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ASYMPTOTIC EXPANSIONS OF THE LEBESGUE CONSTANTS FOR JACOBI SERIES

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Explicit expressions are obtained for the implied constants in the two O-terms in Lorch's asymptotic expansions of the Lebesgue constants associated with Jacobi series [Amer. J. Math., 81 (1959), 875–888]. In particular, a question of Szegö concerning asymptotic monotonicity of the Lebesgue constants for Laplace series is answered. Our method differs from that of Lorch, and makes use of some recently obtained uniform asymptotic expansions for the Jacobi polynomials and their zeros.

1. Introduction and summary. The nth partial sum of the Fourier series of an arbitrary function can be written in the form of an integral involving the Dirichlet kernel. The integral of the absolute value of this kernel is known as the nth *Lebesgue constant*, and is usually denoted by

$$L_n = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{\sin(t/2)}\,dt; \tag{1.1}$$

see [**19**, p. 172]. The behavior of the sequence $\{L_n\}$ is closely connected with convergence and divergence properties of Fourier series, and the importance of this sequence has led many mathematicians to be concerned not only with just its asymptotic formula but also with its full asymptotic expansion. First, Fejer [**1**] showed that

$$L_n = \frac{4}{\pi^2}\log n + c_0 + \frac{c_1}{n} + \frac{\alpha(n)}{n^2}, \tag{1.2}$$

where c_0 and c_1 are constants and $\alpha(n) = O(1)$ as $n \to \infty$. An explicit expression was given for c_0 but not for c_1. Later, infinite asymptotic expansions were derived by Gronwall [**4**], Watson [**18**] and Hardy [**7**].

In an entirely analogous manner, the nth partial sum of the expansion of an arbitrary function in terms of Jacobi polynomials can be written as an integral involving a kernel; see, e.g., [**17**, p. 39]. The nth Lebesgue constant in this case has the integral representation

$$L_n(\alpha,\beta) = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \cdot \int_0^{\pi} \left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1} \left|P_n^{(\alpha+1,\beta)}(\cos\theta)\right| d\theta. \tag{1.3}$$

This result is due to Rau [**12**]. In view of the identity [**17**, p. 60]

$$(1.4)\qquad P_n^{(1/2,-1/2)}(\cos\theta) = \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)\Gamma(\frac{3}{2})}\frac{\sin(n+\frac{1}{2})\theta}{(2n+1)\sin(\theta/2)},$$

Equation (1.3) gives $L_n(-\frac{1}{2}, -\frac{1}{2}) = L_n$, contrary to a statement made in [**8**, footnote 7].

Rau [**12**] was the first to show that for $\alpha > -1/2$ and $\beta > -1$,

$$(1.5)\qquad L_n(\alpha,\beta) = A_{\alpha\beta}n^{\alpha+1/2} + o(n^{\alpha+1/2}), \qquad n\to\infty,$$

where

$$(1.6)\qquad A_{\alpha\beta} = \frac{2}{\pi^{3/2}}\frac{\Gamma(\alpha/2+1/4)\Gamma(\beta/2+3/4)}{\Gamma(\alpha+1)\Gamma([\alpha+\beta]/2+1)}.$$

Later Szegö [**16**] had an alternative proof of (1.5), and furthermore showed that

$$(1.7)\qquad L_n(-\tfrac{1}{2},\beta) = \frac{4}{\pi^2}\log n + o(\log n)$$

for $\beta > -1$, and that for $-1 < \alpha < -\frac{1}{2}$ and $\beta > -1$

$$(1.8)\qquad L_n(\alpha,\beta) = \frac{2^{-\alpha}}{\Gamma(\alpha+1)}\int_0^\infty \theta^\alpha |J_{\alpha+1}(\theta)|\,d\theta + o(1),$$

where $J_{\alpha+1}(\theta)$ is the Bessel function of first kind.

The above results have been sharpened by Lorch [**8**, **9**], particularly in the cases $\alpha = -\frac{1}{2}$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$. For $-\frac{1}{2} < \alpha << \frac{1}{2}$, $\alpha - \beta < 1$ and $\beta > -1$, Lorch's result can be stated as follows:

$$(1.9)\qquad L_n(\alpha,\beta) = A_{\alpha\beta}n^{\alpha+1/2} + B_\alpha + O(n^{\alpha-1/2}) + O(n^{\alpha-\beta-1})$$

where

$$(1.10)\qquad B_\alpha = \frac{2^{-\alpha}}{\Gamma(\alpha+1)}\Bigg\{-M_1(\alpha) + \int_0^{j_1} x^\alpha J_{\alpha+1}(x)\,dx + 2\alpha\sum_{k=1}^\infty (-1)^k \int_{j_k}^{j_{k+1}} x^{\alpha-1}J_\alpha(x)\,dx + 2\sum_{k=1}^\infty \left[M_k(\alpha) - \frac{\sqrt{2}}{\pi^{3/2}}\int_{j_{k-1}}^{j_k} x^{\alpha-1/2}\,dx\right]\Bigg\},$$

both infinite series being absolutely convergent. (Equation (4) in [**9**] contains two misprints; $M_{k+1}(\alpha)$ should be replaced by $M_k(\alpha)$ and $M_1(\alpha)$ should have a minus sign.) In (1.10), $j_n = j_{\alpha+1,n}$ is the nth positive zero of

$J_{\alpha+1}(x)$, $n = 1,2,\ldots$, $j_0 = 0$, and

(1.11) $M_k(\alpha) \equiv (-1)^k (j_{\alpha+1,k})^{\alpha} J_{\alpha}(j_{\alpha+1,k}) > 0, \qquad k = 1,2,\ldots.$

For $\alpha = -\frac{1}{2}$ and $\beta > -1$, Lorch also obtained the result that as $n \to \infty$,

$$L_n(-\tfrac{1}{2},\beta) = \frac{4}{\pi^2}\log n + C_\beta + O(n^{-1}\log n) + O(n^{-\beta-3/2}), \tag{1.12}$$

where

$$C_\beta = \frac{8}{\pi^2}\log 2 + \frac{2}{\pi}\int_0^1 \theta^{-1}\sin\theta\, d\theta - \frac{4}{\pi^2}\int_0^{\pi/2} \frac{1-(\cos\theta)^{\beta+1/2}}{\sin\theta}\, d\theta - \frac{2}{\pi}\int_1^{\infty} \theta^{-1}\left\{\frac{2}{\pi} - |\sin\theta|\right\} d\theta, \tag{1.13}$$

the last integral being convergent. (There is a typographical error in [**9**, (8)]; the factor in front of $\log 2$ should be $8/\pi^2$ and not $4/\pi^2$.)

Lorch's investigation [**10**] was motivated by a question raised by Szegö concerning asymptotic monotonicity of the sequence $\{L_n(0,0)\}$; see also the editor's comment at the end of [**13**]. The result in (1.9), however, fails to answer the question of Szegö. Lorch thus posed to us in 1980 the problem of replacing the O-terms in (1.9) and (1.12) by explicitly determined expressions plus terms of lower asymptotic order. The following results provide a solution to his problem, and were announced in [**3**]. The detailed proofs of these results are the contents of the present paper. The fact that $\{L_n(0,0)\}$ is an asymptotically increasing sequence is an immediate consequence of the result given in (1.18) below.

First, for the restricted range $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $-\frac{1}{2} < \beta < \frac{1}{2}$, we have

$$L_n(\alpha,\beta) = A_{\alpha\beta} n^{\alpha+1/2} + B_\alpha + C_{\alpha\beta} A_{\alpha\beta} n^{\alpha-1/2} + \frac{1}{\Gamma(\alpha+1)} D_\beta n^{\alpha-\beta-1} + O(n^{\alpha-3/2}), \tag{1.14}$$

where

$$C_{\alpha\beta} = \frac{(\alpha+\beta+2)(\alpha+1/2)}{2} \tag{1.15}$$

and

$$D_\beta = 2^{-\beta} \sum_{k=1}^{\infty} \left[\hat{M}_k(\beta) - \frac{\sqrt{2}}{\pi^{3/2}} \int_{j'_{k-1}}^{j'_k} x^{\beta+1/2}\,dx - \frac{\beta+1/2}{\sqrt{2\pi}} \int_{j'_{k-1}}^{j'_k} x^{\beta-1/2}\,dx \right]. \tag{1.16}$$

In (1.16), $j'_n \equiv j_{\beta,n}$ is the nth positive zero of $J_\beta(x)$, $n = 1, 2, \ldots$, $j'_0 = 0$ and

$$\begin{aligned} \hat{M}_k(\beta) &= (-1)^k \left(j_{\beta,k}\right)^{1+\beta} J_{\beta-1}\left(j_{\beta,k}\right) \\ &= (-1)^{k+1} \left(j_{\beta,k}\right)^{1+\beta} J_{\beta+1}\left(j_{\beta,k}\right). \end{aligned} \tag{1.17}$$

In the important particular case of Laplace series (i.e., the series in terms of Legendre polynomials at the end point $x = 1$), $\alpha = \beta = 0$, and (1.14) becomes

(1.18)

$$\begin{aligned} L_n(0,0) = {} & \frac{2^{3/2}}{\sqrt{\pi}} n^{1/2} \\ & + \left[1 + 2 \sum_{k=1}^{\infty} \left\{ M_k(0) - \frac{2^{3/2}}{\pi} \left[k^{1/2} - (k-1)^{1/2} \right] \right\} \right] \\ & + \sqrt{\frac{2}{\pi}}\, n^{-1/2} + n^{-1} \sum_{k=1}^{\infty} \left\{ \hat{M}_k(0) - \frac{2^{3/2}}{3} \left[k^{3/2} - (k-1)^{3/2} \right] \right. \\ & \qquad \left. - 2^{-3/2} \left[k^{1/2} - (k-1)^{1/2} \right] \right\} \\ & + O(n^{-3/2}). \end{aligned}$$

The principal term in (1.18) was first given by Gronwall [**5**, **6**], and later by Szegö [**14**, **15**] with simpler proofs.

An improved version of (1.12) is

$$\begin{aligned} L_n\left(-\tfrac{1}{2}, \beta\right) = {} & \frac{4}{\pi^2} \log n + C_\beta + E_\beta n^{-1} + \frac{1}{\sqrt{\pi}} D_\beta n^{-\beta-3/2} \\ & + O\left(n^{-2} \log n\right), \end{aligned} \tag{1.19}$$

valid for $-\frac{1}{2} < \beta < \frac{1}{2}$, where D_β is as given in (1.16) and

$$E_\beta = \frac{4}{\pi^2} \binom{\beta + \frac{3}{2}}{2}. \tag{1.20}$$

Finally, we turn to the case $-1 < \alpha < -\frac{1}{2}$. Under the additional restrictions $\alpha - \beta > -1$ and $-\frac{1}{2} < \beta < \frac{1}{2}$, we have the following sharpened form of (1.8):

$$L_n(\alpha,\beta) = C_0 + C_1 n^{\alpha+1/2} + C_2 n^{\alpha-1/2} + C_3 n^{\alpha-\beta-1} + O(n^{-2}), \tag{1.21}$$

where

$$C_0 = \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha |J_{\alpha+1}(x)|\, dx \tag{1.22}$$

$$\begin{aligned} C_1 = {} & \frac{4}{\Gamma(\alpha+1)\pi^{3/2}} \\ & \cdot \left\{ \frac{1}{\alpha+1/2}\left(\frac{\pi}{2}\right)^{\alpha+1/2} \right. \\ & \left. + \int_0^{\pi/2} \left[(\sin\theta)^{\alpha-1/2}(\cos\theta)^{\beta+1/2} - \theta^{\alpha-1/2} \right] d\theta \right\} \end{aligned} \tag{1.23}$$

$$C_2 = \tfrac{1}{2}(\alpha+\beta+2)(\alpha+\tfrac{1}{2})C_1 \tag{1.24}$$

$$C_3 = \frac{1}{\Gamma(\alpha+1)} D_\beta, \tag{1.25}$$

D_β again being the same constant given in (1.16).

Lorch's method essentially consists of replacing the Jacobi polynomial in (1.3) by its asymptotic formula of "Hilb's type" [**17**, p. 197], and splitting the interval of integration (O, π) at the points $j_{\alpha+1,k}/N$, $k = 1, \ldots, [N]$, where $N = n + \frac{1}{2}(\alpha+\beta+2)$. Our approach differs from that of Lorch. We first split the interval $(0, \pi)$ at the exact zeros of the Jacobi polynomial and then apply recently obtained uniform asymptotic expansions for the Jacobi polynomials and their zeros [**2**]. Our method may also be extended to give higher order approximations when desired.

2. Sketch of the procedure. For simplicity of presentation, we restrict our attention to the case $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $-\frac{1}{2} < \beta < \frac{1}{2}$. Let θ_k denote the kth zero of $P_n^{(\alpha+1,\beta)}(\cos\theta)$, put $\bar{n} = [n/2]$, and write

$$\begin{aligned} & \frac{\Gamma(\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} L_n(\alpha,\beta) \\ & \qquad = \int_0^{\theta_{\bar{n}}} + \int_{\theta_{\bar{n}}}^{\pi} \equiv L_n^{(1)}(\alpha,\beta) + L_n^{(2)}(\alpha,\beta). \end{aligned} \tag{2.1}$$

We shall first be concerned with the constant $L_n^{(1)}(\alpha,\beta)$. The evaluation of $L_n^{(2)}(\alpha,\beta)$ proceeds in a similar manner. For convenience, we set

$\theta_0 \equiv 0$ and define

$$(2.2) \qquad I_k = \int_{\theta_k}^{\theta_{k+1}} \left(\sin\frac{\theta}{2}\right)^{2\alpha+1} \left(\cos\frac{\theta}{2}\right)^{2\beta+1} P_n^{(\alpha+1,\beta)}(\cos\theta)\, d\theta.$$

Since $\cos 0 = 1$ and

$$(2.3) \qquad P_n^{(\alpha+1,\beta)}(1) = \binom{n+\alpha+1}{n} = \frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)} > 0,$$

we have

$$(2.4) \qquad L_n^{(1)}(\alpha,\beta) = \sum_{k=0}^{\bar{n}-1} (-1)^k I_k.$$

To evaluate I_k, we shall use the following results given in [**2**]; see, in particular, the main theorem, Corollary 2, and the first paragraph in §5 of that reference.

LEMMA 1. *For $\alpha + 1 > -\frac{1}{2}$ and $\alpha + \beta + 1 \geq -1$, we have*

$$(2.5) \qquad \left(\sin\frac{\theta}{2}\right)^{\alpha+1} \left(\cos\frac{\theta}{2}\right)^{\beta} P_n^{(\alpha+1,\beta)}(\cos\theta) = \frac{\Gamma(n+\alpha+2)}{n!}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\left[\sum_{l=0}^{m-1} A_l(\theta)\frac{J_{\alpha+l+1}(N\theta)}{N^{\alpha+l+1}} + \sigma_m\right],$$

where

$$(2.6) \qquad N = n + \tfrac{1}{2}(\alpha+\beta+2)$$

and

$$(2.7) \qquad \sigma_m = \theta^m O(N^{-m-\alpha-1}),$$

the O-term being uniform with respect to $\theta \in [0, \pi - \varepsilon]$, $\varepsilon > 0$. The coefficients $A_l(\theta)$ are analytic functions in $0 \leq \theta \leq \pi - \varepsilon$, and are $O(\theta^l)$ in that interval. In particular, $A_0(\theta) = 1$ and

$$(2.8) \qquad A_1(\theta) = \left[(\alpha+1)^2 - \frac{1}{4}\right]\left(\frac{1-\theta\cot\theta}{2\theta}\right) - \frac{(\alpha+1)^2-\beta^2}{4}\tan\frac{\theta}{2}.$$

LEMMA 2. *Let $\alpha + 1 > -\frac{1}{2}$, $\alpha + \beta + 1 \geq -1$ and let $0 < \theta_1 < \theta_2 < \cdots < \theta_n < \pi$ be the zeros of $P_n^{(\alpha+1,\beta)}(\cos\theta)$. Then, as $n \to \infty$,*

$$(2.9) \qquad \theta_l = \frac{j_{\alpha+1,l}}{N} + \frac{1}{N^2}\left\{\left[(\alpha+1)^2 - \frac{1}{4}\right]\frac{1-t\cot t}{2t} - \frac{(\alpha+1)^2-\beta^2}{4}\tan\frac{t}{2}\right\} + t^2 O(N^{-3}),$$

where $j_{\alpha+1,l}$ is the l-th positive zero of the Bessel function $J_{\alpha+1}(x)$ and $t = j_{\alpha+1,l}/N$. The O-term is uniformly bounded for all values of $l = 1, 2, \ldots, [\gamma n]$, where $\gamma \in (0,1)$ is a constant.

Taking $m = 3$ in (2.5) and substituting the resulting expression in (2.2) gives

$$I_k = \frac{\Gamma(n+\alpha+2)}{n! N^{\alpha+1}} \left[I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)} \right], \tag{2.10}$$

where

$$I_k^{(l+1)} = \frac{1}{N^l} \int_{\theta_k}^{\theta_{k+1}} A_l^*(\theta) J_{\alpha+l+1}(N\theta)\, d\theta, \qquad l = 0, 1, 2, \tag{2.11}$$

and

$$I_k^{(4)} = \frac{1}{N^3} O\left\{ \int_{\theta_k}^{\theta_{k+1}} \theta^3 \left(\sin\frac{\theta}{2}\right)^{\alpha} \left(\cos\frac{\theta}{2}\right)^{\beta+1} \left(\frac{\theta}{\sin\theta}\right)^{1/2} d\theta \right\} \tag{2.12}$$

with

$$A_l^*(\theta) = \left(\frac{\theta}{2}\cot\frac{\theta}{2}\right)^{1/2} \left(\sin\frac{\theta}{2}\right)^{\alpha} \left(\cos\frac{\theta}{2}\right)^{\beta} A_l(\theta). \tag{2.13}$$

Note that the implied constant in the O-symbol in (2.12) is independent of k. It is easily seen that

$$\sum_{k=0}^{\bar{n}-1} (-1)^k I_k^{(4)} = O(N^{-3}). \tag{2.14}$$

Using the identity

$$\frac{d}{dz}\left(z^{-\nu} J_\nu(z)\right) = -z^{-\nu} J_{\nu+1}(z), \tag{2.15}$$

we have by integration by parts

$$I_k^{(2)} = T_1 + T_2, \tag{2.16}$$

where

$$T_1 = \frac{1}{N^2} \left[A_1^*(\theta_k) J_{\alpha+1}(N\theta_k) - A_1^*(\theta_{k+1}) J_{\alpha+1}(N\theta_{k+1}) \right] \tag{2.17}$$

$$T_2 = \frac{1}{N^2} \int_{\theta_k}^{\theta_{k+1}} \left[\theta^{\alpha+1} A_1^*(\theta)\right]' \theta^{-\alpha-1} J_{\alpha+1}(N\theta)\, d\theta. \tag{2.18}$$

For fixed ν, it is well-known that

$$j_{\nu,k} = \left(k + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \frac{4\nu^2 - 1}{8\left(k + \frac{1}{2}\nu - \frac{1}{4}\right)\pi} + O\left(\frac{1}{k^3}\right). \tag{2.19}$$

Thus, by Lemma 2, we have

$$(2.20)\qquad N\theta_k = j_{\alpha+1,k} + O(k/N^2),$$

where the implied constant in the O-symbol is independent of k for $k = 1, \dots, \bar{n}$. Note that a combination of (2.6), (2.19) and (2.20) shows that for large n, we have $\theta_{\bar{n}} < \pi/2$. From (2.20), it also follows that

$$(2.21)\qquad J_{\alpha+1}(N\theta_k) = J_{\alpha+1}(j_{\alpha+1,k}) + J'_{\alpha+1}(\xi)O(k/N^2),$$

where ξ lies between $N\theta_k$ and $j_{\alpha+1,k}$ and hence is $O(k)$. Since $J_{\alpha+1}(j_{\alpha+1,k}) = 0$ and $J'_{\alpha+1}(\xi) = O(\xi^{-1/2})$ as $\xi \to \infty$, (2.21) yields

$$(2.22)\qquad J_{\alpha+1}(N\theta_k) = O(k^{1/2}/N^2).$$

Observe that $A_1^*(\theta) = O(\theta^{\alpha+1})$ and $\theta_k = O(k/N)$. Thus, by coupling (2.17) and (2.22), we obtain

$$(2.23)\qquad T_1 = O(k^{\alpha+3/2}/N^{\alpha+5}).$$

Furthermore, since $[\theta^{\alpha+1}A_1^*(\theta)]' = O(\theta^{2\alpha+1})$, it is easily seen that

$$(2.24)\qquad T_2 = \frac{1}{N^{\alpha+3}}O\left\{\int_{N\theta_k}^{N\theta_{k+1}} y^\alpha |J_{\alpha+1}(y)|\,dy\right\}.$$

The integral inside of the curly bracket is clearly a bounded function of N if $k = 0$, in view of (2.20). For $k = 1, \dots, \bar{n} - 1$, we use (2.19) and the fact $J_{\alpha+1}(y) = O(y^{-1/2})$ to conclude that this integral is $O(k^{\alpha-1/2})$. Now we recall the well-known expansions [**11**, p. 292]

$$(2.25)\qquad \sum_{k=1}^{n-1} k^\alpha - \zeta(-\alpha) \sim \frac{n^{\alpha+1}}{\alpha+1}\sum_{s=0}^{\infty}\binom{\alpha+1}{s}\frac{B_s}{n^s} \qquad (\alpha \neq -1),$$

where $\zeta(\mu)$ is the Riemann-Zeta function. A combination of (2.23), (2.24) and (2.25) gives

$$(2.26)\qquad \sum_{k=0}^{\bar{n}-1} (-1)^k I_k^{(2)} = O(N^{-5/2}).$$

An entirely similar argument leads to

$$(2.27)\qquad \sum_{k=0}^{\bar{n}-1} (-1)^k I_k^{(3)} = O(N^{-5/2}).$$

Here we have used the fact that

$$(2.28)\qquad \bar{n} = \frac{n}{2}\left[1 + O\left(\frac{1}{n}\right)\right], \quad \text{as } n \to \infty.$$

From (2.4) and (2.10), it now follows that as $n \to \infty$,

$$(2.29)\qquad L_n^{(1)}(\alpha,\beta) = \frac{\Gamma(n+\alpha+2)}{n!N^{\alpha+1}}\left[\sum_{k=0}^{\bar{n}-1}(-1)^k I_k^{(1)} + O(N^{-5/2})\right].$$

In the next section, it will be shown that

$$(2.30)\quad \sum_{k=0}^{\bar{n}-1}(-1)^k I_k^{(1)} = \frac{1}{2^\alpha N^{\alpha+1}}\left\{\int_0^{j_1} y^\alpha J_{\alpha+1}(y)\,dy + M_1(\alpha) + \sqrt{\frac{2}{\pi}}\,2^{(\alpha-\beta-1)/2} n^{\alpha-1/2} + 2S_n + 2\alpha R_n + O(N^{\alpha-3/2})\right\},$$

where

$$(2.31)\quad S_n = \sum_{k=2}^{\bar{n}-1} g_{\alpha\beta}(\theta_k) M_k(\alpha),$$

$$(2.32)\quad R_n = \sum_{k=1}^{\bar{n}-1}(-1)^k \int_{j_k}^{j_{k+1}} y^{\alpha-1} J_\alpha(y)\,dy$$

and

$$(2.33)\quad g_{\alpha\beta}(\theta) = \left(\frac{\theta}{2}\cot\frac{\theta}{2}\right)^{1/2}\left[\frac{\sin\theta/2}{\theta/2}\right]^\alpha\left(\cos\frac{\theta}{2}\right)^\beta.$$

To proceed further, we need the following two lemmas, whose proofs are given in §4.

LEMMA 3. *As $n \to \infty$, the sum in* (2.31) *has the asymptotic approximation*

$$(2.34)\quad S_n \approx B_\alpha^{(1)} + \frac{\sqrt{2}}{\pi^{3/2}} N^{\alpha+1/2}\left[\int_0^{\pi/2} g_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta - \int_0^{j_1/N} g_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta - \int_{j_{\bar{n}-1}/N}^{\pi/2} g_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta\right] + \frac{n^{\alpha-1/2}}{\sqrt{2\pi}} 2^{(\alpha-\beta-1)/2} + O(n^{\alpha-3/2}),$$

where $B_\alpha^{(1)}$ is a constant given by

$$(2.35)\quad B_\alpha^{(1)} = \sum_{k=2}^{\infty}\left[M_k(\alpha) - \frac{\sqrt{2}}{\pi^{3/2}}\int_{j_{k-1}}^{j_k} x^{\alpha-1/2}\,dx\right].$$

LEMMA 4. *The asymptotic behavior of the sum in* (2.32) *is given by*

$$R_n = B_\alpha^{(2)} + O(n^{\alpha-3/2}), \tag{2.36}$$

where

$$B_\alpha^{(2)} = \sum_{k=1}^{\infty} (-1)^k \int_{j_k}^{j_{k+1}} x^{\alpha-1} J_\alpha(x)\, dx. \tag{2.37}$$

The constants $B_\alpha^{(1)}$ and $B_2^{(2)}$ are related to the constant B_α given in (1.10). The asymptotic approximation of $L_n^{(1)}(\alpha,\beta)$ is obtained by inserting (2.34) and (2.36) in (2.30), and combining the resulting expression with (2.29). It is anticipated that the terms involving the last two integrals in (2.34) will combine with similar terms from $L_n^{(2)}(\alpha,\beta)$.

Evaluation of $L_n^{(2)}(\alpha,\beta)$ proceeds as follows. By definition we have

$$\begin{aligned} L_n^{(2)}&(\alpha,\beta) \\ &= \int_{\theta_{\bar n}}^{\pi} \left(\sin\frac{\theta}{2}\right)^{2\alpha+1} \left(\cos\frac{\theta}{2}\right)^{2\beta+1} \left|P_n^{(\alpha+1,\beta)}(\cos\theta)\right| d\theta, \end{aligned} \tag{2.38}$$

In the above integral, we replace θ by $\pi-\theta$. The result is

$$\begin{aligned} L_n^{(2)}&(\alpha,\beta) \\ &= \int_{0}^{\pi-\theta_{\bar n}} \left(\sin\frac{\theta}{2}\right)^{2\beta+1} \left(\cos\frac{\theta}{2}\right)^{2\alpha+1} \left|P_n^{(\beta,\alpha+1)}(\cos\theta)\right| d\theta, \end{aligned} \tag{2.39}$$

on account of the identity [**17**, p. 59]

$$P_n^{(\alpha,\beta)}(\cos\theta) = (-1)^n P_n^{(\beta,\alpha)}(-\cos\theta). \tag{2.40}$$

From (2.40), it also follows that there is a one-to-one relationship between the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\beta,\alpha)}$. Suppose the zeros of $P_n^{(\alpha,\beta)}(\cos\theta)$ are arranged in the order:

$$0 < \theta_{n,1}^{(\alpha,\beta)} < \theta_{n,2}^{(\alpha,\beta)} < \cdots < \theta_{n,n}^{(\alpha,\beta)} < \pi.$$

Since $\theta_{n,1}^{(\alpha,\beta)}$ is the smallest zero of $P_n^{(\alpha,\beta)}(\cos\theta)$, it follows that $\pi - \theta_{n,1}^{(\alpha,\beta)}$ is the largest zero of $P_n^{(\beta,\alpha)}(\cos\theta)$. Thus, $\pi - \theta_{n,1}^{(\alpha,\beta)} = \theta_{n,n}^{(\beta,\alpha)}$. In general, we have $\pi - \theta_{n,p}^{(\alpha,\beta)} = \theta_{n,n-p+1}^{(\beta,\alpha)}$, or equivalently

$$\theta_{n,p}^{(\alpha,\beta)} + \theta_{n,n-p+1}^{(\beta,\alpha)} = \pi \tag{2.41}$$

for $p = 1,\ldots,n$. Since $\theta_{\bar n} \equiv \theta_{n,\bar n}^{(\alpha+1,\beta)}$, (2.41) gives $\pi - \theta_{\bar n} = \theta_{n,n-\bar n+1}^{(\beta,\alpha+1)}$. Set $\bar m = n - \bar n + 1$ and $\theta_{\bar m} \equiv \theta_{n,\bar m}^{(\beta,\alpha+1)}$. Equation (2.39) now becomes

$$\begin{aligned} L_n^{(2)}&(\alpha,\beta) \\ &= \int_{0}^{\theta_{\bar m}} \left(\sin\frac{\theta}{2}\right)^{2\beta+1} \left(\cos\frac{\theta}{2}\right)^{2\alpha+1} \left|P_n^{(\beta,\alpha+1)}(\cos\theta)\right| d\theta. \end{aligned} \tag{2.42}$$

Despite the fact that $\theta_{\bar{m}} > \pi/2$ while $\theta_n < \pi/2$ for large values of n, computation of $L_n^{(2)}(\alpha, \beta)$ proceeds in the same manner as $L_n^{(1)}(\alpha, \beta)$. A brief summary of this calculuation is given in §5. Asymptotic expansion (1.14) is obtained by adding the results for $L_n^{(1)}(\alpha, \beta)$ and $L_n^{(2)}(\alpha, \beta)$ together. This is done in §6. Since the derivations of expansions (1.19) and (1.21) are similar to that of (1.14), they will not be presented here.

3. Calculation of $I_k^{(1)}$. From (2.11) and (2.13) we have

$$I_k^{(1)} = \int_{\theta_k}^{\theta_{k+1}} A_0^*(\theta) J_{\alpha+1}(N\theta)\, d\theta \tag{3.1}$$

where

$$A_0^*(\theta) = \left(\frac{\theta}{2}\cot\frac{\theta}{2}\right)^{1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha}\left(\cos\frac{\theta}{2}\right)^{\beta}. \tag{3.2}$$

As in (2.16), integration by parts gives

$$I_k^{(1)} = F_1 + F_2, \tag{3.3}$$

for $k \neq 0$, where

$$F_1 = \frac{1}{N}\left[A_0^*(\theta_k) J_\alpha(N\theta_k) - A_0^*(\theta_{k+1}) J_\alpha(N\theta_{k+1})\right] \tag{3.4}$$

and

$$F_2 = \frac{1}{N}\int_{\theta_k}^{\theta_{k+1}} \left[A_0^*(\theta)\theta^{\alpha}\right]' \theta^{-\alpha} J_\alpha(N\theta)\, d\theta. \tag{3.5}$$

In terms of the function $g_{\alpha\beta}(\theta)$ defined in (2.33) (and also used in [**9**, (30)]), (3.4) becomes

$$\begin{aligned} F_1 = \frac{1}{2^{\alpha} N^{\alpha+1}}\Big[& g_{\alpha\beta}(\theta_k)(N\theta_k)^{\alpha} J_\alpha(N\theta_k) \\ & - g_{\alpha\beta}(\theta_{k+1})(N\theta_{k+1})^{\alpha} J_\alpha(N\theta_{k+1})\Big]. \end{aligned} \tag{3.6}$$

By (2.20), Taylor's theorem gives

$$J_\alpha(N\theta_k) = J_\alpha(j_{\alpha+1,k}) + J_\alpha'(j_{\alpha+1,k}) O\left(\frac{k}{N^2}\right) + J_\alpha''(\xi) O\left(\frac{k^2}{N^4}\right), \tag{3.7}$$

where ξ lies between $N\theta_k$ and $j_{\alpha+1,k}$. Since $j_{\alpha+1,k} \sim \pi k$ by (2.19), $\xi = O(k)$ and hence $J_\alpha''(\xi) = O(k^{-1/2})$. From the identity $zJ_\alpha'(z) + zJ_{\alpha+1}(z) = \alpha J_\alpha(z)$, we have

$$J_\alpha'(j_{\alpha+1,k+1}) = \frac{\alpha}{j_{\alpha+1,k+1}} J_\alpha(j_{\alpha+1,k}) = O(k^{-3/2}). \tag{3.8}$$

Therefore, (3.7) becomes

$$(3.9)\qquad J_\alpha(N\theta_k) = J_\alpha(j_{\alpha+1,k}) + O\left(\frac{1}{k^{1/2}N^2}\right) + O\left(\frac{k^{3/2}}{N^4}\right).$$

Since (2.19) and (2.20) imply

$$(3.10)\qquad (N\theta_k)^\alpha = j^\alpha_{\alpha+1,k} + O\left(\frac{k^\alpha}{N^2}\right),$$

it follows from (3.6) that

$$(3.11)\qquad F_1 = \frac{(-1)^k}{2^\alpha N^{\alpha+1}}\left[g_{\alpha\beta}(\theta_k)M_k(\alpha) + g_{\alpha\beta}(\theta_{k+1})M_{k+1}(\alpha)\right] + O\left(\frac{k^{\alpha-1/2}}{N^{\alpha+3}}\right),$$

where $M_k(\alpha)$ is as given in (1.11). In deriving (3.11), we have also used the fact that $g_{\alpha\beta}(\theta) = 1 + O(\theta^2)$ for $0 \le \theta \le \pi/2$.

We now evaluate F_2 given in (3.5). From (3.2), we obtain

$$(3.12)\qquad \left[A_0^*(\theta)\theta^\alpha\right]' = \frac{\alpha}{2^{\alpha-1}}\theta^{2\alpha-1} + O(\theta^{2\alpha+1}).$$

Inserting (3.12) in (3.5) gives

$$(3.13)\qquad F_2 = \frac{\alpha}{2^{\alpha-1}N^{\alpha+1}}Q + R,$$

where

$$(3.14)\qquad Q = \int_{N\theta_k}^{N\theta_{k+1}} y^{\alpha-1}J_\alpha(y)\,dy$$

and

$$(3.15)\qquad R = \frac{1}{N^{\alpha+3}}O\left\{\int_{N\theta_k}^{N\theta_{k+1}} y^{\alpha+1}|J_\alpha(y)|\,dy\right\}.$$

In view of (2.19), (2.20) and the behavior of the Bessel function, it is easily seen that the integral in (3.15) is equal to

$$\left|\int_{j_k}^{j_{k+1}} y^{\alpha+1}J_\alpha(y)\,dy\right| + O\left(\frac{k^{\alpha+3/2}}{N^2}\right).$$

Since $x^{\alpha+1}J_\alpha(x) = [x^{\alpha+1}J_{\alpha+1}(x)]'$, the last integral is zero. Thus

$$(3.16)\qquad R = O\left(k^{\alpha+3/2}/N^{\alpha+5}\right).$$

By the same argument, we also have

$$(3.17)\qquad Q = \int_{j_k}^{j_{k+1}} y^{\alpha-1}J_\alpha(y)\,dy + O\left(\frac{k^{\alpha-1/2}}{N^2}\right).$$

Now substitute (3.16) and (3.17) in (3.13), and add the resulting equation to (3.11). From (3.3) it follows that the sum is

$$(3.18)\qquad I_k^{(1)} = \frac{(-1)^k}{2^\alpha N^{\alpha+1}}\left[g_{\alpha\beta}(\theta_k)M_k(\alpha) + g_{\alpha\beta}(\theta_{k+1})M_{k+1}(\alpha)\right] + \frac{\alpha}{2^{\alpha-1}N^{\alpha+1}}\int_{j_k}^{j_{k+1}} y^{\alpha-1}J_\alpha(y)\,dy + O\left(\frac{k^{\alpha-1/2}}{N^{\alpha+3}}\right)$$

for $k \neq 0$.

Note that (3.2) gives

$$(3.19)\qquad A_0^*(\theta) = \left(\frac{\theta}{2}\right)^\alpha\left[1 + O(\theta^2)\right].$$

Inserting this in (3.1), we get

$$(3.20)\quad I_0^{(1)} = \frac{1}{2^\alpha}\int_0^{\theta_1} \theta^\alpha J_{\alpha+1}(N\theta)\,d\theta + O\left\{\int_0^{\theta_1} \theta^{\alpha+2}J_{\alpha+1}(N\theta)\,d\theta\right\}.$$

Since (2.20) implies $\theta_1 = O(N^{-1})$, the second integral in (3.20) is $O(N^{-\alpha-3})$. By a similar argument, it can easily be shown that the first integral is equal to

$$\frac{1}{N^{\alpha+1}}\int_0^{j_1} y^\alpha J_{\alpha+1}(y)\,dy + O\left(\frac{1}{N^{\alpha+3}}\right).$$

Thus we have

$$(3.21)\qquad I_0^{(1)} = \frac{1}{2^\alpha N^{\alpha+1}}\int_0^{j_1} y^\alpha J_{\alpha+1}(y)\,dy + O\left(\frac{1}{N^{\alpha+3}}\right).$$

From (2.25), (3.18) and (3.21), it now follows that

$$(3.22)\quad \sum_{k=0}^{\bar{n}-1}(-1)^k I_k^{(1)} = \frac{1}{2^\alpha N^{\alpha+1}}\sum_{k=1}^{\bar{n}-1}\left[g_{\alpha\beta}(\theta_k)M_k(\alpha) + g_{\alpha\beta}(\theta_{k+1})M_{k+1}(\alpha)\right] + \frac{1}{2^\alpha N^{\alpha+1}}\int_0^{j_1} y^\alpha J_{\alpha+1}(y)\,dy + \frac{\alpha}{2^{\alpha-1}N^{\alpha+1}}R_n + O\left(\frac{1}{N^{\alpha+3}}\right) + O(N^{-5/2})$$

where R_n is given in (2.32). The sum on the right-hand side of (3.22) can be written as

$$(3.23)\qquad \sum_{k=1}^{\bar{n}-1}\left[g_{\alpha\beta}(\theta_k)M_k(\alpha) + g_{\alpha\beta}(\theta_{k+1})M_{k+1}(\alpha)\right] = g_{\alpha\beta}(\theta_1)M_1(\alpha) + g_{\alpha\beta}(\theta_{\bar{n}})M_{\bar{n}}(\alpha) + 2S_n$$

with S_n being given in (2.31). Since $g_{\alpha\beta}(\theta) = 1 + O(\theta^2)$ and $\theta_1 = j_1/N + O(N^{-3})$, we obtain

$$(3.24) \qquad g_{\alpha\beta}(\theta_1)M_1(\alpha) = M_1(\alpha) + O(N^{-2}).$$

Inserting (2.19) in the asymptotic expansion of $J_\alpha(x)$ gives

$$(3.25) \quad M_k(\alpha) = \sqrt{\frac{2}{\pi}}\left[j_{\alpha+1,k}^{\alpha-1/2} - \tfrac{1}{4}\left(\alpha+\tfrac{1}{2}\right)\left(\alpha+\tfrac{3}{2}\right)j_{\alpha+1,k}^{\alpha-5/2} + O\left(j_{\alpha+1,k}^{\alpha-7/2}\right)\right];$$

cf. (1.11). This in particular gives

$$(3.26) \qquad M_{\bar{n}}(\alpha) = \left(\frac{\pi}{2}\right)^{\alpha-1} n^{\alpha-1/2} + O(n^{\alpha-3/2}).$$

Since $g_{\alpha\beta}(\theta)$ is analytic in $[0,\pi)$ (see (2.33)), we may expand it at $\theta = \pi/2$. Thus,

$$(3.27) \quad g_{\alpha\beta}(\theta_{\bar{n}}) = g_{\alpha\beta}\left(\frac{\pi}{2} + O(n^{-1})\right) = \left(\frac{\pi}{4}\right)^{1/2-\alpha} 2^{-(\alpha+\beta)/2} + O(n^{-1}),$$

and consequently

$$(3.28) \qquad g_{\alpha\beta}(\theta_{\bar{n}})M_{\bar{n}}(\alpha) = \sqrt{\frac{2}{\pi}}\, 2^{(\alpha-\beta-1)/2} n^{\alpha-1/2} + O(n^{\alpha-3/2}).$$

A combination of (3.22), (3.23), (3.24) and (3.28) yields the desired result (2.30).

4. Proofs of Lemmas 3 and 4. We first give the proof of Lemma 4, which is considerably simpler than that of Lemma 3. Using the identity $[x^{\alpha+1}J_{\alpha+1}(x)]' = x^{\alpha+1}J_\alpha(x)$, we have by integration by parts

$$\int_{j_k}^{j_{k+1}} x^{\alpha-1}J_\alpha(x)\,dx = 2\int_{j_k}^{j_{k+1}} x^{\alpha-2}J_{\alpha+1}(x)\,dx \qquad (k \neq 0),$$

the integrated term vanishing since j_k and j_{k+1} are zeros of $J_{\alpha+1}(x)$. In view of the asymptotic behavior of $J_\alpha(x)$, the last integral is $O(k^{\alpha-5/2})$. Thus

$$\sum_{k=\bar{n}}^{\infty} (-1)^k \int_{j_k}^{j_{k+1}} x^{\alpha-1}J_\alpha(x)\,dx = O(n^{\alpha-3/2}).$$

This establishes Lemma 4.

Now we begin the proof of Lemma 3. By using (2.19) and (3.25), it can be shown that for $k \neq 1$,

$$(4.1) \quad M_k(\alpha) = \frac{2^{1/2}}{\pi^{3/2}}\int_{j_{k-1}}^{j_k} x^{\alpha-1/2}\,dx + \frac{\alpha-1/2}{\sqrt{2\pi}}\int_{j_{k-1}}^{j_k} x^{\alpha-3/2}\,dx + O(k^{\alpha-5/2}).$$

Thus

$$(4.2)\qquad \sum_{k=\bar{n}}^{\infty}\left[M_k(\alpha)-\frac{2^{1/2}}{\pi^{3/2}}\int_{j_{k-1}}^{j_k} x^{\alpha-1/2}\,dx - \frac{\alpha-1/2}{\sqrt{2\pi}}\int_{j_{k-1}}^{j_k} x^{\alpha-3/2}\,dx\right]=O(n^{\alpha-3/2}).$$

Set

$$(4.3)\qquad S_{n,1}=\sum_{k=2}^{\infty}\left[M_k(\alpha)-\frac{2^{1/2}}{\pi^{3/2}}\int_{j_{k-1}}^{j_k} x^{\alpha-1/2}\,dx - \frac{\alpha-1/2}{\sqrt{2\pi}}\int_{j_{k-1}}^{j_k} x^{\alpha-3/2}\,dx\right]$$

$$(4.4)\qquad S_{n,2}=\frac{2^{1/2}}{\pi^{3/2}}\sum_{k=2}^{\bar{n}-1}\int_{j_{k-1}}^{j_k} g_{\alpha\beta}(\theta_k)x^{\alpha-1/2}\,dx$$

and

$$(4.5)\qquad S_{n,3}=\frac{\alpha-1/2}{(2\pi)^{1/2}}\sum_{k=2}^{\bar{n}-1}\int_{j_{k-1}}^{j_k} g_{\alpha\beta}(\theta_k)x^{\alpha-3/2}\,dx.$$

Since $g_{\alpha\beta}(\theta_k)=1+O(k^2/N^2)$, we can now express S_n as

$$(4.6)\qquad S_n=S_{n,1}+S_{n,2}+S_{n,3}+O(n^{\alpha-3/2}).$$

Note that the series $S_{n,1}$ converges absolutely in view of (4.1), and is a constant independent of n.

(A) *Evaluation of* $S_{n,2}$. Make the change of variable $x=\theta N$ in (4.4), and write

$$(4.7)\qquad g_{\alpha\beta}(\theta_k)=g_{\alpha\beta}(\theta)+g'_{\alpha\beta}(\theta)(\theta_k-\theta)+\tfrac{1}{2}g''_{\alpha\beta}(\xi)(\theta_k-\theta)^2,$$

where ξ is between θ_k and θ. Since $\theta\in[j_{k-1}/N, j_k/N]$, by (2.20) we have $\theta_k-\theta=O(1/N)$. Furthermore, since $g_{\alpha\beta}(\theta)$ is analytic in $[0,\pi/2]$, $g''_{\alpha\beta}(\xi)$ is bounded. Thus the remainder term $\frac{1}{2}g''_{\alpha\beta}(\xi)(\theta_k-\theta)^2$ is $O(n^{-2})$, and contributes to $S_{n,2}$ a term of order

$$O\left(n^{-2}\sum_{k=2}^{\bar{n}-1}\int_{j_{k-1}}^{j_k} x^{\alpha-1/2}\,dx\right)=O\left(n^{-2}\sum_{k=2}^{\bar{n}-1}k^{\alpha-1/2}\right)=O(n^{\alpha-3/2}).$$

Inserting (4.7) in (4.4) then leads to

$$(4.8)\qquad S_{n,2}=\frac{2^{1/2}}{\pi^{3/2}}\left[S_{n,2}^{(1)}+S_{n,2}^{(2)}+S_{n,2}^{(3)}\right]+O(n^{\alpha-3/2}),$$

where

$$(4.9)\qquad S_{n,2}^{(1)}=N^{\alpha+1/2}\sum_{k=2}^{\bar{n}-1}\int_{j_{k-1}/N}^{j_k} g_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta$$

$$(4.10)\qquad S_{n,2}^{(2)} = N^{\alpha+1/2} \sum_{n=2}^{n-1} \theta_k \int_{j_{k-1}/N}^{j_k/N} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta$$

and

$$(4.11)\qquad S_{n,2}^{(3)} = -N^{\alpha+1/2} \sum_{k=2}^{\bar{n}-1} \int_{j_{k-1}/N}^{j_k/N} g'_{\alpha\beta}(\theta)\theta^{\alpha+1/2}\,d\theta.$$

Clearly, $S_{n,2}^{(1)}$ can be written as

$$(4.12)\qquad S_{n,2}^{(1)} = N^{\alpha+1/2}\left[\int_0^{\pi/2} - \int_0^{j_1/N} - \int_{j_{\bar{n}-1/N}}^{\pi/2}\right] g_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta,$$

which is exactly the second term on the right-hand side of (2.34) (except for the constant factor $\sqrt{2}/\pi^{3/2}$).

In (4.11), we let $g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2} = f(\theta)$. With an appropriate change of variable, each integral there can be written as

$$(4.13)\qquad \int_{j_{k-1}/N}^{j_k/N} \theta f(\theta)\,d\theta = \frac{1}{2N}(j_k + j_{k-1})\int_{j_{k-1}/N}^{j_k/N} f(\theta)\,d\theta + \int_0^{(j_k-j_{k-1})/2N} \theta\left[f\left\{\frac{1}{2N}(j_k+j_{k-1})+\theta\right\} - f\left\{\frac{1}{2N}(j_k+j_{k-1})-\theta\right\}\right]d\theta.$$

By the mean value theorem the second integral on the right is equal to

$$(4.14)\qquad \int_0^{(j_k-j_{k-1})/2N} 2\theta^2 f'(\xi_\theta)\,d\theta$$

for some ξ_θ satisfying

$$\frac{1}{2N}(j_k+j_{k-1}) - \theta < \xi_\theta < \frac{1}{2N}(j_k+j_{k-1}) + \theta.$$

Since $0 \le \theta \le (j_k - j_{k-1})/2N$ in (4.14), we have $j_{k-1}/N \le \xi_\theta \le j_k/N$ so that, by (2.19), $\xi_\theta = O(k/N)$. From (2.19) it also follows that $(j_k - j_{k-1})/2N = O(1/N)$. Note that the implied constants in the last two O-terms are independent of k. Since $f'(\theta) = O(\theta^{\alpha-1/2})$, a combination of the above results shows that the integral in (4.14) is $O(k^{\alpha-1/2}/N^{\alpha+5/2})$. By using this estimate and (4.13), the addition of (4.10) and (4.11) yields

$$(4.15)\qquad S_{n,2}^{(2)} + S_{n,2}^{(3)} = N^{\alpha+1/2}\sum_{k=2}^{\bar{n}-1}\left\{\left[\theta_k - \frac{1}{2N}(j_k+j_{k-1})\right]\int_{j_{k-1}/N}^{j_k/N} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta\right\} + O(N^{\alpha-3/2}),$$

where we have also made use of (2.25). From (2.19) and (2.20), it now follows that for $k \geq 1$,

$$(4.16) \qquad \theta_k - \frac{1}{2N}(j_k + j_{k-1}) = \frac{\pi}{2N} + O\left(\frac{1}{k^2 N}\right) + O\left(\frac{k}{N^3}\right).$$

Since $g_{\alpha\beta}(\theta) = 1 + O(\theta^2)$ for $0 \leq \theta \leq \pi/2$, each integral under the summation sign in (4.15) is $O(k^{\alpha+1/2}/N^{\alpha+3/2})$, where again the implied constant in the last O-term is independent of k. Since $-\frac{1}{2} < \alpha < \frac{1}{2}$, a combination of (4.15) with (4.16), (2.25) and the last estimate yields

$$(4.17) \qquad S_{n,2}^{(2)} + S_{n,2}^{(3)} = \frac{\pi}{2} N^{\alpha-1/2} \sum_{k=2}^{\bar{n}-1} \int_{j_{k-1}/N}^{j_k/N} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta$$
$$+ O(N^{\alpha-3/2}).$$

The sum in (4.17) may now be written as

$$(4.18) \qquad \int_0^{\pi/2} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta - \int_0^{j_1/N} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta$$
$$- \int_{j_{\bar{n}-1}/N}^{\pi/2} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta.$$

The second integral in (4.18) is $O(N^{-\alpha-3/2})$ and, since $j_{\bar{n}-1}/N = \pi/2 + O(N^{-1})$, the third integral is $O(N^{-1})$. Coupling (4.17) and (4.18) results in

$$(4.19) \quad S_{n,2}^{(2)} + S_{n,2}^{(3)} = \frac{\pi}{2} N^{\alpha-1/2} \int_0^{\pi/2} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta + O(N^{\alpha-3/2}).$$

Finally, using (4.8), the addition of (4.12) and (4.19) gives

$$(4.20) \quad S_{n,2} = \frac{2^{1/2}}{\pi^{3/2}} N^{\alpha+1/2} \left[\int_0^{\pi/2} - \int_0^{j_1/N} - \int_{j_{\bar{n}-1/N}}^{\pi/2}\right] g_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta$$
$$+ \frac{1}{\sqrt{2\pi}} N^{\alpha-1/2} \int_0^{\pi/2} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta + O(n^{\alpha-3/2}).$$

(B) *Evaluation of* $S_{n,3}$. The analysis here parallels that given for $S_{n,2}$, and is in fact simpler. We first make the change of variable $x = \theta N$ in (4.5), and then substitute (4.7) in (4.5). The result is

$$(4.21) \qquad S_{n,3} = \frac{\alpha - 1/2}{(2\pi)^{1/2}} N^{\alpha-1/2}\left[S_{n,3}^{(1)} + S_{n,3}^{(2)}\right] + O(n^{-2}),$$

where

$$(4.22) \qquad S_{n,3}^{(1)} = \int_{j_1/N}^{j_{\bar{n}-1}/N} g_{\alpha\beta}(\theta)\theta^{\alpha-3/2}\,d\theta$$

and

$$(4.23) \qquad S_{n,3}^{(2)} = \sum_{k=2}^{\bar{n}-1} \int_{j_{k-1}/N}^{j_k/N} g'_{\alpha\beta}(\theta)(\theta_k - \theta)\theta^{\alpha-3/2}\,d\theta;$$

cf. (4.8). To the integral in (4.22), we apply integration by parts. By using (3.27) and the fact that $g_{\alpha\beta}(\theta) = 1 + O(\theta^2)$ as $\theta \to 0^+$, it can be shown that the integrated term is equal to

$$\frac{1}{\alpha - 1/2}\left[2^{(\alpha-\beta-1)/2} - (j_1/N)^{\alpha-1/2} + O(n^{-1})\right].$$

The other term can be replaced by

$$\frac{1}{\alpha - 1/2}\int_0^{\pi/2} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta + O(n^{-1}),$$

as was done in the case of $S_{n,2}$, cf. (4.18). Thus

$$(4.24)\quad S_{n,3}^{(1)} = \frac{1}{\alpha - 1/2}\left[2^{(\alpha-\beta-1)/2} - (j_1/N)^{\alpha-1/2} - \int_0^{\pi/2} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta\right] + O(n^{-1}).$$

Since $g'_{\alpha\beta} = O(\theta)$ and $\theta_k - \theta = O(n^{-1})$ for $j_{k-1}/N \le \theta \le j_k/N$, the $(k-1)$th integral in (4.23) is $O(k^{\alpha-1/2}/N^{\alpha+3/2})$. This implies

$$(4.25)\qquad S_{n,3}^{(2)} = O(N^{-1}).$$

The sum of $S_{n,3}^{(1)}$ and $S_{n,3}^{(2)}$ gives, using (4.21),

$$(4.26)\quad S_{n,3} = -\frac{j_1^{\alpha-1/2}}{\sqrt{2\pi}} + \frac{N^{\alpha-1/2}}{\sqrt{2\pi}}\left[2^{(\alpha-\beta-1)/2} - \int_0^{\pi/2} g'_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta\right] + O(n^{\alpha-3/2}),$$

thus completing the evaluation of $S_{n,3}$.

Observing that the sum $S_{n,1}$ in (4.3) can be written as

$$(4.27)\qquad S_{n,1} = B_\alpha^{(1)} + \frac{j_1^{\alpha-1/2}}{\sqrt{2\pi}},$$

where $B_\alpha^{(1)}$ is given in (2.35), the result in (2.34) now follows immediately from (4.6), (4.20) and (4.26). This proves Lemma 3.

5. Evaluation of $L_n^{(2)}(\alpha,\beta)$. In what follows we shall use the same notation as we did in the evaluation of $L_n^{(1)}(\alpha,\beta)$. There should be no confusion resulting from this, when care is taken to distinguish the zeros of $P_n^{(\alpha+1,\beta)}(\cos\theta)$ and $P_n^{(\beta,\alpha+1)}(\cos\theta)$. Thus we again let $\theta_0 \equiv 0$, θ_k denote the kth positive zero of $P_n^{(\beta,\alpha+1)}(\cos\theta)$, and

$$(5.1)\quad I_k = \int_{\theta_k}^{\theta_{k+1}} \left(\sin\frac{\theta}{2}\right)^{2\beta+1}\left(\cos\frac{\theta}{2}\right)^{2\alpha+1} P_n^{(\beta,\alpha+1)}(\cos\theta)\,d\theta.$$

Since $\cos 0 = 1$ and

$$P_n^{(\beta,\alpha+1)}(1) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)} > 0, \tag{5.2}$$

we have from (2.42)

$$L_n^{(2)}(\alpha,\beta) = \sum_{k=0}^{\overline{m}-1} (-1)^k I_k. \tag{5.3}$$

As before, we now use the asymptotic expansion of $P_n^{(\beta,\alpha+1)}(\cos\theta)$ for large n. The result corresponding to (2.29) is

$$L_n^{(2)}(\alpha,\beta) = \frac{\Gamma(n+\beta+1)}{n!N^\beta}\left[\sum_{k=0}^{\overline{m}-1} (-1)^k I_k^{(1)} + O(n^{-5/2})\right], \tag{5.4}$$

where

$$\begin{aligned} \sum_{k=0}^{\overline{m}-1} (-1)^k I_k^{(1)} &= \frac{1}{2^{\beta+1}N^{\beta+2}}\left\{\int_0^{j_{\beta,1}} y^{\beta+1}J_\beta(y)\,dy + \hat{M}_1(\beta)\right. \\ &\qquad \left. + \sqrt{\frac{2}{\pi}}\, 2^{(\beta-\alpha+1)/2} n^{\beta+1/2} + 2\hat{S}_n + O(n^{\beta-1/2})\right\}, \end{aligned} \tag{5.5}$$

$$\hat{S}_n = \sum_{k=2}^{\overline{m}-1} \hat{g}_{\alpha\beta}(\theta_k)\hat{M}_k(\beta), \tag{5.6}$$

$\hat{M}_k(\beta)$ being as given in (1.17), and

$$\hat{g}_{\alpha\beta}(\theta) = \left(\frac{2}{\theta}\tan\frac{\theta}{2}\right)^{1/2}\left(\frac{2}{\theta}\sin\frac{\theta}{2}\right)^{\beta}\left(\cos\frac{\theta}{2}\right)^{\alpha}. \tag{5.7}$$

Note that $\hat{R}_n$, the contribution to $L_n^{(2)}(\alpha,\beta)$ corresponding to R_n in (2.32), is absent in (5.5), because

$$\int_{j_{\beta,k}}^{j_{\beta,k+1}} y^\beta J_{\beta-1}(y)\,dy = \int_{j_{\beta,k}}^{j_{\beta,k+1}} \left[y^\beta J_\beta(y)\right]'dy = 0.$$

An analogue of Lemma 3 is the following approximation:

$$\begin{aligned} \hat{S}_n &= \frac{\sqrt{2}}{\pi^{3/2}} N^{\beta+3/2}\left(\int_0^{\pi/2} - \int_0^{j_{\beta,1}/N} - \int_{j_{\beta,\overline{m}-1}/N}^{\pi/2}\right)\hat{g}_{\alpha\beta}(\theta)\theta^{\beta+1/2}\,d\theta \\ &\quad + \sqrt{\frac{2}{\pi}}\, 2^{(\beta-\alpha-1)/2} N^{\beta+1/2} + 2^\beta D_\beta - \hat{M}_1(\beta) \\ &\quad + \frac{2^{1/2}}{\pi^{3/2}(\beta+\frac{3}{2})} j_{\beta,1}^{\beta+3/2} + O(N^{\beta-1/2}), \end{aligned} \tag{5.8}$$

where D_β is given in (1.16). Note that the sum in D_β starts with $k = 1$, whereas the sum in $B_\alpha^{(1)}$, the corresponding constant in S_n given in (2.35), starts with $k = 2$. Since $\hat{g}_{\alpha\beta}(\theta) = 1 + O(\theta^2)$, the term involving the second integral on the right of (5.8) cancels with the third from the last term in the same equation. Thus

$$(5.9)\qquad \hat{S}_n = \frac{\sqrt{2}}{\pi^{3/2}} N^{\beta+3/2}\left(\int_0^{\pi/2} - \int_{j_{\beta,\bar{m}-1/N}}^{\pi/2}\right)\hat{g}_{\alpha\beta}(\theta)\theta^{\beta+1/2}\,d\theta$$
$$+\sqrt{\frac{2}{\pi}}\,2^{(\beta-\alpha-1)/2}N^{\beta+1/2} + 2^\beta D_\beta - \hat{M}_1(\beta)$$
$$+O(N^{\beta-1/2}).$$

Now we insert (5.9) in (5.5), and observe that

$$(5.10)\qquad \int_0^{j_{\beta,1}} x^{\beta+1}J_\beta(x)\,dx = \hat{M}_1(\beta)$$

which follows from (1.17) and the identity $[x^{\beta+1}J_{\beta+1}(x)]' = x^{\beta+1}J_\beta(x)$. The resulting expression from (5.5) coupled with (5.4) gives

$$(5.11)\qquad \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}L_n^{(2)}(\alpha,\beta)$$
$$= \frac{2^{-1-\beta}}{\Gamma(\alpha+1)}\left[n^{\alpha-\beta-1} + \frac{1}{2}(\alpha+\beta+2)(\alpha-\beta-1)n^{\alpha-\beta-2}\right]$$
$$\cdot\left\{2^{\beta+1}D_\beta + 2\sqrt{\frac{2}{\pi}}\,2^{(\beta-\alpha+1)/2}n^{\beta+1/2}\right.$$
$$+\frac{2^{3/2}}{\pi^{3/2}}N^{\beta+3/2}\left(\int_0^{\pi/2} - \int_{j_{\beta,\bar{m}-1/N}}^{\pi/2}\right)$$
$$\left.\cdot 2^{\beta+1/2}\left(\sin\frac{\theta}{2}\right)^{\beta+1/2}\left(\cos\frac{\theta}{2}\right)^{\alpha-1/2}d\theta\right\}$$
$$+O(n^{\alpha-3/2}),$$

where use has been made of (5.7) and the asymptotic expansion

$$(5.12)\qquad \frac{2^{-1-\beta}\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+1)N^{2\beta+2}}$$
$$= \frac{2^{-1-\beta}}{\Gamma(\alpha+1)}n^{\alpha-\beta-1}\left[1 + \frac{1}{2n}(\alpha+\beta+2)(\alpha-\beta-1) + O(n^{-2})\right].$$

6. The sum of $L_n^{(1)}(\alpha,\beta)$ and $L_n^{(2)}(\alpha,\beta)$. From (2.29), (2.30) and the fact that

$$(6.1)\qquad \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)}\frac{\Gamma(n+\alpha+2)}{n!N^{2\alpha+2}}=1+O(n^{-2}),$$

we have

$$(6.2)\qquad \begin{aligned}&\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}L_n^{(1)}(\alpha,\beta)\\ &\quad=\frac{1}{2^\alpha\Gamma(\alpha+1)}\left\{\int_0^{j_1}y^\alpha J_{\alpha+1}(y)\,dy+M_1(\alpha)\right.\\ &\qquad\left.+\sqrt{\frac{2}{\pi}}\,2^{(\alpha-\beta-1)/2}n^{\alpha-1/2}+2S_n+2\alpha R_n\right\}\\ &\quad+O(N^{\alpha-3/2}).\end{aligned}$$

Since $g_{\alpha\beta}(\theta)=1+O(\theta^2)$, the second integral on the right-hand side of (2.34) can be written as

$$(6.3)\qquad \int_0^{j_1/N}g_{\alpha\beta}(\theta)\theta^{\alpha-1/2}\,d\theta=\frac{1}{N^{\alpha+1/2}}\int_0^{j_1}x^{\alpha-1/2}\,dx+O(N^{-\alpha-5/2}).$$

Now write (2.35) as

$$(6.4)\qquad \begin{aligned}B_\alpha^{(1)}=&\sum_{k=1}^{\infty}\left[M_k(\alpha)-\frac{\sqrt{2}}{\pi^{3/2}}\int_{j_{k-1}}^{j_k}x^{\alpha-1/2}\,dx\right]\\ &-M_1(\alpha)+\frac{\sqrt{2}}{\pi^{3/2}}\int_0^{j_1}x^{\alpha-1/2}\,dx,\end{aligned}$$

where we have used the fact that $j_0=0$. Combining (6.1), (2.34), (2.36), (6.2) and (6.3) gives

$$(6.5)\qquad \begin{aligned}&\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}L_n^{(1)}(\alpha,\beta)\\ &\quad=B_\alpha+\frac{1}{2^\alpha\Gamma(\alpha+1)}\\ &\qquad\cdot\left\{\frac{2^{\alpha+1}}{\pi^{3/2}}N^{\alpha+1/2}\int_0^{\pi/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}d\theta\right.\\ &\qquad+\frac{2^{3/2}}{\pi^{3/2}}2^{(\alpha-\beta-1)/2}n^{\alpha-1/2}\\ &\qquad\left.-\frac{2^{\alpha+1}}{\pi^{3/2}}N^{\alpha+1/2}\int_{j_{\bar{n}-1}/N}^{\pi/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}d\theta\right\}\\ &\quad+O(n^{\alpha-3/2}),\end{aligned}$$

where B_α is given in (1.10) and we have used (2.33) and (2.6).

We now note that

$$(6.6)\qquad \left[n^{\alpha-\beta-1}+\tfrac{1}{2}(\alpha+\beta+2)(\alpha-\beta-1)n^{\alpha-\beta-2}\right]N^{\beta+3/2} = n^{\alpha+1/2}+C_{\alpha\beta}n^{\alpha-1/2}+O(n^{\alpha-3/2}),$$

where $C_{\alpha\beta}$ is given in (1.15). Upon adding (6.5) and (5.11) and using (6.6), we obtain

$$(6.7)\qquad L_n(\alpha,\beta)=I_1^*+I_2^*+I_3^*+B_\alpha+\frac{1}{\Gamma(\alpha+1)}D_\beta + O(n^{\alpha-3/2}),$$

where B_α and D_β are given in (1.10) and (1.16) respectively and where

$$(6.8)\qquad I_1^*=\frac{2}{\pi^{3/2}\Gamma(\alpha+1)}\left(n^{\alpha+1/2}+C_{\alpha\beta}n^{\alpha-1/2}\right)\cdot\left\{\int_0^{\pi/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}d\theta + \int_0^{\pi/2}\left(\sin\frac{\theta}{2}\right)^{\beta+1/2}\left(\cos\frac{\theta}{2}\right)^{\alpha-1/2}d\theta\right\} + O(n^{\alpha-3/2}),$$

$$(6.9)\qquad I_2^*=-\frac{2n^{\alpha+1/2}}{\Gamma(\alpha+1)}\left[\int_{j_{\bar{n}-1}/N}^{\pi/2}\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}d\theta + \int_{j_{\beta,\bar{m}-1}/N}^{\pi/2}\left(\sin\frac{\theta}{2}\right)^{\beta+1/2}\left(\cos\frac{\theta}{2}\right)^{\alpha-1/2}d\theta\right],$$

and

$$(6.10)\qquad I_3^*=\frac{4}{\Gamma(\alpha+1)}\sqrt{\frac{2}{\pi}}\,2^{-(\alpha+\beta+1)/2}n^{\alpha-1/2}.$$

By letting $\theta=\pi-\phi$ in the second integral in (6.8), the two integrals there can be combined into the single integral

$$(6.11)\qquad \int_0^{\pi}\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}d\theta=\frac{\Gamma(\frac{1}{2}\alpha+\frac{1}{4})\Gamma(\frac{1}{2}\beta+\frac{3}{4})}{\Gamma(\frac{1}{2}(\alpha+\beta)+1)}.$$

Using the above result in (6.8) yields

$$(6.12)\qquad I_1^*=A_{\alpha\beta}n^{\alpha+1/2}+C_{\alpha\beta}A_{\alpha\beta}n^{\alpha-1/2}+O(n^{\alpha-3/2}),$$

where $A_{\alpha\beta}$ is given in (1.6). Making the change of variable $\theta=\pi-\phi$ in the second integral in (6.9), the two integrals there can also be combined into the single integral

$$(6.13)\qquad \int_{j_{\bar{n}-1}/N}^{\pi-j_{\beta,\bar{m}-1}/N}\left(\sin\frac{\theta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}d\theta.$$

Note that by (2.20) and (2.41), we have

$$\pi-\frac{j_{\beta,\bar{m}-1}}{N}=\theta_{\bar{n}+1}+O(n^{-2}),\qquad \frac{j_{\bar{n}-1}}{N}=\theta_{\bar{n}-1}+O(n^{-2}).$$

From (2.19) and (2.20), it also follows that

$$\theta_{\bar{n}+1}=\frac{\pi}{2}+O(n^{-1}),\qquad \theta_{\bar{n}-1}=\frac{\pi}{2}+O(n^{-1}),$$

and

$$\theta_{\bar{n}+1}-\theta_{\bar{n}-1}=\frac{2\pi}{n}+O(n^{-2}).$$

Expanding the integrand in (6.13) about $\pi/2$ and using the above results, the integral in (6.13) can be shown to be

$$\frac{2\pi}{n}2^{-(\alpha+\beta)/2}+O(n^{-2}).$$

This result coupled with (6.9) gives

$$(6.14)\quad I_2^*=-\frac{4}{\Gamma(\alpha+1)}\sqrt{\frac{2}{\pi}}\,2^{-(\alpha+\beta+1)/2}n^{\alpha-1/2}+O(n^{\alpha-3/2}).$$

A combination of (6.7), (6.10), (6.12) and (6.14) yields our final result (1.14).

To conclude this paper, we consider the particular case of Laplace series given in (1.18). The constant term and the $O(n^{-1})$ term in (1.18) have a somewhat different appearance from those obtained by putting $\alpha=\beta=0$ in (1.14). The transition to the form in (1.18) is made by writing the second infinite series in (1.10) as

$$\begin{aligned}(6.15)\qquad &\lim_{n\to\infty}\left[M_1(0)+\cdots+M_n(0)-\frac{\sqrt{2}}{\pi^{3/2}}\int_0^{j_{1,n}}x^{-1/2}dx\right]\\ &=\lim_{n\to\infty}\left[M_1(0)+\cdots+M_n(0)-\frac{2^{3/2}}{\pi}n^{1/2}\right]\\ &=\lim_{n\to\infty}\sum_{k=1}^{n}\left\{M_k(0)-\frac{2^{3/2}}{\pi}\left(k^{1/2}-[k-1]^{1/2}\right)\right\},\end{aligned}$$

since from (2.19), $(j_{1,n})^{1/2} = (n\pi)^{1/2} + O(n^{-1/2})$. Now using the fact that, for $\alpha = 0$,

$$(6.16) \qquad \int_0^{j_1} x^\alpha J_{\alpha+1}(x)\,dx = \int_0^{j_{1,1}} J_1(x)\,dx$$
$$= J_0(0) - J_0(j_{1,1}) = 1 + M_1(0),$$

the result for the constant term in (1.18) follows. Similarly in (1.16) we write

$$(6.17) \qquad D_0 = \lim_{n\to\infty}\left[\hat{M}_1(0) + \cdots + \hat{M}_n(0) - \frac{\sqrt{2}}{3/2}\int_0^{j_{0,n}} x^{1/2}\,dx - \frac{1}{2^{3/2}\pi^{1/2}}\int_0^{j_{0,n}} x^{-1/2}\,dx\right]$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left\{\hat{M}_k(0) - \frac{2^{3/2}}{3}\left(k^{3/2} - [k-1]^{3/2}\right) - 2^{-3/2}\left(k^{1/2} - [k-1]^{1/2}\right)\right\},$$

where we have used $(j_{0,n})^{3/2} = \pi^{3/2}(n^{3/2} - \frac{3}{8}n^{1/2}) + O(n^{-1/2})$. Grouping these results together yields the form of $L_n(0,0)$ given in (1.18).

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