{1}{N} S_N(F,\alpha) = \frac{1}{\log 2} \sum_{r=1}^{\infty} F(r) \log\left\{1 + \frac{1}{r(r+2)}\right\}$$

holds for almost all α in $(0,1)$. This result has been extended by others ([**2**, §4], [**7**, Theorem 4]). In particular, we note that the Birkhoff Ergodic Theorem implies that (1) holds if its right-hand side is absolutely convergent.

Here we shall establish analogues of (1) for arithmetic functions F which grow more rapidly than is allowed by Khinchin's theorem. In particular we shall consider the case $F(r) = I(r) = r$ and estimate

$$S_N(I,\alpha) = a_1(\alpha) + \cdots + a_N(\alpha).$$

Khinchin noted at the end of his book *Continued Fractions* [**5**] that $S_N(I,\alpha)/N$ could not have a finite limit for most values of α. Indeed, for almost all α the inequality $a_n(\alpha) > n \log n$ holds for an infinite sequence of integers n in consequence of the following result of Borel and Bernstein

[**4**, Theorem 197], [**5**, Theorem 30], [**1**, Theorem 4.1]:

LEMMA 1. *Let* $\varphi(1), \varphi(2), \ldots$ *be a sequence of positive numbers. For almost all* $\alpha \in (0,1)$ *the inequality* $a_n(\alpha) > \varphi(n)$ *has a finite number of solutions* n *if and only if* $\sum_{n=1}^{\infty} 1/\varphi(n) < \infty$.

Khinchin showed in [**6**] that

$$(b_1 + \cdots + b_N)/(N \log N) \to 1/\log 2 \tag{2}$$

in measure as $N \to \infty$, where

$$b_n = b_n(\alpha) = \begin{cases} a_n, & \text{if } a_n < n(\log n)^{4/3} \\ 0, & \text{otherwise.} \end{cases}$$

The limit (2) cannot hold a.e., since for almost all $\alpha \in (0,1)$ the inequality $b_n > n \log n \log\log n$ holds for an infinite sequence of integers n by Lemma 1.

The obstacle to a.e. convergence, as we shall see, is the occurrence of a single large value of a_n. Here we shall establish an analogue of (1) by excluding at most one summand.

THEOREM 1. *Suppose that* F *is a positive valued arithmetic function satisfying the bound*

$$\left\{ \sum_{j \le N} F(j)^2/j^2 \right\} \Big/ \left\{ \sum_{j \le N} F(j)/j^2 \right\}^2 \le N(\log N)^{-3/2-\varepsilon} \tag{3}$$

for some $\varepsilon > 0$. *Then for almost all* $\alpha \in (0,1)$ *and for all* N *exceeding a number* $N_0(\alpha)$, *we have*

$$S_N(f, \alpha) = (1 + o(1)) \frac{N}{\log 2} \sum_{r \le N} F(r) \log\left\{1 + \frac{1}{r(r+2)}\right\}$$
$$+ \vartheta_+ \max_{1 \le n \le N} F(a_n(\alpha)).$$

Here $0 \le \vartheta_+ = \vartheta_+(N, \alpha, F) \le 1$.

If we take $F(r) = I(r) = r$ we obtain

COROLLARY 1. *For almost all* $\alpha \in (0,1)$ *there exists a number* $N_0 = N_0(\alpha)$ *such that*

$$S_N(I, \alpha) = \frac{1 + o(1)}{\log 2} N \log N + \vartheta_+ \max_{1 \le n \le N} a_n(\alpha)$$

holds for all $N \ge N_0$.

An immediate consequence of Lemma 1 and Corollary 1 is

COROLLARY 2. *Let* $0 < \varphi(1) < \varphi(2) < \cdots$ *satisfy* $\sum_{n=1}^{\infty} 1/\varphi(n) < \infty$. *Then*

$$S_N(I, \alpha) = \frac{1 + o(1)}{\log 2} N \log N + \vartheta_+ \varphi(N)$$

holds for almost all $\alpha \in (0, 1)$ *and all* $N \geq N_0(\alpha)$.

There are two main steps in the proof of the theorem. First, we show that for most α there can be at most one "large" $a_n(\alpha)$. Next we estimate the variance of a truncated form of S_N. The theorem follows easily from these estimates.

2. Auxiliary results. Let $[x]$ denote the integer part of a real number x and let $\{x\} = x - [x]$ denote the fractional part. Define T: $(0, 1) \to [0, 1)$ by $Tx = \{1/x\}$. Then the partial quotients of (an irrational number) α are given by the formulas

$$a_1(\alpha) = [1/\alpha], \qquad a_{n+1}(\alpha) = a_1(T^n \alpha), \quad n \geq 1.$$

(Rational numbers have terminating continued fraction expansions and require slight alteration of the formulas. This is not needed here, since the rationals form a set of measure zero.)

The so called *Gauss measure* μ is defined on Borel subsets of $(0, 1)$ by

$$\mu(E) = \frac{1}{\log 2} \int_{t \in E} \frac{dt}{1 + t}.$$

This measure satisfies the *invariance* relation

$$\mu(T^{-1}E) = \mu(E), \qquad E \text{ a Borel set.}$$

Note that μ and Lebesgue measure have the same null sets.

For r and $k_1, k_2, \ldots, k_r$ positive integers, set

$$E_r = E\begin{pmatrix} 1 & 2 \cdots r \\ k_1 & k_2 \cdots k_r \end{pmatrix} = \{\alpha \colon a_1(\alpha) = k_1, \ldots, a_r(\alpha) = k_r\}.$$

E_r is called a *fundamental interval of rank* r. If r and s are positive integers, B is any Borel set, and E_r any fundamental interval of rank r, then μ satisfies the *mixing relation* [**1**, Chapter 1, §4]

$$\mu(E_r \cap T^{-r-s}B) = \mu(E_r)\mu(B)\{1 + O(q^s)\}$$

uniformly in r, s, B, and E_r. Here q is some number in $(0, 1)$. Together the preceding relations imply that

$$\begin{aligned}
&\mu\{\alpha\colon a_r(\alpha) = m \text{ and } a_{r+s}(\alpha) = n\} \\
&\quad = \mu\{\alpha\colon a_1(\alpha) = m \text{ and } a_{1+s}(\alpha) = n\} \\
&\quad = \mu\left\{\left(\frac{1}{m+1}, \frac{1}{m}\right] \cap T^{-s-1}\left(\frac{1}{n+1}, \frac{1}{n}\right]\right\} \\
&\quad = (\log 2)^{-2} \log \frac{(m+1)(m+1)}{m(m+2)} \log \frac{(n+1)(n+1)}{n(n+2)} \{1 + O(q^s)\}.
\end{aligned}$$

LEMMA 2. *Let $c > 1/2$, and for given $N \in \mathbf{Z}^+$ set $N' = N(\log N)^c$. For almost all $\alpha \in (0,1)$ there exist at most finitely many positive integers N for which the inequalities*

$$a_m(\alpha) > N', \qquad a_n(\alpha) > N' \tag{4}$$

hold for two distinct indices $m, n \leq N$.

Proof. Fix $m < n$. By a weak form of the mixing condition we have

$$\begin{aligned}
&\mu\{\alpha \in (0,1)\colon a_m(\alpha) > N', a_n(\alpha) > N'\} \\
&\qquad \ll \mu\{\alpha\colon a_m(\alpha) > N'\} \cdot \mu\{\alpha\colon a_n(\alpha) > N'\} \\
&\qquad = \mu\{\alpha\colon a_1(\alpha) > N'\}^2 \ll (N')^{-2} = N^{-2}(\log N)^{-2c}.
\end{aligned}$$

It follows that the measure of the set on which (4) holds for *some* distinct indices $m, n \leq 2N$ is of order at most $(\log N)^{-2c}$. For $K = 1, 2, \ldots$ let

$$U_k = \bigcup_{k \geq K} \{\alpha \in (0,1)\colon a_m(\alpha) > (2^k)', a_n(\alpha) > (2^k)' \text{ for some distinct } m, n \leq 2^{k+1}\}.$$

Then

$$\mu(U_k) << \sum_{k \geq K} k^{-2c} \to 0 \quad \text{as } K \to \infty.$$

For $\alpha \notin U_k$ and $N \geq 2^K$ there exists at most one index $n \leq N$ for which

$$a_n(\alpha) > N(\log N)^c. \qquad \square$$

3. Proof of the theorem. Given $\varepsilon > 0$ and $N \in \mathbf{N}$, set

$$a_n^* = a_{n,N}^*(\alpha) = \begin{cases} a_n & \text{if } a_n \leq N(\log N)^{1/2+\varepsilon/4} =: N' \\ 0 & \text{otherwise.} \end{cases}$$

Define $F(0) = 0$,

$$S_N^*(\alpha) = S_N^*(F, \alpha) = \sum_{n \le N} F(a_n^*(\alpha)),$$

$$J_N = \int_0^1 S_N^*(\alpha)\, d\mu(\alpha),$$

and

$$V_N = \int_0^1 (S_N^*(\alpha) - J_N)^2\, d\mu(\alpha).$$

We have

$$\begin{aligned} J_N &= \sum_{n=1}^{N} \int_0^1 F(a_n^*(\alpha))\, d\mu(\alpha) = N \int_0^1 F(a_1^*(\alpha))\, d\mu(\alpha) \\ &= N \sum_{j=1}^{N'} F(j) \mu\{\alpha\colon a_1(\alpha) = j\} \\ &= \frac{N}{\log 2} \sum_{j=1}^{N'} F(j) \log\left(1 + \frac{1}{j(j+2)}\right) \asymp N \sum_{j=1}^{N'} F(j)/j^2. \end{aligned}$$

(We say that $f \asymp g$ if $f \le K_1 g$ and $g \le K_2 f$ for suitable K_1 and K_2.)

Next we show that $V_N \ll N \sum_{j \le N'} F(j)^2/j^2$. We begin by writing

$$\begin{aligned} V_N + J_N^2 &= \int_0^1 |S_N^*(\alpha)|^2\, d\mu(\alpha) \\ &= \sum_{m,n=1}^{N} \int_0^1 F(a_m^*(\alpha)) F(a_n^*(\alpha))\, d\mu(\alpha) = \sum_{m,n=1}^{N} b_{mn}, \end{aligned}$$

say. For $1 \le m < n \le N$ we have

$$\begin{aligned} b_{mn} &= \sum_{j,k \le N'} F(j) F(k) \mu\{\alpha\colon a_m(\alpha) = j,\ a_n(\alpha) = k\} \\ &= \sum_{j,k \le N'} F(j) F(k) \mu\left(\frac{1}{j+1}, \frac{1}{j}\right] \mu\left(\frac{1}{k+1}, \frac{1}{k}\right] (1 + O(q^{n-m})) \\ &= J_N^2 N^{-2} (1 + O(q^{n-m})). \end{aligned}$$

The diagonal terms satisfy

$$\begin{aligned} b_{nn} &= \int_0^1 F(a_n^*(\alpha))^2\, d\mu(\alpha) = \int_0^1 F(a_1^*(\alpha))^2\, d\mu(\alpha) \\ &= \sum_{j \le N'} F(j)^2 \mu\{\alpha\colon a_1(\alpha) = j\} \ll \sum_{j \le N'} F(j)^2/j^2. \end{aligned}$$

Thus we have

$$\begin{aligned} V_N = \sum_{m,n=1}^{N} b_{mn} - J_N^2 &\ll J_N^2 N^{-2} \sum_{m \le n \le N} q^{n-m} + N \sum_{j \le N'} F(j)^2/j^2 \\ &\ll J_N^2/N + N \sum_{j \le N'} F(j)^2/j^2 \\ &\ll N\Big(\sum_{j \le N'} F(j)/j^2 \Big)^2 + N \sum_{n \le N'} F(j)^2/j^2 \\ &\ll N \sum_{j \le N'} F(j)^2/j^2. \end{aligned}$$

The last relation follows from the Cauchy-Schwarz inequality.

Now we apply the estimate of V_N to show that

$$S_N^*(\alpha) = (1 + o(1)) J_N$$

for most values of α. Let

$$c(k) = [\exp k^{1-\varepsilon/4}], \qquad k = 1, 2, \ldots.$$

We have

$$\begin{aligned} \int_{\alpha=0}^{1} \sum_{k=1}^{\infty} \big(S_{c(k)}^*(\alpha) - J_{c(k)}\big)^2 \Bigg(c(k) \sum_{j \le c(k)'} \frac{F(j)^2}{j^2} k^{1+\varepsilon/4} \Bigg)^{-1} d\mu(\alpha) \\ \ll \sum_{k=1}^{\infty} k^{-1-\varepsilon/4} < \infty. \end{aligned}$$

It follows that the integrand in the last integral is finite a.e. and hence

$$S_{c(k)}^*(\alpha) - J_{c(k)} = o\Bigg\{ c(k) \sum_{j \le c(k)'} \frac{F(j)^2}{j^2} k^{1+\varepsilon/4} \Bigg\}^{1/2}$$

for almost all α. The hypothesis of Theorem 1 and a small calculation show that

$$c(k) \sum_{j \le c(k)'} \frac{F(j)^2}{j^2} k^{1+\varepsilon/4} \ll J_{c(k)}^2 / (\log c(k))^{\varepsilon/12} = o\big(J_{c(k)}^2\big),$$

provided that $\varepsilon < 1$. Thus

$$S_{c(k)}^*(\alpha) = (1 + o(1)) J_{c(k)} \quad \text{a.e.}$$

Suppose that $c(k-1) < N \le c(k)$. Then

$$S_{c(k-1)}^*(\alpha) \le S_N^*(\alpha) \le S_{c(k)}^*(\alpha),$$

and so, off a set of measure 0,

$$(1 + o(1))J_{c(k-1)} \leq S_N^*(\alpha) \leq (1 + o(1))J_{c(k)}.$$

Now we show that $J_{c(k)} \sim J_{c(k-1)}$. Recall that

$$J_N = \frac{N}{\log 2} \sum_{j \leq N'} F(j)\log\left(1 + \frac{1}{j(j+2)}\right).$$

Another small calculation shows that

$$c(k-1) = \left(1 + O(k^{-\varepsilon/4})\right)c(k),$$

so $c(k-1) \sim c(k)$ as $k \to \infty$. It remains to show that

$$(5) \qquad \sum_{j \leq c(k)'} F(j)\log\left(1 + \frac{1}{j(j+2)}\right) \sim \sum_{j \leq c(k-1)'} F(j)\log\left(1 + \frac{1}{j(j+2)}\right).$$

We shall show (5) and the final approximation of S_N^* by using

LEMMA 3. *Let F satisfy the hypotheses of Theorem* 1. *Then, as* $N \to \infty$,

$$\sum_{N < r \leq N \log N} \frac{F(r)}{r^2} = o\left(\sum_{r \leq N} \frac{F(r)}{r^2}\right).$$

Proof. The Cauchy Schwarz inequality and condition (3) yield

$$\sum_{N < r \leq N \log N} \frac{F(r)}{r^2} \leq \left\{\sum_{r \leq N \log N} \frac{F(r)^2}{r^2}\right\}^{1/2} \left\{\sum_{r > N} \frac{1}{r^2}\right\}^{1/2}$$
$$\leq \sum_{r \leq N \log N} \frac{F(r)}{r^2} (\log N)^{-1/4 - \varepsilon/2}.$$

Thus, as $N \to \infty$,

$$(1 - o(1)) \sum_{N < r \leq N \log N} \frac{F(r)}{r^2} \leq o(1) \sum_{r \leq N} \frac{F(r)}{r^2},$$

and the lemma follows. □

Returning to the proof of Theorem 1, we see first that (5) holds, and hence

$$S_N^*(\alpha) = (1 + o(1))J_N \quad \text{a.e.}$$

The lemma also implies that

$$S_N^*(\alpha) = (1 + o(1))\frac{N}{\log 2} \sum_{j \leq N} F(j)\log\left(1 + \frac{1}{j(j+2)}\right) \quad \text{a.e.}$$

since

$$\sum_{N<j\leq N'} F(j)\log\left(1+\frac{1}{j(j+2)}\right)$$

is negligible.

Finally, we have by Lemma 2, for almost all α and all sufficiently large N,

$$0 \leq S_N(\alpha) - S_N^*(\alpha) \leq F\left(\max_{1\leq n\leq N} a_n(\alpha)\right) \leq \max_{1\leq n\leq N} F(a_n(\alpha)).$$

This inequality and the last estimate of S_N^* establish the theorem. □

It would be interesting to learn whether Theorem 1 could be established by ergodic methods.

4. Further results. In this section we consider cases in which $S_N(I,\alpha)$ can be estimated by $N(\log N)/(\log 2)$ alone and when by $\varphi(N)$ alone, where $\sum_{n=1} 1/\varphi(n) < \infty$.

First we note that for any $\varepsilon > 0$ and *fixed* $N \in \mathbf{Z}^+$ we have

$$\mu\left\{\alpha \in (0,1): \left|\frac{S_N(I,\alpha)}{N\log N} - \frac{1}{\log 2}\right| > \varepsilon\right\} \ll \frac{1}{\varepsilon \log N}. \tag{6}$$

(The implied constant here is absolute.)

This bound is achieved by setting

$$a_n^{**} = a_{n,N,\varepsilon}^{**}(\alpha) = \begin{cases} a_n, & \text{if } a_n < \varepsilon N\log N = =: N'' \\ 0, & \text{otherwise.} \end{cases}$$

We compute the variance of the sum function S_N^{**} as before and apply Chebyshev's estimate to obtain

$$\mu\left\{\alpha: \left|S_N^{**}(\alpha) - \frac{N\log N}{\log 2}\right| > \varepsilon N \log N\right\} \ll \frac{1}{\varepsilon\log N}.$$

Also, for each $n \leq N$,

$$\mu\{\alpha: a_n(\alpha) > \varepsilon N\log N\} \ll \frac{1}{\varepsilon N\log N},$$

and estimate (6) follows.

Next, we show directly a sharp one sided estimate of Pruitt [**9**, Theorem 5.2].

COROLLARY 3. *For* $N \geq 3$ *set* $\beta(N) = \exp(k\log^2 k)k\log^2 k$ *for*

$$\exp((k-1)\log^2(k-1)) < N \leq \exp(k\log^2 k). \tag{7}$$

Then, for almost all $\alpha \in (0,1)$

$$\limsup_{N\to\infty} \frac{S_N(I,\alpha)}{\beta(N)} = \frac{1}{\log 2}.$$

Proof. In Corollary 2 set

$$\varphi(N) = \beta(N)/(\log\log 10k).$$

An easy calculation shows that $\sum 1/\varphi(N) < \infty$. If N satisfies (7), then by Corollary 2

$$S_N(I,\alpha) \le \frac{1+o(1)}{\log 2}\beta(N) + \beta(N)/\log\log 10k \quad \text{a.e.},$$

so $\limsup S_N(I,\alpha)/\beta(N) \le 1/\log 2$ a.e. On the sequence $N_k = \exp(k\log^2 k)$, the ratio $S_N(I,\alpha)/\beta(N)$ converges to $1/\log 2$ a.e. □

In another direction, we show that in Corollary 2, $\varphi(N)$ dominates $N\log N$ for "most" values of N.

LEMMA 4. *Suppose that* $0 < \varphi(1) \le \varphi(2) \le \cdots$ *and* $\sum_{n=1}^{\infty} 1/\varphi(n) < \infty$. *Let*

$$S = \{n \in \mathbf{Z}^+ \colon \varphi(n) < n\log n\}.$$

Then S has logarithmic density zero.

Proof. Let $T = \{\nu \in \mathbf{Z}^+ \colon (2^{\nu-1}, 2^{\nu}] \cap S \ne \varnothing\}$. Suppose $\nu \in T$. Then there exists an integer n such that $2^{\nu-1} < n \le 2^{\nu}$ and $\varphi(n) < n\log n$, so

$$\sum_{n/2<k\le n} \frac{1}{\varphi(k)} \ge \frac{n-[n/2]}{n\log n} \ge \frac{1}{2\log n} \ge \frac{1}{2\nu\log 2}.$$

Thus

$$\sum_{\nu\in T} \frac{1}{\nu} \le 4(\log 2)\sum_{k=1}^{\infty} 1/\varphi(k) < \infty. \tag{8}$$

Also, we have

$$\sum_{\substack{k\in S\\ 2^{\nu-1}<k\le 2^{\nu}}} \frac{1}{k} \le \begin{cases} \log 2, & \nu\in T\\ 0, & \nu\notin T.\end{cases}$$

With $y = (\log x)/\log 2$ we have

$$\delta_x := \frac{1}{\log x}\sum_{\substack{k\le x\\ k\in S}} \frac{1}{k} \le \frac{1}{y\log 2}\sum_{\nu\le y+1}\sum_{\substack{k\in S\\ 2^{\nu-1}<k\le 2^{\nu}}} \frac{1}{k}$$

$$\le \frac{1}{y}\sum_{\substack{\nu\le y+1\\ \nu\in T}} 1 = \frac{T(y+1)}{y}, \quad \text{say}.$$

We have from (8) that

$$\frac{1}{N}(T(N) - T(N/2)) \le \sum_{\substack{\nu \in T \\ n/2 < \nu \le N}} \frac{1}{\nu} \to 0$$

as $N \to \infty$. Thus $T(y) = o(y)$ as $y \to \infty$. Finally,

$$\limsup_{x \to \infty} \delta_x \le \limsup_{y \to \infty} \frac{T(y+1)}{y} = 0. \qquad \square$$

COROLLARY 4. *Suppose that* φ *satisfies the hypotheses of Lemma* 4. *Then for almost all* $\alpha \in (0,1)$

$$S_N(I, \alpha) \ll \varphi(N)$$

holds for all integers N outside a set of logarithmic density zero.

Acknowledgment. We are indebted to Professor Walter Philipp for several helpful conversations.

REFERENCES

[1] P. Billingsley, *Ergodic Theory and Information*, Wiley, New York, 1965.
[2] W. Doeblin, *Remarques sur la théorie métrique des fractions continues*, Compositio Math., **7** (1940), 353–371.
[3] P. Erdös, *Some remarks on diophantine approximation*, J. Indian Math. Soc., **12** (1948), 67–74.
[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th Edition, Clarendon Press, Oxford, 1979.
[5] A. Ya. Khinchin, *Continued Fractions*, U. Chicago Press, Chicago, 1964.
[6] ______, *Metrische Kettenbruchprobleme*, Compositio Math., **1** (1935), 361–382.
[7] W. Phillip, *Some metrical theorems in number theory*, Pacific J. Math., **20** (1967), 109–127.
[8] ______, *Some metrical theorems in number theory*, II, Duke Math. J., **38** (1970), 447–458.
[9] W. E. Pruitt, *General one-sided laws of the iterated logarithm*, Ann. of Probability, **9** (1981), 1–48.

Received August 28, 1984 and in revised form January 22, 1985. Research supported in part by grants from the National Science Foundation.

UNIVERSITY OF ILLINOIS
URBANA, IL 61801

AND

THE UNIVERSITY OF TEXAS
AUSTIN, TX 78712