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THE REVERSING RESULT FOR THE JONES POLYNOMIAL

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A short proof is given, using linear skein theory, of the theorem of V.F.R. Jones that the one variable "Jones" polynomial associated to an oriented link is independent of the choice of strand orientations, up to a multiple of the variable.

The Jones polynomial of an oriented link K is the element $V(K)$ of $\mathbf{Z}[t^{\pm 1/2}]$ defined by

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2} - t^{-1/2})V(K_0) = 0$$
$$V(U) = 1,$$

where U is the unknot and K_+, K_-, and K_0 are oriented links identical except in a small ball where they have positive, negative and null crossings respectively. Details are to be found in **[J]**, **[F-Y-H-L-M-O]** or **[L-M]**. This note gives a short proof of the following theorem of V. F. R. Jones which states, *inter alia*, that, up to multiplication by a unit of $\mathbf{Z}[t^{\pm 1/2}]$, $V(K)$ is independent of the orientation of K. The original proof used the theory of braids and plaits; the proof here is a simple induction together with a neat *illustration* of linear skein theory. The proof fails (as it must) for the general two-variable oriented link polynomial only at the start of the induction.

THEOREM (*V. F. R. Jones*). *Suppose that a component γ of an oriented link K has linking number λ with the union of the other components. Let $\hat{K}$ be K with the direction of γ reversed. Then $t^{3\lambda}V(K) = V(\hat{K})$.*

Proof. The proof is in five sections.

(1) The theorem is true for the two links of Figure 1. This is an easy exercise in computation.

(2) It is well known that if the orientation of every component of K is reversed then $V(K)$ is unchanged. Further, $V(K_1 \# K_2) = V(K_1)V(K_2)$ where $K_1 \# K_2$ is any connected sum of oriented links K_1 and K_2, and also $V(\overline{K}) = \overline{V(K)}$ where $\overline{K}$ is the obverse (reflection) of K and $\overline{f(t)} = f(t^{-1})$. Thus if the Theorem is true for K_1 and K_2 it is true for $\overline{K}_1$ and for $K_1 \# K_2$.

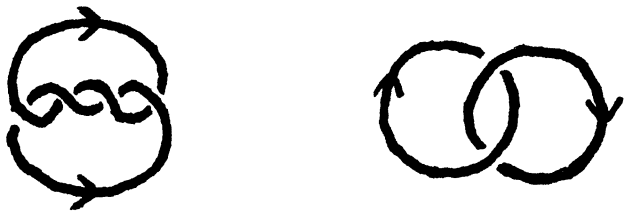

FIGURE 1

(3) Consider the self-crossings of the component γ in some presentation of K. Induction (as repeatedly used in section three of [**L-M**]) on the number of these crossings and on the number of them that have to be switched to unknot γ shows that γ may be assumed to be unknotted.

(4) Let the unknotted component γ bound a disc that meets the remainder of K in n points. Proceed by induction on n. The start of the induction will be given in (5); for the moment assume that $n \geq 4$. Figure 2 depicts a skein triple in which K is K_0. The disc bounded by γ is shown meeting the remainder of K in n points shown as crosses. In K_-, γ has become two unlinked curves γ_1^- and γ_2^- that bound discs that meet the remainder of K_- in n_1 and n_2 points and that link the remainder of K_- with linking numbers λ_1 and λ_2 respectively. The situation of K_+ is exactly similar except that γ_1^+ and γ_2^+ are linked as shown.

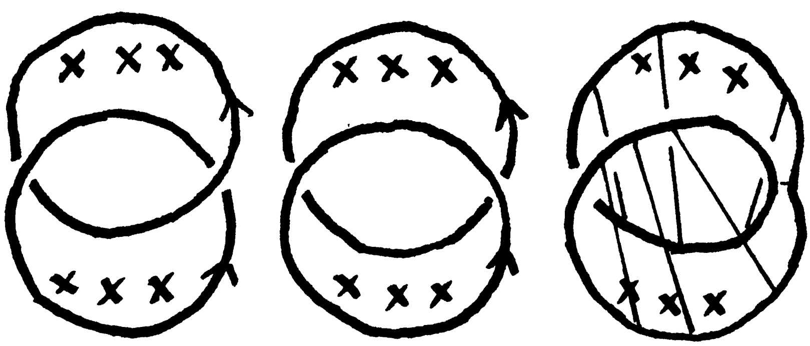

FIGURE 2

Thus $n_1 + n_2 = n$ and $\lambda_1 + \lambda_2 = \lambda$. Choose n_1 and n_2 so that each is at most $n - 2$ (recall $n \geq 4$). Let $\hat{K}_+$, $\hat{K}_-$, and $\hat{K}_0$ be the same links but with the $\gamma_i^{\pm}$ and γ all reversed. Then

$$tV(K_+) - t^{-1}V(K_-) + (t^{1/2} - t^{-1/2})V(K_0) = 0,$$

$$tV(\hat{K}_+) - t^{-1}V(\hat{K}_-) + (t^{1/2} - t^{-1/2})V(\hat{K}_0) = 0.$$

But, by the induction on n, reversing γ_1 and then γ_2^- gives

$$t^{3\lambda_2}t^{3\lambda_1}V(K_-) = V(\hat{K}_1)$$

and reversing γ_1^+ and then γ_2^+ gives

$$t^{3(\lambda_2-1)}t^{3(\lambda_1+1)}V(K_+) = V(\hat{K}_+).$$

It follows immediately that $t^{3\lambda}V(K) = V(\hat{K})$.

This argument extends a little further when $n = 3$. If λ is also 3, choose $n_1 = 1$ and $n_2 = 2$, then the above argument holds if the theorem is known for $n = 3$ and $\lambda = 1$ and for $n \geq 2$. Similarly when $n = 3$, $\lambda = -3$.

(5) Suppose that $n = 3$ and $\lambda = \pm 1$. It is required to show that whatevery tangle is inserted into the room (the rectangle) of Figure 3 to give K, the Theorem holds true and $t^3V(K) = V(\hat{K})$. However, the module over $Z[t^{\pm 1/2}]$ of the linearised skein of this room is generated by the six inhabitants shown in Figure 4.

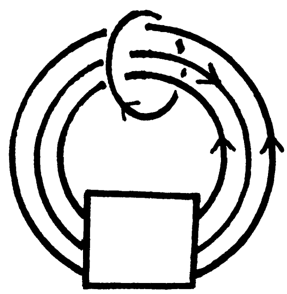

FIGURE 3

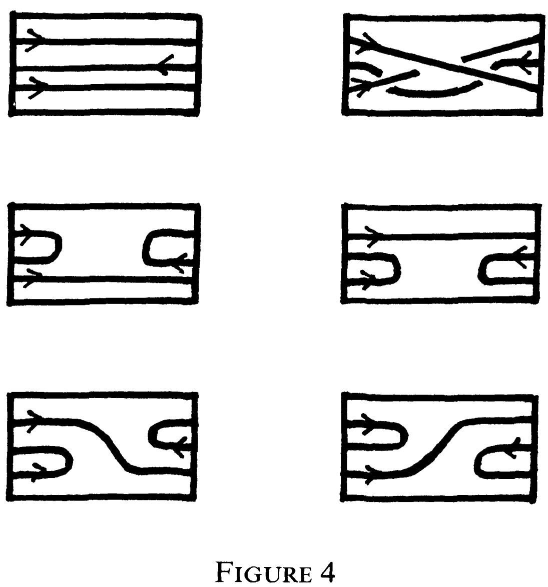

FIGURE 4

Thus all that is required is to check that whichever of these generators is inserted into the room to give K the theorem holds. This follows at once from (1) and (2). A simplified version of this proof works when $n = 2$ there then being only two generators of the analogous rooms (see **[L-M]**). The case $n = 1$ is immediate from (1) and (2) and $n = 0$ is trivial.

This completes the proof; only step (1) fails to generalise to the general (two-variable) link polynomial.

References

[F-Y-H-L-M-O] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc., **12** (2) (1985), 239–246.

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