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ON McCONNELL'S INEQUALITY FOR FUNCTIONALS OF SUBHARMONIC FUNCTIONS

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Recently, McConnell obtained an L^p inequality relating the nontangential maximal function of a nonnegative subharmonic function u and an integral expression involving the Laplacian of u. His result is imposing a restriction on the range of p. In this paper, we show that his inequality holds for all $p \in (0, +\infty)$.

1. Introduction. Let $u(x, t)$ be a nonnegative subharmonic function defined on $R_+^{n+1} = \{(x, t): x \in R^n,\ t > 0\}$. (For the definition of subharmonic functions, see Hayman and Kennedy [5] p. 40.) Let Δu be the Laplacian of u in the sense of distributions. Then, this is a positive measure on R_+^{n+1}. Let

$$N(x) = \sup\{u(y, t): (y, t) \in \Gamma_1(x)\},$$

$$S(x) = \iint_{(y,t)\in\Gamma_1(x)} t^{1-n}\Delta u(y, t),$$

where

$$\Gamma_\alpha(x) = \{(y, t) \in R_+^{n+1}: |x - y| < \alpha t\},$$

$$|x| = |(x_1, \ldots, x_n)| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

If $v(x, t)$ is a real harmonic function defined on R_+^{n+1} and if

$$u(x, t) = v(x, t)^2, \tag{1}$$

then u is nonnegative and subharmonic. In this case, $N^{1/2}$ and $S^{1/2}$ turn out to be the usual nontangential maximal function and the usual area integral of v, respectively. So, the results of Burkholder and Gundy [1] and C. Fefferman and Stein [3] imply that in case of (1) we have

$$\|S\|_{L^p} \leq c(p, n)\|N\|_{L^p} \tag{2}$$

for all $p \in (0, +\infty)$. (Under the additional assumption $\lim_{t\to+\infty} v(x, t) = 0$, they showed also the converse inequality of (2) with other constants $c(p, n)$.)

Recently, McConnell [7] extended the inequality (2) to general nonnegative subharmonic functions.

THEOREM A. *Let u be a nonnegative subharmonic function defined on R_+^{n+1}. There are constants $c(p,n) < +\infty$, depending only on p and n, and a positive constant $p_0(n)$, depending only on n, such that the inequalities*

$$\|S\|_{L^p} \leq c(p,n)\|N\|_{L^p} \tag{3}$$

hold for all p satisfying

$$0 < p < p_0(n) \quad \text{or} \quad 1 \leq p < +\infty; \tag{4}$$

moreover $p_0(1) = 1$.

This theorem in the case $n \geq 2$ is imposing an unnatural restriction (4) on the range of p. In this paper, we remove (4).

THEOREM 1. *Let u be as in Theorem* A. *Let $0 < p < +\infty$. Then, there exist constants $c(p,n) < +\infty$, depending only on p and n, such that* (3) *holds.*

The argument in this paper is an extension of that in our paper [8].

2. Preliminaries. First we prepare notation. The Laplacian Δ and the gradient ∇ in this paper are taken in the sense of distributions. For a measurable subset E of the Euclidean space, let χ_E and $|E|$ be the characteristic function of E and the Lebesgue measure of E, respectively. For $x \in R^n$ and $E \subset R^n$, let $\delta(x,E)$ be the distance of the point x from E. Let $\delta(x,\varnothing) = +\infty$.

For $x \in R^n$, $R > 1$, $\alpha > 0$, and for $u(x,t)$ in Theorem A let

$$\varphi(x) = \max(0, 1 - |x|),$$

$$T_R = \{(x,t) \in R_+^{n+1} : |x| < R,\ 1/R < t < R\},$$

$$N(x;\alpha) = \sup\{u(y,t) : (y,t) \in \Gamma_\alpha(x)\},$$

$$S(x;\alpha) = \iint_{(y,t)\in\Gamma_\alpha(x)} t^{1-n}\Delta u(y,t),$$

$$s(x;\alpha,R) = \iint_{(y,t)\in R_+^{n+1}} \varphi\left(\frac{x-y}{\alpha t}\right) t^{1-n}\Delta u(y,t)\chi_{T_R}(y,t).$$

Note that if $\alpha' > \alpha > 0$, then

$$S(x;\alpha) \leq c(\alpha,\alpha',n) \lim_{R\to+\infty} s(x;\alpha',R). \tag{5}$$

Cubes considered in this paper have sides parallel to the coordinate axes. For a cube I, let $l(I)$ and αI be the side length of I and a cube concentric with I satisfying $l(\alpha I) = \alpha l(I)$, respectively. For a cube I in R^n, let

$$Q(I) = \{(x,t) \in R_+^{n+1} : x \in I,\ t \in (0, l(I))\}.$$

For a nonnegative measure μ on R_+^{n+1} let

$$\|\mu\|_c = \sup_I \mu(Q(I))/|I|,$$

where the supremum is taken over all cubes I in R^n. If $\|\mu\|_c < +\infty$, then μ is called a Carleson measure.

For the proof of Theorem 1 we need the following.

LEMMA 1. *Let u be as in Theorem* A. *Let $\lambda > 0$, $\alpha > \beta > 0$,*

$$\Omega = \{x \in R^n : N(x;\alpha) \le \lambda\}, \tag{6}$$

$$W = \{(x,t) \in R_+^{n+1} : \delta(x,\Omega) < \beta t\}. \tag{7}$$

Then

$$\|t\Delta u \chi_W\|_c \le C\lambda, \tag{8}$$

where C is a constant depending only on α, β and n.

LEMMA 2. *Let u be as in Theorem* A. *Let $\lambda > 0$, $R > 1$, $\gamma > 1$ and $\alpha > \beta > 0$. Then*

$$|\{x \in R^n : s(x;\beta,R) > \gamma\lambda,\ N(x;\alpha) \le \lambda\}| \le Ce^{-c\gamma}|\{x \in R^n : s(x;\beta,R) > \lambda\}| \tag{9}$$

where C and c are positive constants depending only on α, β and n.

3. Proof of Lemma 1.

LEMMA 3. *Let $m \ge 2$ be an integer. Let $r > 0$,*

$$B = \{X \in R^m : |X| < r\},$$

$$0.5B = \{X \in R^m : |X| < 0.5r\}.$$

Let $U(X)$ be a subharmonic function defined on B such that

$$0 \le U(X) \le 1 \quad \textit{for all } X \in B.$$

Then

(i) ΔU *in the sense of distributions satisfies*

$$\int_{X \in 0.5B} \Delta U(X) \leq C r^{m-2},$$

(ii) ∇U *in the sense of distributions is locally integrable on* B *and satisfies*

$$\int_{X \in 0.5B} |\nabla U(X)| \leq C r^{m-1},$$

where C *is a constant depending only on* m.

Proof. We may assume $r = 1$. Let $G(X, Y)$ be the Green function of $B = \{X \in R^m : |X| < 1\}$. Namely, for $(X, Y) \in (B \times B) \setminus \{(X, X) : X \in B\}$, let

$$G(X, Y) = \begin{cases} |X - Y|^{2-m} - \big||Y|X - Y/|Y|\big|^{2-m}, & Y \neq 0, \\ |X|^{2-m} - 1, & Y = 0, \end{cases}$$

if $m \geq 3$ and let

$$G(X, Y) = \begin{cases} \log \dfrac{\big||Y|X - Y/|Y|\big|}{|X - Y|}, & Y \neq 0, \\ \log \dfrac{1}{|X|}, & Y = 0, \end{cases}$$

if $m = 2$. For $Y \in B$ let

$$V(Y) = \frac{1}{\sigma_m} \int_{X \in 0.6B} G(X, Y) \Delta U(X),$$

where

$$\sigma_m = \frac{2\pi^{m/2} \max(1, m - 2)}{\Gamma(m/2)}.$$

Since $U + V$ is nonnegative on B, harmonic on $0.6B$, subharmonic on B and

$$\lim_{\varepsilon \to +0} \sup\{V(Y) : Y \in R^m,\ |Y| = 1 - \varepsilon\} = 0,$$

we have

$$0 \leq U(Y) + V(Y) \leq 1 \quad \text{on } B, \tag{10}$$

$$|\nabla(U+V)(Y)| \le C \quad \text{on } 0.5B. \tag{11}$$

Therefore,

$$\begin{aligned} c\int_{X\in 0.6B} \Delta U(X) &\le \int_{X\in 0.6B} G(X,0)\Delta U(X) \\ &= \sigma_m V(0) \le \sigma_m(U(0)+V(0)) \le \sigma_m \end{aligned} \tag{12}$$

by (10) and

$$\begin{aligned} \int_{Y\in B} |\nabla V(Y)| &\le \frac{1}{\sigma_m}\int_{X\in 0.6B} \Delta U(X)\int_{Y\in B} |\nabla_Y G(X,Y)| \\ &\le C\int_{X\in 0.6B} \Delta U(X) \le C \end{aligned} \tag{13}$$

by (12), where $c>0$ and $C<+\infty$ depend only on m. So, (i) follows from (12) and (ii) follows from (11) and (13). □

Now, we begin the proof of Lemma 1. We may assume

$$\lambda = 1 \tag{14}$$

and $\Omega \ne \varnothing$. Let a cube I be given. It is enough to show

$$\iint_{(x,t)\in Q(I)\cap W} t\Delta u(x,t) \le C|I|. \tag{15}$$

Let $\varepsilon > 0$ be a constant such that

$$\varepsilon < \min\left(\frac{\beta}{n^{1/2}}, \frac{1}{n^{1/2}}\right), \tag{16}$$

$$\frac{(1+\varepsilon)(\beta+\varepsilon)+2n\varepsilon}{1-n^{1/2}\varepsilon} < \alpha. \tag{17}$$

For $\eta > 0$, $x \in R^n$ and $t > 0$, let

$$\psi_\eta(x) = \max(\delta(x,\Omega), \delta(x,I), \eta), \tag{18}$$

$$\varphi_\eta(x,t) = \begin{cases} 1 & \text{if } \frac{1}{\beta}\psi_\eta(x) < t, \\ \dfrac{\beta((\beta+\varepsilon)t-\psi_\eta(x))}{\varepsilon\psi_\eta(x)} & \text{if } \frac{1}{\beta+\varepsilon}\psi_\eta(x) < t \le \frac{1}{\beta}\psi_\eta(x), \\ 0 & \text{if } t \le \frac{1}{\beta+\varepsilon}\psi_\eta(x), \end{cases} \tag{19}$$

$$(20)\qquad h(t)=\begin{cases}0 & \text{if } (1+\varepsilon)l(I)<t,\\ \dfrac{(1+\varepsilon)l(I)-t}{\varepsilon l(I)} & \text{if } l(I)<t\le(1+\varepsilon)l(I),\\ 1 & \text{if } t\le l(I),\end{cases}$$

$$(21)\qquad V_\eta(x,t)=\varphi_\eta(x,t)h(t).$$

Let $R_+^{n+1}=\bigcup_{k=1}^\infty Q_k$ be the Whitney decomposition of R_+^{n+1} such that

(22) $\{Q_k\}_{k=1}^\infty$ are dyadic cubes in R_+^{n+1} with disjoint interiors,

$$(23)\qquad \frac{1}{\varepsilon}l(Q_k)\le\left(\text{distance between } Q_k \text{ and } \partial R_+^{n+1}\right)\le\frac{4}{\varepsilon}l(Q_k),$$

(This collection $\{Q_k\}$ can be obtained by taking all the maximal cubes among the closed dyadic cubes in R_+^{n+1} that satisfy (23).) Let $\{Q_{k(j)}\}_{j=1}^N$ be the subcollection of $\{Q_k\}$ such that

$$(24)\qquad Q_{k(j)}\cap\operatorname{supp}\nabla V_\eta\neq\varnothing.$$

In the following part of this section, the letter C denotes various positive constants depending only on α, β, ε and n.

First we accept the following (25)–(30) temporarily;

$$(25)\qquad |\nabla_\eta(x,t)|\le\frac{C}{t},$$

$$(26)\qquad \operatorname{supp}\nabla V_\eta\subset\bigcup_{j=1}^N Q_{k(j)},$$

$$(27)\qquad \iint_{\operatorname{supp}\nabla V_\eta}\frac{1}{t}\,dx\,dt\le C|I|,$$

$$(28)\qquad \bigcup_{j=1}^N 2n^{1/2}Q_{k(j)}\subset\left\{(x,t)\in R_+^{n+1}:u(x,t)\le 1\right\},$$

$$(29)\qquad \sum_{j=1}^N\left(l(Q_{k(j)})\right)^n\le C|I|,$$

$$(30)\qquad \begin{aligned}&\text{the left-hand side of (15)}\\ &\le\lim_{\eta\to+0}\iint_{(x,t)\in R_+^{n+1}}tV_\eta(x,t)\Delta u(x,t).\end{aligned}$$

Then, (28) and Lemma 3 (ii) imply

$$(31)\qquad \iint_{(x,t)\in Q_{k(j)}}|\nabla u(x,t)|\le C\left(l(Q_{k(j)})\right)^n.$$

Thus,

$$\text{(32)} \qquad \left| \iint_{(x,t)\in R_+^{n+1}} tV_\eta(x,t) \frac{\partial^2 u}{\partial x_i^2}(x,t) \right| = \left| -\iint t \frac{\partial V_\eta}{\partial x_i} \frac{\partial u}{\partial x_i} \right|$$

$$\begin{aligned}
&\leq C \iint_{\operatorname{supp} \nabla V_\eta} \left| \frac{\partial u}{\partial x_i} \right| && \text{by (25)} \\
&\leq C \sum_{j=1}^{N} \iint_{Q_{k(j)}} |\nabla u| && \text{by (26)} \\
&\leq C \sum \left(l(Q_{k(j)}) \right)^n && \text{by (31)} \\
&\leq C|I| && \text{by (29)}
\end{aligned}$$

and

$$\begin{aligned}
\left| \iint_{(x,t)\in R_+^{n+1}} tV_\eta \frac{\partial^2 u}{\partial t^2} \right| &= \left| \iint \left(-V_\eta - t\frac{\partial V_\eta}{\partial t} \right) \frac{\partial u}{\partial t} \right| \\
&= \left| \iint \frac{\partial V_\eta}{\partial t} u - \iint t \frac{\partial V_\eta}{\partial t} \frac{\partial u}{\partial t} \right| \\
&= |(33) - (34)|,
\end{aligned}$$

where

$$\begin{aligned}
|(33)| &\leq \iint_{\operatorname{supp} \nabla V_\eta} \frac{C}{t} u(x,t)\, dx\, dt && \text{by (25)} \\
&\leq \iint_{\operatorname{supp} \nabla V_\eta} \frac{C}{t}\, dx\, dt && \text{by (28) and (26)} \\
&\leq C|I| && \text{by (27)}
\end{aligned}$$

and where

$$|(34)| \leq C|I|$$

follows from the same argument as (32). Thus we get

$$\iint tV_\eta(x,t) \Delta u(x,t) \leq C|I|,$$

which combined with (30) implies (15). □

Next, we prove (25)–(30). (25)–(26) are clear. (30) follows from

$$\begin{aligned}
&\left\{ (x,t) \in R_+^{n+1} : V_\eta(x,t) = 1 \right\} \\
&\quad \supset \left\{ (x,t) \in R_+^{n+1} : \frac{1}{\beta} \psi_\eta(x) \leq t \leq l(I) \right\} \\
&\quad \supset Q(I) \cap W \cap \left\{ (x,t) \in R_+^{n+1} : t > \eta \right\}.
\end{aligned}$$

Proof of (28). Since

$$\begin{aligned}\operatorname{supp}\nabla V_\eta &\subset \operatorname{supp} V_\eta \\ &\subset \left\{(x,t)\in R_+^{n+1}\colon \frac{1}{\beta+\varepsilon}\psi_\eta(x)<t\right\} \\ &\subset \left\{(x,t)\in R_+^{n+1}\colon \frac{1}{\beta+\varepsilon}\delta(x,\Omega)<t\right\} \\ &= \bigcup_{x\in\Omega}\Gamma_{\beta+\varepsilon}(x),\end{aligned}$$

for each $Q_{k(j)}$ there exists an $x\in\Omega$ such that

$$Q_{k(j)}\cap\Gamma_{\beta+\varepsilon}(x)\neq\varnothing,$$

which combined with (17) and (23) implies

$$2n^{1/2}Q_{k(j)}\subset\Gamma_\alpha(x).$$

Therefore,

$$\begin{aligned}\bigcup_{j=1}^{N}2n^{1/2}Q_{k(j)} &\subset \bigcup_{x\in\Omega}\Gamma_\alpha(x) \\ &\subset \left\{(x,t)\in R_+^{n+1}\colon u(x,t)\le 1\right\} \quad \text{by (14).}\end{aligned}$$ □

Proof of (27). Let

$$\tilde I=(1+2(\beta+\varepsilon)(1+\varepsilon))I.$$

Then

$$\begin{aligned}\operatorname{supp}\nabla V_\eta &\subset \operatorname{supp} V_\eta \\ &\subset \left\{(x,t)\in R_+^{n+1}\colon \frac{1}{\beta+\varepsilon}\delta(x,I)\le t\le(1+\varepsilon)l(I)\right\} \\ &\subset Q(\tilde I).\end{aligned}\tag{35}$$

Let

$$\begin{aligned}S_1&=\left\{(x,t)\in R_+^{n+1}\colon \frac{1}{\beta+\varepsilon}\psi_\eta(x)\le t\le\frac{1}{\beta}\psi_\eta(x)\right\},\\ S_2&=\left\{(x,t)\in R_+^{n+1}\colon l(I)\le t\le(1+\varepsilon)l(I)\right\}.\end{aligned}$$

Then, by (19)–(21) and (35) we have

$$\operatorname{supp}\nabla V_\eta\subset(S_1\cup S_2)\cap Q(\tilde I),$$

which combined with (25) implies (27). □

Proof of (29). Let

$$\tilde{S}_1 = \left\{(x,t) \in R_+^{n+1}: \right.$$

$$\left. \frac{1}{(1+\varepsilon)(\beta+\varepsilon)+n^{1/2}\varepsilon}\psi_\eta(x) \le t \le \frac{1+\varepsilon}{\beta - n^{1/2}\varepsilon}\psi_\eta(x)\right\},$$

$$\tilde{S}_2 = \left\{(x,t) \in R_+^{n+1}: \frac{1}{1+\varepsilon}l(I) \le t \le (1+\varepsilon)^2 l(I)\right\}.$$

It follows from (23) that

$$\text{if } Q_k \cap S_i \ne \varnothing, \quad \text{then } Q_k \subset \tilde{S}_i, \tag{36}$$

for $i = 1, 2$, respectively. (The case $i = 2$ is clear. The proof for the case $i = 1$ needs the Lipschitz continuity of ψ_η.) Thus, (36) and (24) imply

$$\bigcup_{j=1}^{N} Q_{k(j)} \subset \tilde{S}_1 \cup \tilde{S}_2. \tag{37}$$

On the other hand, (23)–(24) and (35) imply

$$\bigcup_{j=1}^{N} Q_{k(j)} \subset Q((1+2\varepsilon)\tilde{I}). \tag{38}$$

For $(x,t) \in R_+^{n+1}$ let $P(x,t) = x$. Then, by (22)–(23), (37) and by the geometrical properties of $\tilde{S}_1$ and $\tilde{S}_2$, we have

$$\left\| \sum_{j=1}^{N} \chi_{P(Q_{k(j)})}(x) \right\|_{L^\infty(R^n)} \le C,$$

which combined with (38) implies

$$\begin{aligned} \sum_{j=1}^{N} \left(l(Q_{k(j)})\right)^n &= \sum_j |P(Q_{k(j)})| \\ &\le C\left|\bigcup_j P(Q_{k(j)})\right| \le C|I|. \end{aligned}$$ □

4. Proof of Lemma 2. In the rest of this paper, the letter C denotes various positive constants depending only on α, β and n.

We continue to assume (14).

Let

$$\mathscr{S}(x) = \iint_{(y,t)\in R_+^{n+1}} \varphi\left(\frac{x-y}{\beta t}\right) t^{1-n} \Delta u(y,t) \chi_{W\cap T_R}(y,t).$$

Note that

$$(39)\qquad \mathcal{S}(x) = s(x;\beta,R) \quad \text{on } \Omega,$$

$$(40)\qquad \mathcal{S}(x) \le s(x;\beta,R) \quad \text{on } R^n.$$

Since $\mathcal{S}(x) < +\infty$ and since $\mathcal{S}(x)$ is the balayage of the Carleson measure $t\Delta u\chi_{W\cap T_R}$ with respect to the kernel $\varphi(x)$, which has a compact support and which belongs to the Lipschitz class, a well-known estimate of the BMO-norm in terms of the norm of Carleson measure gives us

$$\|\mathcal{S}\|_{\mathrm{BMO}} \le C\|t\Delta u\chi_{W\cap T_R}\|_c,$$

which combined with Lemma 1 and (14) implies

$$(41)\qquad \|\mathcal{S}\|_{\mathrm{BMO}} \le C.$$

Thus, the left-hand side of (9) with (14)

$$\begin{aligned}
&= |\{\mathcal{S}(x) > \gamma,\ N(x;\alpha) \le 1\}| && \text{by (39)}\\
&\le |\{\mathcal{S}(x) > \gamma\}| \le^{(*)} Ce^{-c\gamma}|\{\mathcal{S}(x) > 1\}| &&\\
&\le Ce^{-c\gamma}|\{s(x;\beta,R) > 1\}| && \text{by (40)}\\
&= \text{the right-hand side of (9) with (14)}, &&
\end{aligned}$$

where the inequality $(*)$ follows from (41) and from an easy modification of the result of John-Nirenberg [6]. (See Lemma 2.1 of [8] for details.) □

5. Proof of Theorem 1. Let $\beta' = (\alpha+\beta)/2$. Applying Lemma 2 with β replaced by β' gives us

$$\begin{aligned}
&|\{s(x;\beta',R) > \gamma\lambda\}|\\
&\qquad \le |\{s(x;\beta',R) > \gamma\lambda,\ N(x;\alpha) \le \lambda\}| + |\{N(x;\alpha) > \lambda\}|\\
&\qquad \le Ce^{-c\gamma}|\{s(x;\beta',R) > \lambda\}| + |\{N(x;\alpha) > \lambda\}|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\gamma^{-p}\|s(\cdot;\beta',R)\|_{L^p}^p &= p\int_0^{+\infty} \lambda^{p-1}|\{s(x;\beta',R) > \gamma\lambda\}|\,d\lambda\\
&\le p\int_0^{+\infty} \lambda^{p-1}\big(Ce^{-c\gamma}|\{s(x;\beta',R) > \lambda\}| + |\{N(x;\alpha) > \lambda\}|\big)\,d\lambda\\
&= Ce^{-c\gamma}\|s(\cdot;\beta',R)\|_{L^p}^p + \|N(\cdot;\alpha)\|_{L^p}^p.
\end{aligned}$$

Since $\|s(\cdot;\beta',R)\|_{L^p} < +\infty$, the above inequality with sufficiently large γ implies

$$2^{-1}\gamma^{-p}\|s(\cdot;\beta',R)\|_{L^p}^p \le \|N(\cdot;\alpha)\|_{L^p}^p.$$

Letting $R \to +\infty$ and recalling (5), we get

$$\|S(\cdot\,;\beta)\|_{L^p} \leq C(\alpha,\beta,p,n)\|N(\cdot\,;\alpha)\|_{L^p}. \tag{42}$$

On the other hand, the argument of [3], p. 166, Lemma 1 shows

$$c(\alpha,p,n)\|N\|_{L^p} \leq \|N(\cdot\,;\alpha)\|_{L^p} \leq C(\alpha,p,n)\|N\|_{L^p}. \tag{43}$$

The argument of [2], p. 19, Theorem 3.4 and that of [7], p. 296, Lemma 3.3 show

$$c(\beta,p,n)\|S\|_{L^p} \leq \|S(\cdot\,;\beta)\|_{L^p} \leq C(\beta,p,n)\|S\|_{L^p}, \tag{44}$$

where

$$0 < c(\alpha,p,n),\quad c(\beta,p,n) \quad \text{and}$$
$$C(\alpha,p,n),\quad C(\beta,p,n) < +\infty.$$

Therefore, Theorem 1 follows from (42)–(44).

References

[1] D. L. Burkholder and R. F. Gundy, *Distribution function inequalities for the area integral*, Studia Math., **44** (1972), 527–544.

[2] A. Calderon and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math., **16** (1975), 1–63.

[3] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137–193.

[4] R. Fefferman, R. F. Gundy, M. Silverstein and E. M. Stein, *Inequalities for ratios of functionals of harmonic functions*, Proc. Nat. Acad. Sci. USA, **79** (1982), 7958–7960.

[5] W. K. Hayman and P. B. Kennedy, *Subharmonic Functions*, Academic Press 1976.

[6] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415–426.

[7] T. R. McConnell, *Area integrals and subharmonic functions*, Indiana Univ. Math. J., **33** (1984), 289–303.

[8] T. Murai and A. Uchiyama, *Good λ inequalities for the area integral and the nontangential maximal function*, to appear in Studia Math., **83** (1986), 47–58.

[9] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton, 1970.

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