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ON EXISTENCE CRITERIA FOR CAPILLARY FREE SURFACES WITHOUT GRAVITY

LUEN-FAI TAM

Consider a cylinder of homogenous material closed at one end by a base of general cross section Ω and partly filled with liquid. We want to find conditions under which in the absence of gravity the liquid can cover Ω and is in mechanical equilibrium.

If the liquid can cover Ω, then the liquid surface is a graph over the base. In general, the surface has constant mean curvature and makes constant angle with the bounding wall. Even if Ω is convex analytic, such a surface may not exist. However, it is the case when Ω is piecewise smooth that interests us. In this case, the interior angles at the corners play an important role. It turns out that the existence of the liquid surface as a graph over the base can be characterized by the nonexistence of a certain subsidiary variational problem.

1. Introduction and preliminary results. Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$. The problem of minimizing the following functional has received much attention:

$$(1.1)\quad \Phi(u;\gamma) \equiv \int_\Omega \sqrt{1+|Du|^2} + \int_\Omega H_\gamma u\,dx - \cos\gamma \int_{\partial\Omega} u\,dH_{n-1}$$

for $u \in BV(\Omega)$, the space of functions of bounded variation in Ω, where $\pi/2 > \gamma \geq 0$ is a constant and $H_\gamma = (|\partial\Omega|/|\Omega|)\cos\gamma$.

In order to study the problem, we also consider the subsidiary functional:

$$(1.2)\quad G(A;\gamma) \equiv \int_\Omega |D\phi_A| + \int_\Omega H_\gamma \phi_A\,dH_n - \cos\gamma \int_{\partial\Omega} \phi_A\,dH_{n-1}$$

where $A \subset \Omega$ is a Caccioppoli set (or set of locally finite perimeter), and ϕ_A is the characteristic function of A. The following necessary condition for the existence of (variational) solution of $\Phi(u;\gamma)$ was obtained by Concus-Finn [**2**]:

$$(1.3)\quad G(A;\gamma) > 0 \quad \text{for all } A \subset \Omega,\ A \neq \varnothing \text{ or } \Omega.$$

As for sufficiency, Giusti [**9**] gave the following general criterion: if there exists $\varepsilon > 0$, such that

$$(1.3)'\quad (1-\varepsilon)\int_\Omega |D\phi_A| + \int_\Omega H_\gamma \phi_A\,dH_n - \cos\gamma \int_{\partial\Omega} \phi_A\,dH_{n-1} > 0$$

for all $A \subset \Omega$, $A \neq \emptyset$ or Ω, then $\Phi(u;\gamma)$ has a solution. Finn [5] proved that in some cases we need only the weaker condition (1.3). To be more precise, Finn proved that if $n = 2$, Ω is a piecewise smooth domain, 2α = smallest interior angle and $\pi/2 > \alpha > \pi/2 - \gamma$ at each corner, then (1.3) is also sufficient for the existence of soltion of $\Phi(u;\gamma)$. In fact, he proved that in this case (1.3) implies (1.3)′.

The case $\alpha + \gamma = \pi/2$ is more delicate, because in that case there exists no $\varepsilon > 0$ so that (1.3)′ is true. In this case, Finn [6] introduced some additional conditions on $\partial\Omega$.

Specifically, if Ω satisfies the following hypothesis, then (1.3) still guarantees the existence of a solution:

Hypothesis $\alpha(\gamma)$. At each vertex P with interior angle 2α, it is possible to place a lower hemisphere $v(x;\gamma)$ of radius $R_\gamma = 2H_\gamma^{-1}$, with equatorial circle Q passing through P in such a way that at each point of $\partial\Omega$ interior to Q and to some neighborhood N_P of P there holds $Tv \cdot \nu \geq \cos\gamma$, where ν is the outward normal of $\partial\Omega$.

The results of Finn are restricted to $n = 2$. The methods of proof do not extend readily to higher dimensions. In this paper, it is our aim to generalize those results to higher dimensions by using another method, and at the same time give a new proof of Finn's result. Moreover we shall also show that even if Ω does not satisfy hypothesis $\alpha(\gamma)$ (but satisfies (1.3)), we still can find a solution of

$$(1.4)\qquad \begin{cases} \operatorname{div} Tw = H_\gamma & \text{in } \Omega \\ Tw = \dfrac{Du}{\sqrt{1+|Dw|^2}} & \\ Tw \cdot \nu = \cos\gamma & \text{weakly on } \partial\Omega. \end{cases}$$

We shall use the idea of generalized solution introduced by M. Miranda [16]. This idea has ben employed to deal with different problems, for example see [10], [11], [16] and [17].

The functional $\Phi(u;\gamma)$ is related to the following functional:

$$(1.5)\quad F(U;\gamma) \equiv \int_{\Omega\times\mathbf{R}} |D\phi_U| + \int_{\Omega\times\mathbf{R}} H_\gamma \phi_U \,dx\,dt - \cos\gamma \int_{\partial\Omega\times\mathbf{R}} \phi_U \,dH_n$$

where $U \subset \Omega \times \mathbf{R}$ is a Caccioppoli set.

DEFINITION 1.1. A Caccioppoli set $U \subset \Omega \times \mathbf{R}$ is said to be a solution of $\Phi(u;\gamma)$ if for any $T > 0$, and for any Caccioppoli set $V \subset \Omega \times \mathbf{R}$ with support of $\phi_U - \phi_V$ contained in $\Omega \times [-T, T]$ we have

$$F_T(U;\gamma) \le F_T(V;\gamma) \tag{1.6}$$

where

$$F_T(U;\gamma) \equiv \int_{\Omega\times[-T,T]} |D\phi_U| + \int_{\Omega\times[-T,T]} H_\gamma \phi_U \, dx\, dt - \cos\gamma \int_{\partial\Omega\times[-T,T]} \phi_U \, dH_n.$$

DEFINITION 1.2. A function $u\colon \Omega \to [-\infty, \infty]$ is said to be a generalized solution of $\Phi(u;\gamma)$ if its subgraph $U = \{(x,t) \in \Omega \times \mathbf{R} \mid t < u(x)\}$ is a solution of $F(U;\gamma)$.

DEFINITION 1.3. $A \subset \Omega$ is a solution of $G(A;\gamma)$ if $G(A;\gamma) \le G(E;\gamma)$ for any Caccioppoli set $E \subset \Omega$.

The relation between $\Phi(u;\gamma)$ and $F(U;\gamma)$ is the following theorem by M. Miranda **[16]**, see also **[11]** and **[17]**.

THEOREM 1.1. *Let Ω be a Lipschitz domain and $u \in BV(\Omega)$, then u is a solution of $\Phi(u;\gamma)$ if and only if u is a generalized solution of $\phi(u;\gamma)$.*

REMARK 1.1. In the definitions of $\Phi(u;\gamma)$ and $F(u;\gamma)$, Definition 1.1–1.3 and Theorem 1.1 there is no need to restrict γ so that $0 \le \gamma < \pi/2$.

By Theorem 1.1, in order to find a solution of (1.2) we may first find a generalized solution and then prove that it actually belongs to $BV(\Omega)$.

We need several lemmas. First of all, we say that an open portion Γ_1 of $\partial\Omega$ has Lipschitz constant L if for any point $x \in \Gamma_1$, there exists a ball $B_\rho(x)$ such that $B_\rho(x) \cap \partial\Omega =$ graph g for some Lipschitz function $g\colon A \subset \mathbf{R}^{n-1} \to \mathbf{R}$ with A open and the Lipschitz constant of g is L.

LEMMA 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, Γ_1 is an open portion of $\partial\Omega$ with Lipschitz constant L. Suppose Γ is a closed subset of Γ_1, then there is a constant $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ we can find a constant C_1 depending only on δ, Ω, Γ_1, and Γ such that for any $T > 0$*

$$\int_{\Gamma\times(-T,T)} |f|\, dH_n \le \sqrt{1+L^2} \int_{\Omega_\delta\times(-T,T)} |Df| + C_1 \int_{\Omega\times(-T,T)} |f|\, dx\, dt \tag{1.7}$$

for all $f \in BV(\Omega \times (-T,T))$, where $\Omega_\delta = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \le \delta\}$. Note that δ_0 and C_1 do not depend on T.

The proof of the lemma is the same as the proof of Lemma 1.1 in [3], so we omit it.

LEMMA 1.2. *With the same assumptions as in Lemma* 1.1 *and let* $\{f_j\}$ *be a sequence of functions in* $BV(\Omega \times (-T,T))$, $T > 0$, *such that the functions* f_j *converge in* $L^1(\Omega \times (-T,T))$ *to some function* $f \in BV(\Omega \times (-T,T))$. *If* $\cos\gamma\sqrt{1+L^2} \le 1$ *then*

$$(1.8)\quad \int_{\Omega\times(-T,T)} |Df| - \cos\gamma \int_{\Gamma\times(-T,T)} f\, dH_n \le \liminf_{j\to\infty}\left(\int_{\Omega\times(-T,T)} |Df_j| - \cos\gamma \int_{\Gamma\times(-T,T)} f_j\, dH_n \right).$$

Proof. (See [11].) Define

$$J(u) = \int_{\Omega\times(-T,T)} |Du| - \cos\gamma \int_{\Gamma\times(-T,T)} u\, dH_{n-1}.$$

By Lemma 1.1, there are constants δ_0 and C_1 not depending on j, such that for any $0 < \delta < \delta_0$ if $\Sigma_\delta = \Omega - \Omega_\delta$,

$$\begin{aligned} J(f) - J(f_j) &\le \int_{\Omega\times(-T,T)} |Df| - \int_{\Omega\times(-T,T)} |Df_j| \\ &\quad + \cos\gamma \int_{\Gamma\times(-T,T)} |f - f_j|\, dH_n \\ &\le \int_{\Omega\times(-T,T)} |Df| - \int_{\Omega\times(-T,T)} |Df_j| \\ &\quad + \cos\gamma\sqrt{1+L^2} \int_{\Omega_\delta\times(-T,T)} |Df - Df_j| \\ &\quad + C_1 \cos\gamma \int_{\Omega\times(-T,T)} |f - f_j|\, dx \\ &\le \int_{\Sigma_\delta\times(-T,T)} |Df| - \int_{\Sigma_\delta\times(-T,T)} |Df_j| \\ &\quad + 2\int_{\Omega_\delta\times(-T,T)} |Df| + C_1 \int_{\Omega\times(-T,T)} |f - f_j|\, dx \end{aligned}$$

where we have used the fact $\cos\gamma\sqrt{1+L^2} \leq 1$. Since f_j converge to f in $L^1(\Omega \times (-T,T))$, by semicontinuity Theorem 1.9 of [12] and let $j \to \infty$ we get

$$J(f) - \liminf_{j\to\infty} J(f_j) \leq 2\int_{\Omega_\delta \times(-T,T)} |Df|.$$

Since $f \in BV(\Omega \times (-T,T))$, let $\delta \to 0$ the lemma follows. □

In application, we need a refined version of Lemma 1.2. Let us assume $\partial\Omega$ satisfies

(1.9) there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we can find L_ε such that $\partial\Omega$ has Lipschitz constant L_ε and $\sqrt{1+L_\varepsilon^2}(\cos\gamma - \varepsilon) \leq 1$.

REMARK 1.2. If $\partial\Omega$ has Lipschitz constant L with $\cos\gamma\sqrt{1+L^2} \leq 1$, then $\partial\Omega$ obviously satisfies (1.9). If $\partial\Omega$ is smooth and $\gamma = 0$ then there will be no L such that $\partial\Omega$ has Lipschitz constant L and $\cos\gamma\sqrt{1+L^2} \leq 1$. However $\partial\Omega$ satisfies (1.9). Also if $n = 2$, $\partial\Omega$ is piecewise smooth and the smallest interior angle 2α satisfies $\alpha + \gamma = \pi/2$ (we assume every interior angle is less than π) then $\partial\Omega$ again satisfies condition (1.9).

LEMMA 1.3. *Same assumptions as in Lemma* 1.2 *except that now we only assume* $\Gamma_1 = \partial\Omega$ *satisfying* (1.9). *If* $\int_{\Gamma\times(-T,T)} f_j\,dH_n$ *are uniformly bounded, then the conclusion of Lemma* 1.2 *is still true.*

Proof. By assumption, there is a constant $M > 0$ such that

$$\left|\int_{\Gamma\times(-T,T)} f_j\,dH_n\right| + \left|\int_{\Gamma\times(-T,T)} f\,dH_n\right| \leq M \quad \text{for all } j. \tag{1.10}$$

Let $J(u)$ be the functional defined in Lemma 1.2. For any $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} J(f) - J(f_j) \leq & \int_{\Omega\times(-T,T)} |Df| - (\cos\gamma - \varepsilon)\int_{\Gamma\times(-T,T)} f\,dH_n \\ & - \int_{\Omega\times(-T,T)} |Df_j| + (\cos\gamma - \varepsilon)\int_{\Gamma\times(-T,T)} f_j\,dH_n \\ & + \varepsilon\left(\left|\int_{\Gamma\times(-T,T)} f\,dH_n\right| + \left|\int_{\Gamma\times(-T,T)} f_j\,dH_n\right|\right). \end{aligned}$$

By Lemma 1.2, (1.9) and (1.10), let $j \to \infty$

$$J(f) - \liminf_{j\to\infty} J(f_j) \leq \varepsilon M.$$

Since ε can be arbitrary small, the lemma follows.

2. Existence of generalized solution.

LEMMA 2.1. *Suppose Ω is a bounded Lipschitz domain in $\mathbf{R}^n$ satisfying* (1.3), *then $\Phi(u;\ \beta)$ has a bounded solution u_β for all $\pi/2 > \beta > \gamma$. Furthermore u_β is analytic and satisfies* $\operatorname{div} Tu_\beta = H_\beta$ *in Ω.*

Proof. Since $G(A;\ \gamma) > 0$ for all $A \neq \varnothing$ or Ω, for such an A

$$\int_\Omega |D\phi_A| + \int_\Omega H_\beta \cdot \frac{\cos\gamma}{\cos\beta} \cdot \phi_A\,dx - \cos\beta \int_{\partial\Omega} \frac{\cos\gamma}{\cos\beta} \cdot \phi_A\,dH_{n-1} > 0.$$

Hence,

$$(1-\varepsilon)\int_\Omega |D\phi_A| + \int_\Omega H_\beta \phi_A\,dx - \cos\beta\int_{\partial\Omega}\phi_A\,dH_{n-1} > 0$$

$$\text{for } \varepsilon = 1 - \frac{\cos\beta}{\cos\gamma} > 0.$$

From [9], we conclude that $\Phi(u;\ \gamma)$ has a bounded solution u_β which is in $C^2(\Omega)$ and satisfies $\operatorname{div} Tu_\beta = H_\beta$ in Ω. Since H_β is a constant, therefore u_β is analytic. □

THEOREM 2.1. *Let Ω be a bounded Lipschitz domain satisfying* (1.3) *and* (1.9). *For any sequence $\pi/2 > \gamma_j > \gamma$ and $\gamma_j \searrow \gamma$, if u_j is a solution of $\Phi(u;\ \gamma_j)$ for each j, then we can find a subsequence of u_j which converges pointwise almost everywhere to a generalized solution of $\Phi(u;\ \gamma)$.*

Proof. (See [**11**].) Since Ω satisfies (1.3), the solutions u_j of $\Phi(u;\ \gamma_j)$ exist by Lemma 2.1. Let U_j be the subgraph of u_j, U_j is a solution of $F(U;\ \gamma_j)$ by Theorem 1.1. For any $T > 0$, compare U_j with $U_j - \Omega \times (-T, T)$:

$$\int_{\Omega\times[-T,T]} |D\phi_{U_j}| + \int_{\Omega\times[-T,T]} H_{\gamma_j}\phi_{U_j}\,dx\,dt$$
$$-\cos\gamma_j \int_{\partial\Omega\times[-T,T]} \phi_{U_j}\,dH_n \le 2|\Omega|.$$

$$\int_{\Omega\times[-T,T]} |D\phi_{U_j}| \le 2|\Omega| + \cos\gamma_j\int_{\partial\Omega\times[-T,T]}\phi_{U_j}\,dH_n \le 2|\Omega| + 2T|\partial\Omega|.$$

Also $\int_{\Omega\times[-T,T]} \phi_{U_j}\,dx\,dt \le 2T|\Omega|$. Hence

$$\int_{\Omega\times[-T,T]} |D\phi_{U_j}| + \int_{\Omega\times[-T,T]} \phi_{U_j}\,dx\,dt \le (2+2T)|\Omega| + 2T|\partial\Omega| \quad \text{for all } j. \tag{2.1}$$

By compactness Theorem 1.19 of **[12]**, we can select a subsequence of ϕ_{U_j} which converges almost everywhere in $\Omega \times (-T, T)$ to the characteristic function of some set. Take a sequence $T_k \to \infty$, and use diagonal process, we can find a subsequence of ϕ_{U_j}, which we also call it ϕ_{U_j}, converging almost everywhere to the characteristic function of some set w, which may take the valeus $-\infty$ or $+\infty$, and $\lim_{j\to\infty} u_j = w$ almost everywhere in Ω. By (2.1) and semicontinuity, we know that W is a Caccioppoli set.

It remains to prove that w is a generalized solution of $\Phi(u; \gamma)$. Note that for almost all $T > 0$,

(i) the traces of ϕ_{U_j} and ϕ_W on $\Omega \times \{-T, T\}$ are ϕ_{U_j} and ϕ_W H_n-almost everywhere;

(ii) $\int_{\Omega\times\{-T,T\}} |D\phi_{U_j}| = \int_{\Omega\times\{-T,T\}} |D\phi_W| = 0$; and

(iii) $\lim_{j\to\infty} \int_{\Omega\times\{-T,T\}} |\phi_{U_j} - \phi_W|\, dH_n = 0$.

(i) and (ii) follow from the fact that U_j and W are Caccioppoli sets and (iii) follows from the fact that $\lim_{j\to\infty} \int_{\Omega\times[-T,T]} |\phi_{U_j} - \phi_W|\, dH_n = 0$ for all $T > 0$.

For any $T > 0$ satisfying (i), (ii) and (iii), and for any Caccioppoli set $V \subset \Omega \times \mathbf{R}$, $V = W$ outside $\Omega \times [-T, T]$, define

$$V_j = \begin{cases} U_j & \text{outside } \Omega \times [-T, T], \\ V & \text{in } \Omega \times [-T, T] \end{cases}.$$

Then

$$\int_{\Omega\times[-T,T]} |D\phi_{U_j}| + \int_{\Omega\times[-T,T]} H_{\gamma_j}\phi_{U_j}\, dx\, dt - \cos\gamma_j \int_{\partial\Omega\times[-T,T]} \phi_{U_j}\, dH_n$$
$$\leq \int_{\Omega\times[-T,T]} |D\phi_{V_j}| + \int_{\Omega\times[-T,T]} H_{\gamma_j}\phi_{V_j}\, dx\, dt$$
$$- \cos\gamma_j \int_{\partial\Omega\times[-T,T]} \phi_{V_j}\, dH_n.$$

Therefore

$$\begin{aligned} (2.2)\quad & \int_{\Omega\times[-T,T]} |D\phi_{U_j}| + \int_{\Omega\times[-T,T]} H_{\gamma_j}\phi_{U_j}\, dx\, dt \\ & - \cos\gamma \int_{\partial\Omega\times[-T,T]} \phi_{U_j}\, dH_n + (\cos\gamma - \cos\gamma_j) \int_{\partial\Omega\times[-T,T]} \phi_{U_j}\, dH_n \\ \leq & \int_{\Omega\times[-T,T]} |D\phi_V| + \int_{\Omega\times[-T,T]} H_{\gamma_j}\phi_V\, dx\, dt \\ & - \cos\gamma_j \int_{\partial\Omega\times[-T,T]} \phi_V\, dH_n + \int_{\Omega\times\{-T,T\}} |\phi_W - \phi_{U_j}|\, dH_n. \end{aligned}$$

Since T satisfies (ii) and (iii), by Lemma 1.3 with $\Gamma = \Gamma_1 = \partial\Omega$ and the fact that $\lim_{j\to\infty}\gamma_j = \gamma$, let $j \to \infty$ in (2.2):

$$(2.3)\quad \int_{\Omega\times[-T,T]} |D\phi_W| + \int_{\Omega\times[-T,T]} H_\gamma \phi_W \, dx\, dt - \cos\gamma \int_{\partial\Omega\times[-T,T]} \phi_W \, dH_n \le \int_{\Omega\times[-T,T]} |D\phi_V| + \int_{\Omega\times[-T,T]} H_\gamma \phi_V \, dx\, dt - \cos\gamma \int_{\partial\Omega\times[-T,T]} \phi_V \, dH_n.$$

(2.3) is true for almost all $T > 0$, so it is easy to see that (2.3) holds for all $T > 0$. Therefore w is a generalized solution of $\Phi(u; \gamma)$. □

Take any sequence $\gamma_j \searrow \gamma$, since the solution u_j of $\Phi(u; \gamma_j)$ is unique up to an additive constant, we normalize u_j in Theorem 2.1 by

$$(2.4)\quad \begin{aligned} &\left|\{x \in \Omega \,|\, u_j(x) \ge 0\}\right| \ge |\Omega|/4 \quad \text{and} \\ &\left|\{x \in \Omega \,|\, u_j(x) \le 0\}\right| \ge |\Omega|/4. \end{aligned}$$

By passing to a subsequence, $\lim_{j\to\infty} u_j = w$ almost everywhere in Ω where w is a generalized solution of $\Phi(u; \gamma)$. *In the remaining part of this paper we always assume Ω satisfies* (1.3) *and* (1.9), *and we fix the sequence u_j and the function w described above unless otherwise specified.*

Let $P = \{x \in \Omega \,|\, w(x) = +\infty\}$, and $N = \{x \in \Omega \,|\, w(x) = -\infty\}$. Then we have:

LEMMA 2.2. *P is a solution of $G(A; \gamma)$ and N is a solution of $G(A; \pi - \gamma)$.*

Proof. For any positive integer j, $w - j$ is also a generalized solution of $\Phi(u; \gamma)$. As in the proof of Theorem 2.1, we can find a subsequence of $w - j$ which converges to a generalized solution w' of $\Phi(u; \gamma)$. But the subgraph of w' is $P \times \mathbf{R}$. Therefore $P \times \mathbf{R}$ is a solution of $F(U; \gamma)$. Since $P \times \mathbf{R}$ is a cylinder, so P is a solution of $G(A; \gamma)$. By considering $-w$ which is a generalized solution of $\Phi(u; \pi - \gamma)$, one can similarly prove that N is a solution of $G(A; \pi - \gamma)$. □

COROLLARY 2.1. *w is finite almost everywhere.*

Proof. By Lemma 2.2, $G(P;\gamma) \le G(\Omega;\gamma) = 0$. By (1.3) we conclude $P = \varnothing$ or Ω. But each u_j satisfies (2.4) so $P = \varnothing$. Similarly we can prove $N = \varnothing$. □

3. Boundedness of w: case when $\cos\gamma\sqrt{1+L^2} < 1$.

LEMMA 3.1. *Let U be a solution of $F(U;\beta)$ and Γ_1 be an open portion of $\partial\Omega$. Assume* (i) *$\Gamma_1 = \varnothing$ or Γ_1 has a Lipschitz constant L with $|\cos\beta\sqrt{1+L^2}| \le a < 1$; and* (ii) *$|H_\beta| \le b$. For any subdomain Ω' of Ω with* $\operatorname{dist}(\Omega', \partial\Omega - \Gamma_1) > \sigma > 0$, *we can find constants $C_3 > 0$ and $r_0 > 0$, which depend only on n, b, σ, and Ω if $\Gamma_1 = \varnothing$ and depend also on Γ_1, L and a if $\Gamma_1 \ne \varnothing$, such that for all $(x_0, t_0) \in \overline{\Omega}' \times \mathbf{R}$, the following are true*:

(3.1) *if $|U_r(x_0,t_0)| = |C_r(x_0,t_0) \cap U| > 0$ for all $r > 0$, then*

$$|U_r(x_0,t_0)| \ge C_3 r^{n+1} \quad \textit{for all } r \le r_0;$$

(3.2) *if $|U_r'(x_0,t_0)| = |C_r(x_0,t_0) - U| > 0$ for all $r > 0$, then*

$$|U_r'(x_0,t_0)| \ge C_3 r^{n+1} \quad \textit{for all } r \le r_0,$$

where $C_r(x_0,t_0) = \{(x,t) \in \mathbf{R}^{n+1} \mid |x - x_0| < r \text{ and } |t - t_0| < r\}$.

Note. We define the distance between a set and the empty set to be $+\infty$.

Proof. The proof is essentially the same as the proof of Theorem 3.2 in [**11**]. Let us first assume $\Gamma_1 \ne \varnothing$. For simplicity we write $_r = U_r(x_0,t_0)$, $C_r = C_r(x_0,t_0)$ and $H = H_\beta$. Now compare U with $U - C_r$,

$$\text{(3.3)} \quad \int_{(\Omega\times\mathbf{R})\cap C_r} |D\phi_U| + \int_{(\Omega\times\mathbf{R})\cap C_r} H\phi_U\,dx\,dt - \cos\beta\int_{(\partial\Omega\times\mathbf{R})\cap C_r} \phi_U\,dH_n \le \int_{\partial C_r} \phi_U\,dH_n.$$

So for almost all $r > 0$,

$$\text{(3.4)} \quad \int_{\Omega\times\mathbf{R}} |D\phi_{U_r}| + \int_{\Omega\times\mathbf{R}} H\phi_{U_r}\,dx\,dt - \cos\beta\int_{\partial\Omega\times\mathbf{R}} \phi_{U_r}\,dH_n \le 2\int_{\partial C_r} \phi_U\,dH_n.$$

Let $\Gamma = \{x \in \Gamma_1 \,|\, \mathrm{dist}(x, \partial\Omega - \Gamma_1) \geq \sigma/2\}$. Suppose $0 < r < \sigma/2$, then by Lemma 1.1,

$$(3.5)\quad \int_{\partial\Omega\times\mathbf{R}} \phi_{U_r}\, dH_n = \int_{\Gamma\times\mathbf{R}} \phi_{U_r}\, dH_n \leq \sqrt{1+L^2}\int_{\Omega\times\mathbf{R}} |D\phi_{U_r}| + C_1|U_r|$$

$$\leq \sqrt{1+L^2}\int_{\Omega\times\mathbf{R}} |D\phi_{U_r}| + C_1 k(n+1)|C_r|^{1/(n+1)}\int |D\phi_{U_r}|,$$

where C_1 is the constant in Lemma 1.1 which depends only on σ, Ω and Γ_1, and $k(n+1)$ is the isoperimeteric constant in $\mathbf{R}^{n+1}$. In the last inequality we have used the isoperimetric inequality Corollary 1.29 in [12]. Now choose $r_1 > 0$ small enough such that

$$(3.6)\qquad r_1 \leq \sigma/2 \quad\text{and}\quad C_1 \cdot k(n+1)\left|C_{r_1}\right|^{1/(n+1)} < 1/2.$$

Since $\int |D\phi_{U_r}| = \int_{\Omega\times\mathbf{R}} |D\phi_{U_r}| + \int_{\partial\Omega\times\mathbf{R}} \phi_{U_r}\, dH_n$, so from (3.5) and (3.6) we have

$$(3.7)\quad \int_{\partial\Omega\times\mathbf{R}} \phi_{U_r}\, dH_n \leq \frac{\sqrt{1+L^2} + C_1 k(n+1)|C_r|^{1/(n+1)}}{1 - C_1 k(n+1)|C_r|^{1/(n+1)}} \int_{\Omega\times\mathbf{R}} |D\phi_{U_r}|,$$

and

$$(3.8)\qquad \int |D\phi_{U_r}| = \int_{\Omega\times\mathbf{R}} |D\phi_{U_r}| + \int_{\partial\Omega\times\mathbf{R}} \phi_{U_r}\, dH_n$$

$$\leq \frac{1+\sqrt{1+L^2}}{1 - C_1 k(n+1)|C_r|^{1/(n+1)}} \int_{\Omega\times\mathbf{R}} |D\phi_{U_r}|$$

$$\leq 2\left(1+\sqrt{1+L^2}\right)\int_{\Omega\times\mathbf{R}} |D\phi_{U_r}|,$$

for all $0 < r \leq r_1$.

By Lemma 2.1 of [11]:

$$(3.9)\quad \int_{\Omega\times\mathbf{R}} H\phi_{U_r}\, dx\, dt = \int_{U_r} H\, dx\, dt \geq -k(n)\cdot b \cdot |C_r|^{1/n}\int |D\phi_{U_r}|.$$

Combining (3.7)–(3.9):

$$(3.10)\quad \int_{\Omega\times\mathbf{R}} H\phi_{U_r}\, dx\, dt - \cos\beta \cdot \int_{\partial\Omega\times\mathbf{R}} \phi_{U_r}\, dH_n$$

$$\geq -\left\{2b\left(1+\sqrt{1+L^2}\right)\cdot k(n)|C_r|^{1/n}\right.$$

$$\left. + \cos\beta\left(\frac{\sqrt{1+L^2} + C_1 k(n+1)|C_r|^{1/(n+1)}}{1 - C_1 k(n+1)|C_r|^{1/(n+1)}}\right)\right\} \cdot \int_{\Omega\times\mathbf{R}} |D\phi_{U_r}|,$$

for almost all $0 < r \leq r_1$.

The expression in the whole bracket above tends to $\cos\beta\sqrt{1+L^2}$ as $r \to 0$. Since $\cos\beta\sqrt{1+L^2} \le a < 1$, we may choose $0 < r_0 \le r_1$ such that the expression in the bracket is less than $(1+a)/2$ for all $0 < r \le r_0$. We see that r_0 depends only on n, a, b, σ, L, Ω, and Γ_1. By (3.4), (3.6), (3.8) and (3.10) we get

$$\begin{aligned} 2\frac{d}{dr}|U_r| &= 2\int_{\partial C_r} \phi_U \, dH_n \ge \int_{\Omega\times\mathbf{R}} |D\phi_{U_r}| - \frac{1+a}{2}\int_{\Omega\times\mathbf{R}} |D\phi_{U_r}| \\ &\ge \frac{1-a}{4\left(1+\sqrt{1+L^2}\right)}\int |D\phi_{U_r}| \\ &\ge \frac{1-a}{4\cdot k(n+1)\left(1+\sqrt{1+L^2}\right)}|U_r|^{n/(n+1)}, \end{aligned}$$

for almost all $0 < r \le r_0$. If $|U_r| > 0$ for all $r > 0$, then we have $|U_r| \ge C_3 r^{n+1}$ for all $r \le r_0$ where C_3 depends only n, a, and L, which can be computed explicitly.

If $\Gamma_1 = \varnothing$, then there is no need to estimate boundary term $\int_{\partial\Omega\times\mathbf{R}} \phi_{U_r} \, dH_n$. So in this case we do not need the inequality $\cos\beta\sqrt{1+L^2} \le a < 1$, the rest of the proof is similar. Hence (3.1) is true. Similarly by considering $U' = \Omega\times\mathbf{R} - U$, we can also prove (3.2). □

THEOREM 3.1. *Suppose Ω is a bounded Lipschitz domain satisfying* (1.3) *and* (1.9). *Let γ_j, u_j and w be as before, and let Γ_1 be an open portion of $\partial\Omega$ such that either $\Gamma_1 = \varnothing$ or Γ_1 has a Lipschitz constant L with $\cos\gamma\sqrt{1+L^2} < 1$. Then for any subdomain $\Omega' \subset \Omega$ with* $\operatorname{dist}(\Omega', \partial\Omega - \Gamma_1) > 0$, *there exists a constant C_4 independent of j such that*

$$\sup_{\Omega'} |u_j| \le C_4 \quad \textit{and} \tag{3.11}$$

$$\sup_{\Omega'} |w| \le C_4. \tag{3.12}$$

Proof. We assume $\Gamma_1 \ne \varnothing$, the case when $\Gamma_1 = \varnothing$ can be proved similarly. Suppose the functions u_j are not uniformly bounded above in Ω'. Since each u_j is bounded in Ω, by passing to a subsequence if necessary, we can find $x_j \in \Omega'$, $\lim_{j\to\infty} x_j = x_0 \in \overline{\Omega}'$ and $\lim_{j\to\infty} u_j(x_j) = +\infty$. Let C_3 and r_0 be the constants in Lemma 3.1 corresponding to $a = \cos\gamma\sqrt{1+L^2}$ and $b = H_\gamma$. For any $t > 0$, we hae $u_j(x_j) > t$ and $|x_j - x_0| < r_0/2$ if j is large enough. Let U_j be the subgraph of u_j. Then

U_j is a solution of $F(U;\ \gamma_j)$, and if j is large enough $|U_{j,r}(x_j, t)| = |U_j \cap C_r(x_j, t)| > 0$ for all $r > 0$, because u_j is regular in Ω. Since $\cos\gamma_j\sqrt{1+L^2} < a$ and $0 < H_{\gamma_j} < b$, by Lemma 3.1 if j is large enough then

$$\left|U_{j,r_0}(x_0, t)\right| \geq \left|U_{j,r_0/2}(x_j, t)\right| \geq C_3\left(\frac{r_0}{2}\right)^{n+1}.$$

Let $j \to \infty$, we conclude that

$$\left|W_{r_0}(x_0, t)\right| \geq C_3\left(\frac{r_0}{2}\right)^{n+1},$$

where W is the subgraph of w. Since t is arbitrary, this contradicts Corollary 2.1 that w is finite almost everywhere. Therefore u_j are uniformly bounded above in Ω'. Similarly we can prove that u_j are uniformly bounded below in Ω', and (3.11) is proved. (3.12) is an immediate consequence of (3.11) because u_j converge to w almost everywhere in Ω. □

THEOREM 3.2. *Let Ω be a bounded Lipschitz domain having Lipschitz constant L with* $\cos\gamma\sqrt{1+L^2} < 1$. *If Ω satisfied* (1.3), *then $\Phi(u;\ \gamma)$ has a bounded solution which is analytic in Ω.*

Proof. By Theorem 3.1 with $\Gamma_1 = \partial\Omega$, we conclude that w is bounded and hence is a bounded solution of $\Phi(u;\ \gamma)$. As in the proof of Lemma 2.1, we see that w is analytic in Ω. □

If Ω only satisfies (1.9) (and (1.3)) then w may be unbounded. For example, if Ω is smooth and $\gamma = 0$, then w may be unbounded, see **[10]**. However we have the following:

THEOREM 3.3. *Let Ω be a bounded Lipschitz domain satisfying* (1.3) *and* (1.9), *then $\Phi(u;\ \gamma)$ has a locally bounded generalized solution which is analytic and satisfies* $\operatorname{div} Tw = H_\gamma$ *in Ω.*

Proof. By Theorem 3.1, for any subdomains $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, there exists a constant C_4 not depending of j such that $\sup_{\Omega''}|u_j| \leq C_4$ for all j.

Since each u_j is analytic and satisfies $\operatorname{div} Tu = H_{\gamma_j}$ in Ω. By Corollary 15.7 of **[8]**, we can find a constant C_5 not depending on j such that

$$\sup_{\overline{\Omega}'}\left|Du_j\right| + \sup_{\overline{\Omega}'}\left|D^2u_j\right| + \sup_{\overline{\Omega}'}\left|D^3u_j\right| \leq C_5,$$

for all j. By passing to a subsequence, u_j, Du_j and D^2u_j converge uniformly in Ω'. Hence $w \in C^2(\Omega')$ and satisfies $\operatorname{div} Tw = H_\gamma$, so w is analytic in Ω'. Since Ω' can be any subdomain of Ω such that $\overline{\Omega}' \subset \Omega$, the theorem follows. □

Now we are going to investigate the boundary behavior of w. Following [7] and [13] we have the following definitions:

DEFINITION 3.1. We say a family of domains Ω_k exhausting Ω if $\partial\Omega_k \in C^1$, $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\bigcup_k \Omega_k = \Omega$.

If Ω is Lipschitz we can always find such an exhausting sequence by **[14]**.

DEFINITION 3.2. Let Ω be a Lipschitz domain, $u \in C^2(\Omega)$ is said to satisfy $Tu \cdot \nu = \cos\gamma$ weakly on $\partial\Omega$ if for any exhausting sequence Ω_k of Ω,

$$\lim_{k\to\infty} \int_{\partial\Omega_k} fTu \cdot \nu_k \, dH_{n-1} = \int_{\partial\Omega} f\cos\gamma \, dH_{n-1} \tag{3.13}$$

for any function f belonging to the Sobolev space $W^{1,1}(\Omega)$, wher ν_k is the outward normal of $\partial\Omega_k$.

THEOREM 3.4. *If Ω satisfies* (1.3) *and* (1.9), *and if u_j and w are as before, then w satisfies $Tw \cdot \nu = \cos\gamma$ weakly on $\partial\Omega$.*

Proof. For each j, u_j is a variational solution of $\Phi(u;\ \gamma_j)$. By the proof of Lemma 2 in [7], for all $f \in W^{1,1}(\Omega)$, we have

$$\int_\Omega Df \cdot \frac{Du_j}{\sqrt{1+|Du_j|^2}} \, dx + \int_\Omega fH_{\gamma_j} \, dx - \int_{\partial\Omega} f\cos\gamma_j \, dH_{n-1} = 0. \tag{3.14}$$

From the proof of Theorem 3.3, by passing to a subsequence $\lim_{j\to\infty} Du_j = Dw$ in Ω. As $|Du_j/\sqrt{1+|Du_j|^2}| < 1$ in Ω for all j, noting that $\lim_{j\to\infty}\cos\gamma_j = \cos\gamma$ and $\lim_{j\to\infty} H_{\gamma_j} = H_\gamma$, let $j \to \infty$ in (3.14), by Lebesgue dominated convergence theorem, we get

$$\int_\Omega Df \cdot \frac{Dw}{\sqrt{1+|Dw|^2}} \, dx + \int_\Omega fH_\gamma \, dx - \int_{\partial\Omega} f\cos\gamma \, dH_{n-1} = 0. \tag{3.15}$$

Let Ω_k be an exhausting sequence of Ω. Multiply $\operatorname{div} Tw = H_\gamma$ by f and integrating by parts over Ω_k:

$$\int_{\Omega_k} Df \cdot \frac{Dw}{\sqrt{1+|Dw|^2}} \, dx + \int_{\Omega_k} fH_\gamma \, dx - \int_{\partial\Omega_k} fTw \cdot \nu_k \, dH_{n-1} = 0. \tag{3.16}$$

By Lebesgue dominated convergence theorem again,

$$\lim_{k\to\infty}\int_{\Omega_k} Df\cdot\frac{Dw}{\sqrt{1+|Dw|^2}}\,dx=\int_\Omega Df\cdot\frac{Dw}{\sqrt{1+|Dw|^2}}\,dx \quad\text{and}$$

$$\lim_{k\to\infty}\int_{\Omega_k} fH_\gamma\,dx=\int_\Omega fH_\gamma\,dx.$$

Therefore combine (3.15) and (3.16), we have

$$\lim_{k\to\infty}\int_{\partial\Omega_k} fTw\cdot\nu_k\,dH_{n-1}=\int_{\partial\Omega} f\cos\gamma\,dH_{n-1}. \qquad \square$$

4. Boundedness of w**: general case.** As we noted before, if there is no L with $\cos\gamma\sqrt{1+L^2}<1$ such that Ω has Lipschitz constant L, then the generalized solution w may fail to be bounded. However, if we impose some reasonable assumptions on $\partial\Omega$, we can still prove that w is bounded and hence w is a solution of $\Phi(u;\gamma)$.

As before we assume Ω satisfies (1.3) and (1.9). In addition we assume $\partial\Omega=\Gamma'\cup\Gamma''$ such that

(4.1) Γ' is an open portion of $\partial\Omega$ with Lipschitz constant L, $\cos\gamma\sqrt{1+L^2}<1$;

(4.2) $\Gamma''=\Gamma^1\cup\Gamma^2\cup\cdots\cup\Gamma^N$, each Γ^k is closed;

(4.3) for each $k=1,\ldots,N$, there is an open set $O_k\subset\mathbf{R}^n$ containing Γ^k such that $\operatorname{dist}(O_k,\bigcup_{m\neq k}\Gamma^m)>0$, $O_k\cap\Omega$ is Lipschitz and connected; also there are functions $v_k^{(1)}$ and $v_k^{(2)}$ belonging to $C^2(\mathbf{R}^n)$ and satisfying:

(i) $\operatorname{div} Tv_k^{(1)}=\operatorname{div} Tv_k^{(2)}=H_\gamma$ in $O_k\cap\Omega$; and

(ii) $Tv_k^{(1)}\cdot\nu\geq\cos\gamma\geq Tv_k^{(2)}\cdot\nu$ H_{n-1}-almost everywhere on $O_k\cap\partial\Omega$.

$v_k^{(1)}$ and $v_k^{(2)}$ will serve as upper and lower comparison surfaces for w respectively. In some cases, for example, $n=2$ and Ω is piecewise smooth, lower comparison surfaces always exist, see [6]. Our assumptions on $\partial\Omega$ is a generalization of the hypothesis $\alpha(\gamma)$ introduced in §1 to higher dimensions.

THEOREM 4.1. *Suppose* Ω *satisfies* (1.3), (1.9) *and* (4.1)–(4.3). *Let* u_j *and* w *be as before, then* w *is bounded and is a variational solution of* $\Phi(u;\gamma)$.

Proof. By Theorem 3.1 and (4.1), we know that w is bounded in every subdomain $\Omega'\subset\Omega$ with $\operatorname{dist}(\Omega',\partial\Omega-\Gamma')>0$. It remains to prove that w is bounded in $O_k\cap\Omega$ for $k=1,\ldots,N$. Firstly, we want to prove that w

is bounded above. For simplicity, we write $O = O_k$, $\Gamma = \Gamma^k$ and $v = v_k^{(1)}$. Let $O' \subset\subset O$ be an open set containing Γ, then $\text{dist}(O - O', \Gamma'') > 0$ by assumption (4.3), and so $\text{dist}((O - O') \cap \Omega, \Gamma'') > 0$. By Theorem 3.1, the functions u_j are uniformly bounded in $(O - O') \cap \Omega$. By adding a constant to v, we may assume the $u_j < v$ on $(O - O') \cap \Omega$. We assert that $w \le v$ on $O \cap \Omega$.

From the proof of Theorem 3.1, we know that by passing to a subsequence u_j converge to w and Du_j converge to Dw, both uniformly on compact subsets of Ω. If $v(x_0) < w(x_0)$ for some $x_0 \in O \cap \Omega$, then there exist positive numbers ρ, M, and a positive integer j_0 such that $\overline{B_\rho(x_0)} \subset \Omega$, and we have $0 < u_j - v < M$ in $B_\rho(x_0)$, for $j \ge j_0$. Define

$$w_j = \begin{cases} 0 & u_j - v \le 0 \\ u_j - v & 0 < u_j - v < M \quad \text{in } O \cap \Omega. \\ M & M \le u_j - v \end{cases}$$

Extend w_j to be zero in $\Omega - O$. Since $w_j = 0$ in $(O - O') \cap \Omega$ and $\text{dist}(\Omega - O, O') > 0$, so $w_j \in W^{1,1}(\Omega)$. From the proof of Lemma 2 of [7]:

$$(4.4) \qquad \int_{O\cap\Omega} Dw_j \cdot \frac{Du_j}{\sqrt{1 + |Du_j|^2}}\, dx$$

$$= -\int_{O\cap\Omega} H_{\gamma_j} w_j \, dx + \int_{O\cap\partial\Omega} w_j \cos\gamma_j \, dH_{n-1},$$

bearing in mind that $w_j = 0$ in $\Omega - O$. On the other hand by assumption (4.3)(i) and (ii),

$$(4.5) \qquad \int_{O\cap\Omega} Dw_j \cdot \frac{Dv}{\sqrt{1 + |Dv|^2}}\, dx$$

$$\ge -\int_{O\cap\Omega} H_\gamma w_j \, dx + \int_{O\cap\partial\Omega} w_j \cos\gamma \, dH_{n-1}.$$

Subtracting (4.5) from (4.4):

$$(4.6) \quad \int_{O\cap\Omega} Dw_j \cdot \left(\frac{Du_j}{\sqrt{1 + |Du_j|^2}} - \frac{Dv}{\sqrt{1 + |Dv|^2}} \right)$$

$$\le \int_{O\cap\Omega} \left(H_\gamma - H_{\gamma_j}\right) w_j \, dx + \int_{O\cap\partial\Omega} w_j (\cos\gamma_j - \cos\gamma) \, dH_{n-1}.$$

By the definition of w_j and the particular structure of the operator Tu, it is easy to see

$$Dw_j \cdot \left(\frac{Du_j}{\sqrt{1+|Du_j|^2}} - \frac{Dv}{\sqrt{1+|Dv|^2}} \right) \geq 0.$$

By (4.6), for $j \geq j_0$:

$$(4.7) \quad \int_{B_\rho(x_0)} D(u_j - v) \cdot \left(\frac{Du_j}{\sqrt{1+|Du_j|^2}} - \frac{Dv}{\sqrt{1+|Dv|^2}} \right) dx$$

$$\leq \int_{O\cap\Omega} Dw_j \cdot \left(\frac{Du_j}{\sqrt{1+|Du|^2}} - \frac{Dv}{\sqrt{1+|Dv|^2}} \right) dx$$

$$\leq \int_{O\cap\Omega} (H_\gamma - H_{\gamma_j}) w_j \, dx + \int_{O\cap\partial\Omega} w_j(\cos\gamma_j - \cos\gamma)\, dH_{n-1}.$$

Let $j \to \infty$, noting that $0 \leq w_j \leq M$ for all j,

$$(4.8) \quad \int_{B_\rho(x_0)} D(w - v) \cdot \left(\frac{Dw}{\sqrt{1+|Dw|^2}} - \frac{Dv}{\sqrt{1+|Dv|^2}} \right) \leq 0.$$

But

$$D(w - v) \cdot \left(\frac{Dw}{\sqrt{1+|Dw|^2}} - \frac{Dv}{\sqrt{1+|Dv|^2}} \right)$$

$$\geq |Dw - Dv|^2 \cdot \int_0^1 \frac{dt}{\left(1+|Dv_t|^2\right)^{3/2}},$$

where $v_t = tw + (1-t)v$. Therefore $Dw - Dv = 0$ in $B_\rho(x_0)$ and $w = v + K$ there, where $K = w(x_0) - v(x_0) > 0$.

Let $A = \{x \in O \cap \Omega \mid w(x) = v(x) + K\}$. A is obviously closed in $O \cap \Omega$ since w and v are continuous. Also from the proof above, we see that A is open. Since $A \neq \varnothing$, and by assumption $O \cap \Omega$ is connected we conclude that $A = O \cap \Omega$. This is impossible because $u_j < v$ in $(O - O')$ $\cap\, \Omega$ which is not empty. Hence $w \leq v$ in $O \cap \Omega$, and w is bounded above in Ω. Similarly, we can prove that w is bounded below. By Theorem 1.1, w is a solution of $\Phi(u; \gamma)$. □

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PURDUE UNIVERSITY
WEST LAFAYETTE, IN 47907