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CHARACTERIZING REDUCED WITT RINGS OF HIGHER LEVEL

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Mulcahy's Spaces of Signatures (SOS) is an abstract setting for the reduced Witt rings of higher level of Becker and Rosenberg just as Marshall's Spaces of Orderings is an abstract setting for the ordinary reduced Witt ring. Finitely constructible SOS's are those built up in a finite number of steps from the smallest SOS using 2 operations. We show that finitely constructible SOS's are precisely those that arise from preordered fields (subject to a certain finiteness condition). This allows us to give an inductive construction for the reduced Witt rings of higher level for certain preordered fields, which generalizes a result of Craven for the ordinary reduced Witt ring. We also obtain a generalization of Bröcker's results on the possible number of orderings of a field.

1. Preliminaries. For a field K we set $\dot{K} = K \setminus \{0\}$. The symbol $\sqcup$ stands for disjoint union. We begin by recalling some of the theory of preorders of higher level from [**4**]:

A subset T of K is a preorder if $\dot{T} = T \setminus \{0\}$ is a subgroup of $\dot{K}$ and $\dot{T} + \dot{T} \subseteq \dot{T}$. We assume throughout that all preorders are of finite exponent, i.e., $K^m \subseteq T$ for some $m \in \mathbf{N}$. Since $-1 \notin \dot{T}$, the exponent of the group $\dot{K}/\dot{T}$ is even, say $2n$. We call n the level of T.

Let $\mu = \{z \in \mathbf{C}\colon z^r = 1 \text{ for some } r \in \mathbf{N}\}$. For an abelian group G let G^* denote $\operatorname{Hom}(G, \mu)$, with the usual compact-open topology. $\chi \in (\dot{K})^*$ is called a signature if $\ker\chi + \ker\chi \subseteq \ker\chi$. We write $\operatorname{Sgn}(K)$ for the set of signatures of K. For a preorder T let $X_T = \{\chi \in \operatorname{Sgn}(K)\colon \chi(\dot{T}) = 1\}$. Then $\dot{T} = \bigcap_{\chi \in X_T} \ker\chi$ [**4**, 1.4].

We make extensive use of Krull valuations: If $v\colon \dot{K} \to \Gamma$ is a valuation, we denote the valuation ring by A, the group of units by U, the maximal ideal by I and the residue class field by $\mathscr{k}$. If $\mathscr{k}$ is formally real, we say v is a real valuation.

An element $\chi \in (\dot{K})^*$ is "compatible" with a valuation v, written $v \sim \chi$, if $1 + I \subseteq \ker\chi$. In this case, the equation $\bar{\chi}(u + I) = \chi(u)$ defines an element $\bar{\chi} \in (\dot{\mathscr{k}})^*$, called the pushdown of χ along v, and $\chi \in \operatorname{Sgn}(K)$ iff $\bar{\chi} \in \operatorname{Sgn}(\mathscr{k})$ [**3**, 1.12, 2.5]. A preorder T of K is "fully compatible", written $v \sim_f T$, if each $\chi \in X_T$ is compatible with v, i.e., if $1 + I \subseteq T$. In that case the image of $A \cap T$ in $\mathscr{k}$, denoted $\bar{T}$, is a preorder of $\mathscr{k}$.

For $\chi \in \mathrm{Sgn}(K)$, let $A(\chi) = \{a \in K$: there is an $n \in \mathbf{N}$ with $n + a$ and $n - a \in \ker\chi\}$. Then $A(\chi)$ is the smallest valuation ring such that $A(\chi) \sim \chi$ and the pushdown of χ along $A(\chi)$ is archimedean [**3**, 2.7].

DEFINITION 1.1 (cf. [**13**], [**17**]). Let G be an abelian group of exponent $2s$, for some $s \in \mathbf{N}$. Let $X \subseteq G^*$ be nonempty. An *n-dimensional form f over* (X, G) is an n-tuple $\langle a_1, \ldots, a_n\rangle$ with $a_i \in G$. We write $\dim f = n$ and, for $\sigma \in X$, $\sigma(f) = \sum_{i=1}^{n}\sigma(a_i)$. Two forms f and g are said to be *equivalent* if $\sigma(f) = \sigma(g)$ for all $\sigma \in X$. We write $f \equiv g$. f and g are *isometric*, written $f \cong g$, if $f \equiv g$ and $\dim f = \dim g$. We say a form f *represents* $x \in G$ if $f \cong \langle x, x_2, \ldots, x_n\rangle$ for some $x_i \in G$. $D(f)$ denotes the set of all elements of G represented by f. The sum, $f \oplus g$, and the product, $f \otimes g$, are defined in the usual way (see [**13**]) and the form $\langle a\rangle \otimes f$, $a \in G$, is denoted af. We use the convention that there is an empty form $\langle\ \rangle$, i.e., a form with no entries. $\mathrm{Dim}(\langle\ \rangle) = 0$ and $\sigma(\langle\ \rangle) = 0$.

The pair (X, G) is called a *Space of Signatures*, or *SOS*, when the following axioms hold:

S_0: For any $\sigma \in X$ and any odd integer k, $\sigma^k \in X$.

S_1: X is closed in G^*.

S_2: There is an $e \in G$ such that $\sigma(e) = -1$ for all $\sigma \in X$.

S_3: $X^\perp = \{a \in G$: $\sigma(a) = 1$ for all $\sigma \in X\} = 1$.

S_4: If f and g are forms over (X, G) and $z \in D(f \oplus g)$, then there is an $x \in D(f)$ and a $y \in D(g)$ such that $z \in D(\langle x, y\rangle)$.

S_5: If $\sigma \in G^* \setminus \{1\}$ is such that $D(\langle 1, x\rangle) \subseteq \ker\sigma$, for all $x \in \ker\sigma$, then $\sigma \in X$.

REMARKS 1.2.

(i) By S_3, the e of S_2 is unique and we denote it by -1.

(ii) If $G^2 = 1$, then a Space of Signatures is precisely what Marshall called a Space of Orderings [**13, 14, 15, 16**].

(iii) If T is a preorder of K, then $(X_T, \dot{K}/\dot{T})$ is an SOS [**17**, 1.10(iii)].

(iv) If (X_1, G_1) and (X_2, G_2) are two SOS's and there is an isomorphism α: $G_1 \to G_2$ such that $\alpha^*(X_2) = X_1$, where α^* denotes the dual map, we will write $(X_1, G_1) = (X_2, G_2)$.

DEFINITION 1.3. An SOS (X, G) is *realizable* if there exists a field K and a preorder T such that $(X_T, \dot{K}/\dot{T}) = (X, G)$.

EXAMPLES 1.4. (i) Let $G = \{\pm 1\}$ be the 2-element group. Let $X \subseteq G^*$ consist of the character that sends -1 to -1. We denote the pair (X, G) by $\mathscr{C}_2$. Obviously $\mathscr{C}_2$ is realizable, for example by $K = \mathbf{R}$ and $T = \mathbf{R}^2$.

(ii) (cf. [**17**, 1.10(iii)]). Let G be any group of finite even exponent, and fix an element -1 of order 2. Set $X = \{\chi \in G^*\colon \chi(-1) = -1\}$. By analogy with [**4**, p. 448] we call (X, G) a *fan*. We will see later that all fans are realizable SOS's.

2. Group extensions and direct sums. In this section we examine two ways in which a SOS can be built up out of "smaller" SOS's. Our main goal is to show that if we start with realizable SOS's our new SOS is also realizable.

DEFINITION 2.1 (cf. [**15**, 3.6]). Suppose (X_0, G_0) is a SOS and G is an abelian group of finite exponent with $G_0 \subset G$. Set $X = \{\chi \in G^*\colon \chi|_{G_0} \in X_0\}$. The pair (X, G) is called a *group extension of* (X_0, G_0). We shall say that (X, G) is a *group extension* if there exists (X_0, G_0) such that (X, G) is a group extension of (X_0, G_0). If (X, G) is a group extension, then it is a SOS with $-1_G = -1_{G_0}$ [**17**, 2.6].

REMARKS 2.2. (i) Group extensions of a given SOS (X_0, G_0) correspond to abelian group extensions of G_0: If (X, G) is a group extension of (X_0, G_0) then we have an exact sequence of abelian groups $1 \to G_0 \to G \to G/G_0 \to 1$. Conversely, given an exact sequence $1 \to G_0 \to G \to H \to 1$ where H is abelian of finite exponent, then setting $X = \{\chi \in G^*\colon \chi|_{G_0} \in X_0\}$ and identifying G_0 with its image in G, we see that the pair (X, G) is a group extension of (X_0, G_0).

(ii) Suppose (X, G) is a group extension of (X_0, G_0). Since μ is divisible, it is $\mathbf{Z}$-injective [**7**, Prop. 5.1, p. 134] and thus there is a dual exact sequence

$$1 \to (G/G_0)^* \to G^* \xrightarrow{\text{res}} G_0^* \to 1,$$

where res is the restriction map. Then $\chi \in X$ iff $\text{res}(\chi) \in X_0$, and thus there is a non-canonical bijection $X \leftrightarrow X_0 \times (G/G_0)^*$.

(iii) Clearly (X, G) is a fan iff (X, G) is a group extension of $\mathscr{C}_2$ or $\mathscr{C}_2$ itself.

(iv) Equivalent abelian group extensions, in the sense of [**10**, p. 211], give rise to the "same" SOS, in the sense of [1.2, (v)].

PROPOSITION 2.3. *Let K be a field, $T \subseteq K$ a preorder and v a valuation such that $v \sim_f T$. If $\Gamma/v(\dot{T}) = 1$, then $(X_T, \dot{K}/\dot{T}) = (X_{\bar{T}}, \dot{\bar{k}}/\dot{\bar{T}})$. If $\Gamma/v(\dot{T}) \neq 1$, then $(X_T, \dot{K}/\dot{T})$ is a group extension of $(X_{\bar{T}}, \dot{\bar{k}}/\dot{\bar{T}})$.*

Proof. By [**3**, 2.6] the sequence

$$1 \to \dot{\bar{k}}/\dot{\bar{T}} \xrightarrow{\alpha} \dot{K}/\dot{T} \xrightarrow{\beta} \Gamma/v(\dot{T}) \to 1$$

is exact, where $\alpha(\overline{u\dot{T}}) = u\dot{T}$ and $\beta(a\dot{T}) = v(a)v(\dot{T})$. By [**4**, 2.7] $\alpha^*(\chi) \in X_{\bar{T}}$ iff $\chi \in X_T$.

COROLLARY 2.4. *Given a field K, a preorder T and a valuation v such that $v \sim_f T$. Then T is a fan iff $\bar{T}$ is a fan.*

Proof. This follows easily from [2.3] if we note the following: If (X, G) is a group extension of (X_0, G_0), then for any $\chi \in G^*$, $\chi(-1) = -1$ iff $\chi|_{G_0}(-1) = -1$.

Now we fix a field F, a preorder $T \subseteq F$ and a SOS (X, G) which is a group extension of $(X_T, \dot{F}/\dot{T})$. Thus we have an exact sequence

$$(1) \qquad 1 \to \dot{F}/\dot{T} \to G \to H \to 1.$$

We want to show that (X, G) is realizable. To do this we will construct a field K, a preorder $Q \subseteq K$, and a valuation $v \sim_f Q$ such that the exact sequence of [2.3] is equivalent to (1).

PROPOSITION 2.5. (*With the above notation*), (X, G) *is realizable.*

Proof. Fix λ': $H \times H \to \dot{F}/\dot{T}$, a factor set corresponding to the exact sequence (1). Since H is abelian of finite exponent, $H \cong \Gamma/\Delta$ where Γ is a direct sum of copies of $\mathbf{Z}$ [**10**, 15.2]. Order Γ lexicographically. By [**10**, 51.3], there is a factor set λ: $\Gamma \times \Gamma \to F$ such that $\lambda(\gamma_1, \gamma_1)\dot{T} = \lambda'(\gamma_1\Delta, \gamma_2\Delta)$ for all $\gamma_1, \gamma_2 \in \Gamma$.

Let K be the formal power series of Γ over F: This is the set of functions Ω: $\Gamma \to \dot{F}$ with $S(\Omega)$, the support of Ω, well-ordered. Addition is defined in the obvious way and multiplication is given by

$$(\Omega_1\Omega_2)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} \lambda(\gamma_1, \gamma_2)\Omega_1(\gamma_1)\Omega_2(\gamma_2)$$

for all $\Omega_i \in K$ and $\gamma \in \Gamma$. Then K is a field and the map v: $K \to \Gamma$, defined by $v(\Omega)$ = least element of $S(\Omega)$, is a valuation with value group Γ and residue class field F [**20**, 5.3 and 5.4].

Let $Q = \{\Omega \in K\colon\ v(\Omega) \in \Delta \text{ and } \Omega(v(\Omega)) \in T\}$. It is straightforward to check that Q is a preorder, $v \sim_f Q$, $\bar{Q} = T$, and $v(\dot{Q}) = \Delta$.

We want to show that $\dot{K}/\dot{Q} \cong G$. Since λ' is a factor set corresponding to (1), we can identify G with pairs $(a\dot{T}, h)$, $a \in \dot{F}$ and $h \in H$, where

$$(a_1\dot{T}, h_1)(a_2\dot{T}, h_2) = (\lambda'(h_1, h_2)a_1a_2\dot{T}, h_1h_2).$$

Define θ: $\dot{K} \to G$ via $\theta(\Omega) = (\Omega(v(\Omega))\dot{T}, v(\Omega)\Delta)$. Since the "twisting" in K is via λ and that of G is via λ', a straightforward check shows that θ is a homomorphism. It is clear that θ is surjective and $\ker\theta = \dot{Q}$. Hence $\dot{K}/\dot{Q} \cong G$.

By [2.3], $(X_Q, \dot{K}/\dot{Q})$ is the group extension of $(X_T, \dot{F}/\dot{T})$ arising from the exact sequence

$$1 \to \dot{F}/\dot{T} \to \dot{K}/\dot{Q} \xrightarrow{\bar{v}} H \to 1 \tag{2}$$

where $\bar{v}(\Omega + \dot{Q}) = v(\Omega) + \Delta$. An easy check shows that (1) and (2) are equivalent exact sequences and thus $(X, G) = (X_Q, \dot{K}/\dot{Q})$.

REMARK 2.6. For $G^2 = 1$ and $T = \sum F^2$ [2.5] is easy (see [**8**, 2.4]). In this case the field K is an iterated power series field $F((t_1))((t_2))\ldots$, where the number of variables is equal to $\dim F_2(H)$.

DEFINITION 2.7. Let (X_1, G_1) and (X_2, G_2) be two SOS's and set $G = G_1 \times G_2$. Then $G^* \cong G_1^* \times G_2^*$ and thus $X_1 \times \{1\}$ and $\{1\} \times X_2$ embed in G^*. Set $X = (X_1 \times \{1\}) \cup (\{1\}) \times X_2)$. This union is clearly disjoint and thus we write $X = X_1 \sqcup X_2$. The pair (X, G) is a SOS [**17**, 2.3], called the *direct sum of* (X_1, G_1) *and* (X_2, G_2) and we write $(X, G) = (X_1, G_1) \oplus (X_2, G_2)$.

THEOREM 2.8. *Suppose K_1 and K_2 are fields with preorders $T_i \subseteq K_i$. Then $(X_{T_1}, \dot{K}_1/\dot{T}_1) \oplus (X_{T_2}, \dot{K}_2/\dot{T}_2)$ is realizable.*

The case of $T_i = \sum K_i^2$ is proven (in a different form) by Craven [**8**, 2.4]. The proof uses a construction of Bröcker [**6**] (see also [**19**, §1]). To prove [2.8] we need to construct a field F and a preorder T such that $\dot{F}/\dot{T} \cong \dot{K}_1/\dot{T}_1 \times \dot{K}_2/\dot{T}_2$ and $X_T = X_{T_1} \sqcup X_{T_2}$. The construction of F and T will follow along the lines of Bröcker's construction. However, we will need new methods to show that $X_T = X_{T_1} \sqcup X_{T_2}$: We make use of an equivalence relation on X_T defined in [**4**]. Before proving [2.8], we need several lemmas.

LEMMA 2.9 (*cf.* [**19**, 1.1]). *Let K be a field and $v_1, \ldots, v_r$ pairwise independent (non-trivial) valuations on K. Suppose for each i, $1 \le i \le r$, we have (K_i, v_i') an immediate extension of (K, v_i)* [**9**, *p.* 24] *together with a preorder T_i of K_i such that $v_i' \sim_f T_i$. Then the diagonal map $\dot{K} \to \dot{K}_1/\dot{T}_1 \times \cdots \times \dot{K}_r/\dot{T}_r$ is onto.*

Proof. Let y in $\dot{K}_1/\dot{T}_1 \times \cdots \times \dot{K}_r/\dot{T}_r$ be $(m_1\dot{T}_1, \ldots, m_r\dot{T}_r)$. Since (K_i, v_i') is an immediate extension of (K, v_i), for each i there is an $a_i \in \dot{K}$ such that $v_i(a_i) = v_i'(m_i)$, thus $m_i/a_i = u_i'$ for some unit u_i' in K_i. Then there is a unit u_i in (K, v_i) such that $u_i' + I_i' = u_i + I_i$. Hence $m_i = (a_iu_i)(1 + x_i)$ for some $x_i \in I_i'$.

Let $n = \max\{\text{level}\, T_i\}$. By the Approximation Theorem for Independent Valuations [**5**, Chap. 6, §7, No. 2, Thm. 1], there is an $a \in K$ such that $v_i(a - (a_iu_i)^{2n-1}) > v_i((a_iu_i)^{2n-1})$, hence $a = (a_iu_i)^{2n-1}(1 + y_i)$ for some $y_i \in I_i$. Thus we have $am_i = (a_iu_i)^{2n}(1 + x_i)(1 + y_i)$ and hence $am_i \in T_i$. Thus $1/a$ maps to y and we are done.

LEMMA 2.10. *Given a field K and preorder T of level $\leq n$. Suppose v is a valuation on K such that the value group Γ is $2n$-divisible and $v \sim_f T$. Then $(X_T, \dot{K}/\dot{T}) = (X_{\bar{T}}, \dot{k}/\dot{\bar{T}})$.*

Proof. An easy generalization of [**12**, 3.7] shows that $v(\sum K^{2n}) = 2n\Gamma$. Thus $2n\Gamma \subseteq v(\dot{T})$ and hence $\Gamma/v(\dot{T}) = 1$. The lemma now follows from [2.3].

LEMMA 2.11. *Let v be a henselian valuation on K with residue class field k. Then any preorder of K is fully compatible with v and given any preorder Q of k there is a preorder T of K such that $\bar{T} = Q$.*

Proof. An easy generalization of [**12**, 4.16] shows that a henselian valuation is fully compatible with $\sum K^{2n}$ for any n, and hence with any preorder. Let $S = \sum K^{2n}$, where $n = \text{level}\, Q$, then clearly $\bar{S} \subseteq Q$. We let T be the "wedge product" $S \wedge Q = S \cdot \{u \in U\colon \bar{u} \in Q\}$. By the first statement, $v \sim_f T$ and by [**4**, 2.5], $\bar{T} = Q$.

The following is from [**4**, §5]:

DEFINITION 2.12. (i) We define the equivalence relation of "dependency" on X_T as follows: If χ is archimedean, then $\chi \sim \chi'$ when $\chi = \chi'$. Otherwise $\chi \sim \chi'$ when $A(\chi)A(\chi') \neq K$.

(ii) We set $A[\chi] = \prod_{\chi' \sim \chi} A(\chi')$, clearly a real valuation ring, and we set $T[\chi] = (\bigcap_{\chi' \sim \chi} \ker \chi') \cup \{0\}$, clearly a preorder of finite exponent. If X_T has only 1 dependency class, we write A_T for $A[\chi]$ (χ any element of X_T).

Becker and Rosenberg prove results on $A[\chi]$ and $T[\chi]$ when T is of finite index. However, their proofs do not use the full strength of this assumption and thus more general statements hold.

LEMMA 2.13 (*cf.* [**4**, 5.6 *and* 5.7]). *Let K be a field and T a preorder such that X_T has only finitely many dependency classes $[\chi_1], \ldots, [\chi_s]$. We write $T_i = T[\chi_i]$. Further suppose that for all non-archimedean χ_i, $A[\chi_i] \neq K$. Then*

(i) $X_{T_i} = [\chi_i]$

(ii) *The natural map $\dot{K}/\dot{T} \to \prod_1^s \dot{K}/\dot{T}_i$ is an isomorphism, and hence $(X_T, \dot{K}/\dot{T}) = \oplus_{i=1}^s (X_{T_i}, \dot{K}/\dot{T}_i)$.*

Proof. If χ_i is archimedean, (i) is clear. For non-archimedean χ_i, the proof of (i) is exactly that of [**4**, 5.6(i)]. The proof of (ii) is exactly that of [**4**, 5.7].

Proof of Theorem 2.8. Let α_i be the transcendence degree of K_i over $\mathbf{Q}$. We first show that w.l.o.g we can assume $\alpha_1 = \alpha_2$. Suppose $\alpha_1 < \alpha_2$.

Let v be the x-adic valuation on $K_1(x)$, then $\mathscr{k}_v = K_1$ and $\Gamma_v = \mathbf{Z}$. Let $n =$ level T_1 and let $\Gamma' = \{k/(2n)^m\colon k, m \in \mathbf{Z}\}$

By [**9**, 28.1], there is an extension of valued fields $(L, \omega) \supseteq (K_1(x), v)$ such that L is algebraic over $K_1(x)$, $\Gamma_\omega = \Gamma'$ and $\mathscr{k}_\omega = K_1$. In particular, Γ_ω is $2n$-divisible.

Let (K, v') be a henselization of (L, ω). By [2.11], there is a preorder T of K such that $\overline{T} = T_1$. Then, by [2.10], $(X_T, \dot{K}/\dot{T}) = (X_{T_1}, \dot{K}_1/\dot{T}_1)$ and clearly the transcendence degree of K over $\mathbf{Q}$ is $\alpha_1 + 1$.

Using Zorn's lemma and the above construction, we see that we can replace K_1 with a field of transcendence degree α_2 over $\mathbf{Q}$ without changing the SOS. Thus w.l.o.g. we can assume $\alpha_1 = \alpha_2$.

Let L be a purely transcendental extension of $\mathbf{Q}$ of degree α_1, then K_1 and K_2 are algebraic over L. Let v be the x-adic valuation on $L(x)$ and let ω be the degree valuation on $L(x)$. As above, by [**9**, 28.1] there is an algebraic extension of $L(x)$, L', and valuations v' and ω' extending v and ω such that $\mathscr{k}_{v'} = K_1$, $\mathscr{k}_{\omega'} = K_2$, $\Gamma_{v'}$ is $2n_1$-divisible and $\Gamma_{\omega'}$ is $2n_2$-divisible, where $n_i =$ level T_i.

Let (M_1, v'') be a henselization of (L', v') and let (M_2, ω'') be a henselization of (L', ω'), both in the same algebraic closure of L'. By [2.11], there are preorders Q_i of M_i such that $\overline{Q}_i = T_i$. Hence, by [2.10], $(X_{T_i}, \dot{M}_i/\dot{Q}_i) = (X_{T_i}, \dot{K}_i/\dot{T}_i)$.

Now we let $F = M_1 \cap M_2$ and $T = Q_1 \cap Q_2$. Set $v_0 = v''|_F$ and $\omega_0 = \omega''|_F$. Since $L' \subseteq F$, M_1 and M_2 are algebraic over F and thus, since v and ω are independent, v_0 and ω_0 are independent. Also note that (M_1, v'') is a henselization of (F, v_0) and (M_2, ω'') is a henselization of (F, ω_0). In particular, $(M_1, v'') \supseteq (F, v_0)$ and $(M_2, \omega'') \supseteq (F, \omega_0)$ are

immediate extensions and thus, by [2.9], the diagonal map $\dot{F}/\dot{T} \to \dot{M}_1/\dot{Q}_1 \times \dot{M}_2/\dot{Q}_2$ is an isomorphism.

It remains to show that $X_T = X_{Q_1} \sqcup X_{Q_2}$. First we note that F, M_1 and M_2 have no archimedean orders: If P were an archimedean order on M_1, then since $v'' \sim P$ [2.11], we would have $M_1 = A(P) \subseteq A_{v''}$ [3, 2.7], a contradiction. Similarly, M_2 has no archimedean orders. By [**19**, 1.3] and [**1**, Chap. 1, Lemma 6], any archimedean order on F would lift to one on M_1 or M_2, thus F has no archimedean orders.

Now given $\chi \in X_T$, we claim that $v_0 \sim \chi$ or $\omega_0 \sim \chi$. Let A_1 be the valuation ring of v_0 and let A_2 be the valuation ring of ω_0. First suppose that $A(\chi)$, A_1 and A_2 are pairwise independent. Let (M_3, v_1) be a henselization of $(F, A(\chi))$, then by [2.11] there is a preorder T_3 of M_3 such that $\overline{T}_3 = \ker \chi$. By [2.9] the diagonal map $\dot{F} \to \dot{M}_1/\dot{Q}_1 \times \dot{M}_2/\dot{Q}_2 \times \dot{M}_3/\dot{T}_3$ is onto, which is a contradiction since $\dot{F} \to \dot{M}_1/\dot{Q}_1 \times \dot{M}_2/\dot{Q}_2$ is onto. Thus, since A_1 and A_2 are independent, either $A(\chi)A_1 \neq F$ or $A(\chi)A_2 \neq F$, say $A(\chi)A_1 \neq F$. By our construction of Γ_{v_0}, v_0 is a rank 1 valuation. Then, since $A_1 \subseteq A(\chi)A_1$ we must have $A_1 = A(\chi)A_1$ and so $A(\chi) \subseteq A_1$. Thus $v_0 \sim \chi$. Hence any $\chi \in X_T$ has $v_0 \sim \chi$ or $\omega_0 \sim \chi$.

Fix $\chi_i' \in X_{Q_i}$ and let $\chi_i = \chi_i'|_F$. Then clearly $v_0 \sim \chi_1$ and $\omega_0 \sim \chi_2$. Suppose χ_1 and χ_2 are in the same dependency class in X_T, then $A(\chi_1)A(\chi_2) = A \neq F$. Since valuation rings containing a given one are linearly ordered [**5**, Chap. 6, 4.1], either $A \subseteq A_1$ or $A_1 \subseteq A$.

As above, we must have $A \subseteq A_1$, and similarly $A \subseteq A_2$. But this implies that A_1 and A_2 are linearly ordered which is impossible since they are independent. Thus $[\chi_1] \neq [\chi_2]$.

Since every $\chi \in X_T$ has $v_0 \sim \chi$ or $\omega_0 \sim \chi$, every χ is in $[\chi_1]$ or $[\chi_2]$. Furthermore, $A[\chi_1] \subseteq A_1$ and $A[\chi_2] \subseteq A_2$. Let $S_i = T[\chi_i]$, then, by [2.13], the diagonal map $\dot{F}/\dot{T} \to \dot{F}/\dot{S}_1 \times \dot{F}/\dot{S}_2$ is an isomorphism and $X_T = X_{S_1} \sqcup X_{S_2}$.

We claim that $S_i = Q_i \cap F$. If $\chi(Q_1 \cap F) = 1$, then $v_0 \sim \chi$ and so $\chi \in X_{S_1}$. Hence $X_{Q_1 \cap F} \subseteq X_{S_1}$ and thus $S_1 \subseteq Q_1 \cap F$ [**4**, 1.4]. Similarly, $S_2 \subseteq Q_2 \cap F$. Given $x \in Q_1 \cap F$. By the above, there is a $y\dot{T} \in \dot{F}/\dot{T}$ which maps to $(x\dot{S}_1, \dot{S}_2)$ under the diagonal map. Now, $y \in S_2 \subseteq Q_2$ and $yx^{-1} \in S_1 \subseteq Q_1$, thus $y \in Q_1 \cap Q_2 = T$, since $x \in Q_1$, and thus $x \in S_1$. Hence $Q_1 \cap F \subseteq S_1$ and similarly we see that $Q_2 \cap F \subseteq S_2$. Thus $S_i = Q_i \cap F$.

Given $\chi \in X_T$, $\chi(Q_1 \cap F) = 1$ or $\chi(Q_2 \cap F) = 1$. Suppose $\chi(Q_1 \cap F) = 1$, then since the diagonal map $\dot{F}/\dot{T} \to \dot{M}_1/\dot{Q}_1 \times \dot{M}_2/\dot{Q}_2$ is an isomorphism, χ lifts to a character $\theta \in (\dot{M}_1/\dot{Q}_1)^*$. An easy check shows that $\overline{\chi} = \overline{\theta}$ in the residue class field of v_0 and v'' and hence

$\theta \in X_{Q_1}$ [**3**, 2.7]. Similarly, if $\chi(Q_2 \cap F) = 1$, χ lifts to X_{Q_2}. Thus $X_T = X_{Q_1} \sqcup X_{Q_2}$ and we are done.

3. Finitely constructible spaces of signatures and the reduced Witt ring.

DEFINITION 3.1. We define m-constructible SOS's inductively as follows: (X, G) is 1-constructible iff $(X, G) = \mathscr{C}_2$. For $m > 1$, (X, G) is m-constructible iff one of the following holds:

(a) There exist SOS's (X_1, G_1) and (X_2, G_2) and $k < m$ such that each (X_i, G_i) is k-constructible and $(X, G) = (X_1, G_1) \oplus (X_2, G_2)$.

(b) There exists a SOS (X_0, G_0) and $k < m$ such that (X_0, G_0) is k-constructible and (X, G) is a group extension of (X_0, G_0).

(c) (X, G) is k-constructible for some $k < m$.

We say (X, G) is finitely constructible if it is m-constructible for some $m \in \mathbf{N}$.

DEFINITION AND REMARK 3.2 [**4**, Sec. 5]. The pushdown of a signature χ along $A(\chi)$ is archimedean and thus induces a unique order embedding of $\mathscr{k}(\chi)$ into $\mathbf{R}$. Thus χ leads to a real-valued place $\lambda(\chi)$ with valuation ring $A(\chi)$. Set $M_T = \{\lambda(\chi) \colon \chi \in X_T\}$.

If $\lambda(\chi) = \lambda(\chi')$, then $A(\chi) = A(\chi')$ and thus if χ is non-archimedean, $\chi \sim \chi'$ [2.12]. If χ is archimedean, then $\lambda(\chi) = \lambda(\chi')$ iff $\chi = \chi'$ [**5**, Prop. 5, Sec. 3, No. 2]. Hence the number of dependency classes of X_T is $\leq |M_T|$.

LEMMA 3.3. *If $|M_T| = 1$, then T is a fan.*

Proof. Let $A = A(\chi)$ for some $\chi \in X_T$. By assumption, $A(\chi') = A$ for all $\chi' \in X_T$. Thus $A \sim_f T$ and $\overline{T}$, the pushdown along A, is a fan. Hence, by [2.4], T is a fan.

LEMMA 3.4. *Given a valuation ring v such that $v \sim_f T$. Then $|M_{\overline{T}}| = |M_T|$.*

Proof. Given $\lambda \in M_T$, define $\theta(\lambda) \in M_{\overline{T}}$ via $\theta(\lambda)(u + I) = \lambda(u)$. Given $\lambda' \in M_{\overline{T}}$, define $\theta'(\lambda') \in M_T$ via $\theta'(\lambda')(x) = \infty$ if $x \notin A$ and $\theta'(\lambda')(x) = \lambda'(x + I)$ for $x \in A$. Then clearly $\theta(\theta') =$ identity on $M_{\overline{T}}$ and $\theta'(\theta) =$ identity on M_T. Thus θ is a bijection and so $|M_{\overline{T}}| = |M_T|$.

We are now ready to prove our main theorem.

THEOREM 3.5. *(X, G) is finitely constructible iff (X, G) is realizable by a field K and a preorder T such that $|M_T| < \infty$.*

Proof. If (X, G) is m-constructible we use induction on m and [2.5] and [2.8] to show that (X, G) is realizable. Note that if $X_T = X_{T_1} \sqcup X_{T_2}$, then clearly $|M_T| \leq |M_{T_1}| + |M_{T_2}|$ and thus direct sum preserves finiteness of M_T. By [3.4], our group extension construction does also and hence the preorder T constructed has $|M_T| < \infty$.

Now suppose that T is a preorder with $|M_T| < \infty$. We induct on $|M_T|$. If $|M_T| = 1$ we are done by [3.3], since every fan is $\mathscr{C}_2$ or a group extension of $\mathscr{C}_2$.

Suppose $|M_T| > 1$. By [3.2], X_T has only finitely many dependency classes.

Case 1. The dependency classes of X_T are $X_{T_1}, \ldots, X_{T_s}$ where $s > 1$. By [3.2] we have $M_T = M_{T_1} \sqcup \cdots \sqcup M_{T_s}$ and thus by induction each $(X_{T_i}, \dot{K}/\dot{T}_i)$ is finitely constructible. Hence, by [2.13, ii] $(X_T, \dot{K}/\dot{T})$ is finitely constructible.

Case 2. Suppose X_T has 1 dependency class. Since $|M_T| > 1$, T is non-archimedean. Since $|M_T| < \infty$, there are only finitely many $A(\chi)$'s, $\chi \in X_T$, and thus, as in [**4**, 5.6(i)], we see that $A_T \neq K$. Let $\overline{T}$ be the pushdown of T along A_T, then by [2.3] it suffices to show that $(X_{\overline{T}}, \dot{\mathscr{k}}/\dot{\overline{T}})$ is finitely constructible.

If $X_{\overline{T}}$ has more than one dependency class then, since $|M_{\overline{T}}| = |M_T|$ [3.4], the proof of Case 1 shows that $(X_{\overline{T}}, \dot{\mathscr{k}}/\dot{\overline{T}})$ is finitely constructible. If $X_{\overline{T}}$ has 1 dependency class, then $A_{\overline{T}} = \mathscr{k}$ [**2**, p. 1961–1963]. As above, this implies that $\overline{T}$ is archimedean. In particular, $\overline{T}$ is a fan and we are done.

DEFINITION 3.6. (i) Let (X, G) be a SOS. For a form f over (X, G) denote by $[f]$ its X-equivalence class [1.1]. The set $W(X)$ of equivalence classes of forms over (X, G) carries a natural ring structure as follows: We define $[f] + [g]$ to be $[f \oplus g]$ and we define $[f] \cdot [g]$ to be $[f \otimes g]$. Then if we define $-[f]$ to be $[-f]$, clearly $(W(X), +, \cdot)$ is a commutative ring with $1 = [\langle 1 \rangle]$ and $0 = [\langle 1, -1 \rangle]$, the class of all hyperbolic forms. The ring $W(X)$ is called the *Witt Ring of* (X, G).

(ii) Let $I(X)$ in $W(X)$ be the set of classes of even-dimensional forms. That this is well defined follows from the fact that if f and g are X-equivalent forms, then $\dim f \equiv \dim g \pmod 2$, which can be shown as in the proof of [**4**, 4.8]. It is clear that $I(X)$ is an ideal.

REMARK 3.7. Given a field K and preorder T then W_T, the higher level reduced Witt ring of [**4**, 4.2], is easily seen to be isomorphic to $W(X_T)$.

DEFINITION 3.8 [**18**, p. 13]. (i) Given a commutative ring R, denote by $U(R)$ the abelian group of units of R.

(ii) Given a commutative ring R and an abelian group H. Suppose λ: $H \times H \to U(R)$ is a factor set. We define the *twisted group ring of R over H by λ*, $R^t[H]$, as the set of formal sums $\sum_{h \in H}(r_h, h)$ where $r_h \in R$ and only finitely many r_h's are non-zero. If $\sum_{h \in H}(s_h, h)$ is another element of $R^t[H]$ then addition is defined by $\sum(r_h, h) + \sum(s_h, h) = \sum(r_h + s_h, h)$, and multiplication is defined by $(\sum(r_h, h)) \cdot (\sum(s_h, h)) = \sum(z_h, h)$, where $z_h = \sum_{h_1 h_2 = h} \lambda(h_1, h_2) r_{h_1} s_{h_2}$. Then $R^t[H]$ is a commutative ring with $0 = (0_R, 1_H)$ and $1 = (1_R, 1_H)$.

DEFINITION 3.9 (cf. [**4**, 2.10] and [**17**, 3.11]). Given a SOS (X, G) which is a group extension of (X_0, G_0). Then we have an exact sequence $1 \to G_0 \to G \to G/G_0 \to 1$ which gives a rise to a factor set λ: $G/G_0 \times G/G_0 \to G_0$ such that under this bijection the multiplication in G is given by: $(e_1, h_1) \cdot (e_2, h_2) = (\lambda(h_1, h_2) e_1 e_2, h_1 h_2)$ for all $e_i \in G_0$ and $h_i \in G/G_0$.

Given $f = \langle a_1, \ldots, a_n \rangle$, a form over (X_0, G_0), and $h \in G/G_0$, let $\{f, h\} = \langle (a_1, h), \ldots, (a_n, h) \rangle$, a form over (X, G). Let f be any form over (X, G), then for each $h \in G/G_0$ there is obviously a (possibly empty) form f_h over (X_0, G_0), called the *hth residue form of f*, such that $f = \oplus_{h \in G/G_0} \{f_h, h\}$.

LEMMA 3.10 (*cf.* [**4**, 4.7]). *Let f and g be forms over (X, G). Then $[f] = [g]$ in $W(X)$ iff $[f_h] = [g_h]$ in $W(X_0)$ for all $h \in G/G_0$.*

Proof. This follows easily from [**17**, 2.7].

PROPOSITION 3.11. *Given (X, G) a group extension of (X_0, G_0). Let λ be a factor set arising from the exact sequence $1 \to G_0 \to G \to G/G_0 \to 1$. Then $W(X) \cong W(X_0)^t[G/G_0]$, the twisted group ring of $W(X_0)$ by G/G_0 arising from λ.*

Proof. Define φ: $W(X) \to W(X_0)^t[G/G_0]$ by

$$\varphi([f]) = \sum_{h \in G/G_0} \{[\bar{f}_h], h\},$$

where $\bar{f}_h$ is defined as in [3.9] using λ. By [3.10], φ is well-defined and injective.

A straightforward calculation shows that for a_1, $a_2 \in G$, $\varphi([\langle a_1\rangle] + [\langle a_2\rangle]) = \varphi([\langle a_1\rangle]) + \varphi([\langle a_2\rangle])$. Then, since any $[f]$ is a sum of $[\langle a\rangle]$'s, it follows easily that φ is a homomorphism.

Given a form $\bar{f} = \langle e_1, \ldots, e_k\rangle$ over (X_0, G_0) and $h \in G/G_0$. Let f be the form $\langle (e_1, h), \ldots, (e_k, h)\rangle$ over (X, G), then clearly $\varphi([f]) = \{[\bar{f}], h\}$. Thus φ is surjective and therefore φ is an isomorphism.

The following is implicit in [**4**, §2]:

COROLLARY 3.12. *Given a field K, a preorder T, and v a valuation such that $v \sim_f T$. Then $W(X_T) \cong W(X_{\bar{T}})'[\Gamma/v(\dot{T})]$, where the twisting arises from a factor set of the canonical exact sequence $1 \to \dot{\bar{k}}/\dot{\bar{T}} \to \dot{K}/\dot{T} \to \Gamma/v(\dot{\bar{T}}) \to 1$.*

PROPOSITION 3.13. *Given* SOS's (X, G), (X_1, G_1) *and* (X_2, G_2) *such that* $(X, G) = (X_1, G_1) \oplus (X_2, G_2)$. *Then* $W(X) \cong \mathbf{Z} + (I(X_1) \times I(X_2)) \subseteq W(X_1) \times W(X_2)$, *where* $\mathbf{Z}$ *has the diagonal embedding in* $W(X_1) \times W(X_2)$.

Proof. Given f, a form over (X, G), then $f = \langle (a_1, b_1), \ldots, (a_m, b_m)\rangle$, where each a_i is in G_1 and each b_i is in G_2. Thus we write $f = f_1 \sqcup f_2$ where $f_1 = \langle a_1, \ldots, a_m\rangle$, a form over (X_1, G_1) and $f_2 = \langle b_1, \ldots, b_m\rangle$, a form over (X_2, G_2). Then it follows from the definition of direct sum that $[f_1 \sqcup f_2] = [g_1 \sqcup g_2]$ iff $\sigma(f_1 \sqcup f_2) = \sigma(g_1 \sqcup g_2)$ for all σ in X iff $\sigma_1(f_1) = \sigma_1(g_1)$ for all σ_1 in X_1 and $\sigma_2(f_2) = \sigma_2(g_2)$ for all σ_2 in X_2 iff $[f_1] = [g_1]$ in $W(X_1)$ and $[f_2] = [g_2]$ in $W(X_2)$. Now, define β: $W(X) \to W(X_1) \times W(X_2)$ by

$$\beta([f_1 \sqcup f_2]) = ([f_1], [f_2]).$$

By the above, β is well-defined and injective.

Now we want to show that the image of β is $\mathbf{Z} + I(X_1) \times I(X_2)$. Given f, a form over (X, G), when we write f as $f_1 \sqcup f_2$ we have dimension(f_1) = dimension(f_2). If dim(f_i) is even, then $\beta([f_1 \sqcup f_2]) = ([f_1], [f_2]) \subseteq I(X_1) \times I(X_2)$. If dim$(f_i)$ is odd, then

$$\begin{aligned}\beta([f_1 \sqcup f_2]) &= ([f_1], [f_2]) = ([\langle 1, -1\rangle \oplus f_1], [\langle 1, -1\rangle \oplus f_2]) \\ &= 1 + ([\langle -1\rangle \oplus f_1], [\langle -1\rangle \oplus f_2]) \subseteq \mathbf{Z} + I(X_1) \times I(X_2).\end{aligned}$$

Thus Image $(\beta) \subseteq \mathbf{Z} + I(X_1) \times I(X_2)$.

Given $n + ([g_1], [g_2])$ in $\mathbf{Z} + I(X_1) \times I(X_2)$. Suppose $g_1 = \langle a_1, \ldots, a_{2m} \rangle$ and $g_2 = \langle b_1, \ldots, b_{2k} \rangle$, then w.l.o.g. we can assume $m \leq k$. Let

$$f = n\langle 1 \rangle \oplus \langle (a_1, b_1), \ldots, (a_{2m}, b_{2m}), (1, b_{2m+1}), (-1, b_{2m+2}), \ldots, (-1, b_{2k}) \rangle,$$

a form over (X, G). Then $\beta(f) = n + ([g_1 \oplus (m-k)\langle 1, -1 \rangle], [g_2]) = n + ([g_1], [g_2])$. Hence $\mathbf{Z} + I(X_1) \times I(X_2) \subseteq \mathrm{Image}(\beta)$ and we are done.

DEFINITION 3.14. A ring R is *finitely realizable* if there is a field K and a preorder T with $|M_T| < \infty$ such that $W(X_T) \cong R$.

Using our previous result on finitely constructible SOS's [3.5], we can now give an inductive construction for finitely realizable rings. In the case where $G^2 = 1$ this is due to Craven.

THEOREM 3.15 (*cf.* [**8**, 2.1]). *Finitely realizable rings are precisely those given by the following inductive construction*:

(a) $\mathbf{Z}$ *is finitely realizable.*

(b) *If* R_1 *and* R_2 *are finitely realizable, then* $R = \mathbf{Z} + I_1 \times I_2$ *is also, where* I_1 *and* I_2 *are the ideals of even dimensional forms.*

(c) *If* R_0 *is finitely realizable,* H *is abelian of finite exponent and* λ: $H \times H \to U(R)$ *is a factor set with* $\mathrm{image}(\lambda) \subseteq \{ r \in U(R_0)$: r *has finite exponent*$\}$, *then* $R_0^t[H]$ *is finitely realizable, where the twisting arises from* λ.

Proof. This follows easily from [3.5], using [3.11], [3.13] and the fact that for a preordered field (K, T), $\dot{K}/\dot{T} = \{ r \in U(R)$: r has finite exponent$\}$ [Becker, Rosenberg, unpublished].

We now wish to make use of [3.5] to study the number of signatures in X_T for a preorder T of finite index. In the case where $G^2 = 1$, this was done by Bröcker [**6**].

DEFINITION 3.16. (i) Let A be the semigroup $\oplus_{i=1}^{\infty} \mathbf{Z}^+$ with addition defined in the usual way. For $\alpha = (e_1, \ldots, e_m, 0, \ldots)$ in A, (all entries beyond the mth one are 0) and p a prime number, let $G_{\alpha,p} = e_1 C_p \times e_2 C_{p^2} \times \cdots \times e_m C_{p^m}$, where $C_n = \mathbf{Z}/n\mathbf{Z}$ and $kC_n = C_n \times C_n \times \cdots \times C_n$, k times. Set $B = \oplus_{j=1}^{\infty} A$.

(ii) For $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k, 0, \ldots)$ in B (each α_i is in A and all entries beyond the kth one are 0), set $G_{\bar{\alpha}} = G_{\alpha_1, p_1} \times G_{\alpha_2, p_2} \times \cdots \times G_{\alpha_k, p_k}$, where $\{ p_1 < p_2 < p_3 < \cdots \}$ is the set of prime numbers.

(iii) Given $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ in B, $(\bar{\alpha}, \bar{\beta}) \sim \bar{\gamma}$ if there is an exact sequence $1 \to G_{\bar{\gamma}} \to G_{\bar{\alpha}} \to G_{\bar{\beta}} \to 1$.

(iv) For $\alpha = (e_1, e_2, \dots, e_m, 0, 0, \dots)$ in A, define $|\alpha| = \sum_{j=1}^{m} j \cdot e_j$.

Each part of the following lemma is clear or a direct consequence of the Fundamental Theorem for abelian groups.

LEMMA 3.17. (i) *If* $\alpha = (\alpha_1, \dots, \alpha_k, 0, 0, \dots)$ *is in* B, *then* $|G_{\bar{\alpha}}| = p_1^{|\alpha_1|} \cdot p_2^{|\alpha_2|} \cdot \dots \cdot p_k^{|\alpha_k|}$.

(ii) G *is a finite abelian group iff* $G = G_{\bar{\alpha}}$ *for some* $\bar{\alpha}$ *in* B.

(iii) $\bar{\gamma} = \bar{\alpha} + \bar{\beta}$ *iff* $G_{\bar{\alpha}} \times G_{\bar{\beta}} \cong G_{\bar{\gamma}}$. *Thus if* $\bar{\gamma} = \bar{\alpha} + \bar{\beta}$, *then* $(\bar{\alpha}, \bar{\beta}) \sim \bar{\gamma}$.

DEFINITION 3.18. For $\bar{\alpha}$ in B, we define $S(\bar{\alpha})$, a subset of $\mathbf{N}$, recursively as follows:

$$S(((1, 0, 0, \dots), 0, 0, \dots)) = \{1\}.$$

For all other $\bar{\alpha}$, $a \in S(\bar{\alpha})$ iff there are $\bar{\alpha}_1$ and $\bar{\alpha}_2$ in B such that:

1. $\bar{\alpha} = \bar{\alpha}_1 + \bar{\alpha}_2$ and $a = a_1 + a_2$ for some $a_i \in S(\bar{\alpha}_i)$ or
2. $(\bar{\alpha}, \bar{\alpha}_2) \sim \bar{\alpha}$ and $a = |G_{\bar{\alpha}_2}| \cdot a_1$ for some $a_1 \in S(\bar{\alpha}_1)$.

Note that pairs $(\bar{\alpha}_1 \cdot \bar{\alpha}_2)$ with $(\bar{\alpha}_1 \cdot \bar{\alpha}_2) \sim \bar{\alpha}$ correspond to abelian group extensions of $G_{\bar{\alpha}_2}$ by $G_{\bar{\alpha}_1}$. Since $\mathrm{Ext}(H_1, H_2)$ is finite for finite abelian groups H_1 and H_2 [**10**, p. 222], for a fixed $\bar{\alpha}$ in B there are only finitely many pairs $(\bar{\alpha}_1, \bar{\alpha}_2)$ with $(\bar{\alpha}_1, \bar{\alpha}_2) \sim \bar{\alpha}$. Thus any $S(\bar{\alpha})$ can be "constructed" in a finite number of steps.

THEOREM 3.19 (*cf.* [**6**, 3.21]). *Suppose* $\bar{\alpha} \in B$ *and* K *is a field with a preorder* T *such that* $\dot{K}/\dot{T} \cong G_{\bar{\alpha}}$, *then* $|X_T| \in S(\bar{\alpha})$. *Conversely, given any* $\bar{\alpha} \in B$, *then for any* $a \in S(\bar{\alpha})$ *there is a field* K *and a preorder* T *such that* $\dot{K}/\dot{T} \cong G_{\bar{\alpha}}$ *and* $|X_T| = a$.

Proof. For the first statement, we induct on $|G_{\bar{\alpha}}|$. If $|G_{\bar{\alpha}}| = 2$, then $\bar{\alpha} = ((1, 0, 0, \dots), 0, 0, \dots)$ and $|X_T| = 1$ which is in $S(\bar{\alpha})$.

Suppose $|G_{\bar{\alpha}}| > 2$. Then, by [3.5], $(X_T, \dot{K}/\dot{T})$ is a direct sum or a group extension. If it is a direct sum, we are done by induction and (1) of the definition of $S(\bar{\alpha})$.

Suppose $(X_T, \dot{K}/\dot{T})$ is a group extension of $(X_T, \dot{K}_1/\dot{T}_1)$. Let $H = (\dot{K}/\dot{T})/(\dot{K}_1/\dot{T}_1)$ and say $\dot{K}_1/\dot{T}_1 \cong G_{\bar{\alpha}_1}$ and $H \cong G_{\bar{\alpha}_2}$ [3.17]. Then, by definition, $(\bar{\alpha}_1 \cdot \bar{\alpha}_2) \sim \bar{\alpha}$ and by induction $|X_{T_1}| \in S(\bar{\alpha}_1)$. By [2.2, ii], $|X_T| = |X_{T_1}| \cdot |G_{\bar{\alpha}_2}|$ which is in $S(\bar{\alpha})$ by (2) of the definition.

For the second statement we again induct on $|G_{\bar{\alpha}}|$. For (1) we use [2.8] and for (2) we use [2.5].

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