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EIGENVALUE ESTIMATES WITH APPLICATIONS TO MINIMAL SURFACES

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We study eigenvalue estimates of branched Riemannian coverings of compact manifolds. We prove that if

$$\varphi : M^n \to N^n$$

is a branched Riemannian covering, and $\{\mu_i\}_{i=0}^{\infty}$ and $\{\lambda_i\}_{i=0}^{\infty}$ are the eigenvalues of the Laplace-Beltrami operator on M and N, respectively, then

$$\sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t},$$

for all positive t, where k is the number of sheets of the covering. As one application of this estimate we show that the index of a minimal oriented surface in $\mathbf{R}^3$ is bounded by a constant multiple of the total curvature. Another consequence of our estimate is that the index of a closed oriented minimal surface in a flat three-dimensional torus is bounded by a constant multiple of the degree of the Gauss map.

1. Introduction. Motivated by problems in the theory of minimal surfaces, we study the following question. Let

$$\varphi : M^n \to N^n$$

be a branched Riemannian covering of compact manifolds, which has a singular set of codimension at least two. By this we mean that we endow M^n with the pullback metric

$$\varphi^*(ds_N),$$

where φ^* is singular on a set of codimension at least two. We then want to estimate the eigenvalues of the Laplace-Beltrami operator on M in terms of the corresponding eigenvalues of N. Note that we can speak of the eigenvalues of $(M, \varphi^*(ds_N))$ although the metric is possibly singular, since a singular set of codimension at least two will not affect the integrals of the variational characterization of the eigenvalues.

Our main theorem gives the estimate

$$\sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t},$$

for all $t > 0$, where k is the number of sheets of the covering φ, and $\{\mu_i\}$ and $\{\lambda_i\}$ are the eigenvalues of the Laplace-Beltrami operator on M and N, respectively. We then use this estimate to show that if $M^2 \subseteq \mathbf{R}^3$ is an oriented complete minimal surface of finite total curvature, then the index of M is bounded by a constant multiple of the total curvature. Here, the index of M is defined to be the limit of the indices of an increasing sequence of exhausting compact domains in M. The index of a domain D is the number of negative eigenvalues of the eigenvalue problem

$$(\Delta + |A|^2)\varphi + \lambda\varphi = 0 \quad \text{on } D, \qquad \varphi|_{\partial D} = 0,$$

where A is the second fundamental form of M as a submanifold of $\mathbf{R}^3$. Geometrically, the index of M can be described as the maximum dimension of a linear space of compactly supported deformations that decrease the area up to second order. Finally we also show that the index of a closed oriented minimal surface in a flat three-dimensional torus is bounded by a constant multiple of the degree of the Gauss map.

2. The eigenvalue estimate. Our main result is the following theorem.

THEOREM. *Let*

$$\varphi : M^n \to N^n$$

be a k-sheeted branched Riemannian covering of compact manifolds, which has a singular set of codimensioin at least two. Let $\{\mu_i\}_{i=0}^{\infty}$ *and* $\{\lambda_i\}_{i=0}^{\infty}$ *be the eigenvalues of the Laplace-Beltrami operator on* M^n *and* N^n, *respectively. Then for all* $t > 0$,

$$\sum_{i=0}^{\infty} e^{-\mu_i t} \le k \sum_{i=0}^{\infty} e^{-\lambda_i t}.$$

REMARK. Before proving the theorem we note that the main difficulty is that the fundamental comparison theorems of Cheng [1] do not carry through if the metric has singularities. We instead utilize the heat kernel on M and N to circumvent this difficulty.

Proof. We restrict φ of the theorem to φ_-:

$$\varphi_- : M_- \to N_-,$$

where

$$M_- = M - E(\varepsilon),$$

and $E(\varepsilon)$ is an open set of volume less than ε with smooth boundary, containing the singular set. We then simply define N_- to be the image under φ restricted to M_-.

Now fixing $x \in M_-$, we consider

$$H : y \mapsto H_{N_-}(\varphi(x), \varphi(y), t), \qquad y \in M_-,\ t > 0,$$

where H_{N_-} is the heat kernel on N_-, with Dirichlet boundary conditions. Since φ_- is the local isometry, the function H solves the heat equation on M_-. As t tends to zero we obtain

$$H_{N_-}(\varphi(x), \varphi(y), t) \to \sum_{\varphi(x_i) = \varphi(x)} \delta_{x_i}.$$

On the other hand, for the heat kernel H_{M_-} on M_- with Dirichlet boundary conditions, we have as t tends to zero

$$H_{M_-}(x, y, t) \to \delta_x.$$

Hence, at $t = 0$ we have in the sense of distributions

$$H_{M_-}(x, y, 0) \le H_{N_-}(\varphi(x), \varphi(y), 0).$$

By the maximum principle for the heat equation, we then have

$$H_{M_-}(x, y, t) \le H_{N_-}(\varphi(x), \varphi(y), t), \tag{1}$$

for all $t > 0$. Inequality (1) holds for all x and y in M_- so we can let $x = y$ and integrate over M_-:

$$\int_{M_-} H_{M_-}(x, x, t)\, dV(x) \le \int_{M_-} H_{N_-}(\varphi(x), \varphi(x), t)\, dV(x).$$

Since φ_- is a k-sheeted covering, we have

$$\int_{M_-} H_{N_-}(\varphi(x), \varphi(x), t)\, dV(x) = k \int_{N_-} H_{N_-}(z, z, t)\, dV(z).$$

Again using the maximum principle for the heat equation, we obtain

$$\int_{N_-} H_{N_-}(z, z, t)\, dV(z) \le \int_N H_N(z, z, t)\, dV(z),$$

where H_N denotes the heat kernel of N. We have therefore shown that

$$\int_{M_-} H_{M_-}(x, x, t)\, dV(x) \le k \int_N H_N(z, z, t)\, dV(z).$$

Finally, letting the volume ε of $E(\varepsilon)$ tend to zero, we obtain

$$\int_M H_M(x, x, t)\, dV(x) \le k \int_N H_N(z, z, t)\, dV(z), \tag{2}$$

where H_M is the heat kernel of M. Using separation of variables, one shows that the heat kernels H_M and H_N have the representations

$$H_M(x, y, t) = \sum_{i=0}^{\infty} e^{-\mu_i t}\psi_i(x)\psi_i(y)$$

$$H_N(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y),$$

where

$$\Delta\psi_i + \mu_i\psi_i = 0, \qquad i = 0, 1, 2, \ldots,$$

and

$$\Delta\varphi_i + \lambda_i\varphi_i = 0, \qquad i = 0, 1, 2, \ldots,$$

are the eigenvalues and eigenfunctions of M and N, respectively, normalized so that $\{\psi_i\}_{i=0}^{\infty}$ and $\{\varphi_i\}_{i=0}^{\infty}$ form orthonormal systems. Using these representations in inequality (2), we obtain

$$\sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t},$$

finishing the proof of the theorem.

3. Applications to minimal surfaces. In [**2**], D. Fisher-Colbrie shows that a complete minimal oriented surface M in $\mathbf{R}^3$ has finite index if and only if it has finite total curvature (see the introduction for the definition of index). A natural question to ask then is how the index varies with the total curvature. Using our eigenvalue estimate, we can show that the index is bounded by a constant multiple of the total curvature.

THEOREM. *Let M^2 be a complete oriented minimal surface in $\mathbf{R}^3$. Set*

$$k = \frac{1}{4\pi} \int_M (-K)\, dV,$$

where K is the Gaussian curvature of M. Then

$$\text{index of } M \leq (7.68183) \cdot k.$$

Proof. Without loss of generality, we can assume that k is finite. By Osserman's classical theorem, we then know that M is conformally a compact Riemann surface $\overline{M}$, punctured at a finite set of points. Also, the Gauss map extends to a conformal map

$$G : \overline{M} \to S^2.$$

For a minimal surface in $\mathbf{R}^3$, $|A|^2 = -2K$. Now, the number of negative eigenvalues for

$$\Delta + |A|^2 = \Delta - 2K,$$

on any domain D in M, is the same as the number of negative eigenvalues of the corresponding domain in $\overline{M}$ for the operator

$$\Delta_{\overline{M}} + 2,$$

where we use the pullback metric from S^2 on $\overline{M}$. This follows from the fact that

$$G^*\left(ds^2_{S^2}\right) = (-K) \cdot ds^2_M,$$

and $\Delta_M = (-K)\Delta_{\overline{M}}$. Since the index of M is the limit of the indices of an exhausting sequence of domains D in M, we can conclude, by the domain monotonicity of eigenvalues, that the index of M is bounded by the number of negative eigenvalues of $\Delta_{\overline{M}} + 2$ on $\overline{M}$, or equivalently, by the number of eigenvalues of $\Delta_{\overline{M}}$ that are strictly less than two.

Now, G is a holomorphic mapping so it establishes $\overline{M}$ as a k-sheeted branched cover of S^2. The singular set of this covering is the set of isolated points where $K = 0$. We can therefore apply our eigenvalue estimate and conclude that

$$\sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t}, \quad \text{all } t > 0,$$

where $\{\mu_i\}_{i=0}^{\infty}$ and $\{\lambda_i\}_{i=0}^{\infty}$ are the eigenvalues of $\overline{M}$ and S^2, respectively. Since the index of M is bounded by the number of μ_i's that are strictly less than two, we conclude that

$$(\text{index of } M) \cdot e^{-2t} \leq \sum_{\mu_i < 2} e^{-\mu_i t} \leq \sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t}.$$

Hence

$$\text{index of } M \leq \left(e^{2t} \sum_{i=0}^{\infty} e^{-\lambda_i t} \right) \cdot k.$$

The ith distinct eigenvalue of S^2 is known to be $i(i+1)$, with multiplicity $2i + 1$. Using this, we find that $t = 0.4506\ldots$ gives the smallest possible value of $7.68182\ldots$ for the coefficient of k, proving the theorem.

As another application of our eigenvalue estimate, we consider the case of minimal surfaces in a flat three-dimensional torus. Let N be such a torus, which we know we can write isometrically as

$$N = \mathbf{R}^3/\Lambda,$$

where Λ is a cocompact lattice, and let M be a closed minimal oriented surface immersed in N. We can define the Gauss map

$$G: M \to S^2$$

by viewing M as a minimal surface in $\mathbf{R}^3$, periodic with respect to the lattice Λ.

The index of M is the number of negative eigenvalues of

$$\Delta + |A|^2 = \Delta - 2K$$

on M, where A denotes the second fundamental form of M in N, and K denotes the Gaussian curvature of M. We endow M with the pullback metric from S^2 via G and conclude, using the same argument as in the preceding example, that

$$\text{index of } M \leq (7.68183) \cdot k,$$

where k is the degree of the Gauss map.

References

[1] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z., **143** (1975), 289–297.

[2] D. Fischer-Colbrie, *On complete minimal surfaces with finite index in three manifolds*, preprint.

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