ONE PARAMETER SUBSEMIGROUPS IN LOCALLY COMPLETE DIFFERENTIABLE SEMIGROUPS

J. P. HOLMES

We present here some theorems concerning the existence of one parameter subsemigroups in the generalized differentiable semigroups introduced by Graham. These semigroups are based on a type of generalized differentiable manifold and include as examples many semigroups of matrices and subsemigroups of Lie groups which are not ordinary manifolds. The multiplication function is required to be strongly differentiable (a generalization of Frechet differentiability). We shed some light on a question of Graham by showing that such locally complete C^k monoids which contain C^2 curves starting at 1 must contain C^k one parameter subsemigroups. The argument shows that closed submonoids of Lie groups which contain C^2 curves with one end at 1 contain one parameter subsemigroups.

Preliminaries. The notion of semigroup with differentiable multiplication based on an ordinary differentiable manifold was studied in [**H1, H2**]. If such a C^k semigroup S has an idempotent e then there is an open subgroup of eSe which has e as its identity element [**H2**, Theorem 1]. Thus one may appeal to Lie theory for the existence of one parameter subsemigroups. Indeed, each one parameter subsemigroup is contained in a one parameter subgroup. This is not the case with the differentiable semigroups defined by Graham.

The definitions in this paragraph are due to Graham in [**G1, G2**]. A subset A of the Banach space E is said to be admissible provided that each point of A is a limit point of the interior of A. A function, f, with domain the admissible set A and range in the Banach space F is differentiable at the point p in A provided that there is a continuous linear function T from E to F so that for each positive number c there is a positive number d so that if each of x and y is within d of p then

$$|f(x) - f(y) - T(x - y)| < c|x - y|.$$

In this case T is unique and is denoted by $f'(p)$. As usual, the statement that f is C^1 on A means that f is differentiable at each point of A and the function f' is continuous as a function from A into $L(E, F)$, the space of linear transformations from E to F with the usual norm topology. The statement that f is C^k means that $f^{(k-1)}$ is C^1. A Hausdorff topological space S is a C^k manifold based on the Banach

space E provided that for each point p of S there is a homeomorphism g_p from a neighborhood U of p onto an admissible subset A of E containing 0 so that $g_p(p) = 0$ and the composition $g_p \circ g_q^{-1}$ is C^k on its domain for each choice of p and q in S. The theory of these manifolds including the definition of product manifold is elucidated in [G2]. Finally, according to Graham, a topological semigroup is said to be C^k provided that it is based on a C^k manifold and the multiplication function is C^k as a function from $S \times S$ into S.

This notion of differentiable semigroup includes as examples such things as the closed interval $[0,1]$ under ordinary real multiplication, the unit disk D under ordinary complex multiplication, and certain closed subsemigroups of Lie groups studied in [HL] which were excluded under the definition in [H1].

Much of the calculus on C^k manifolds mimics the standard theory. An exception is the possible nonexistence of solutions to differential equations (vector fields may point outside of the manifold).

EXAMPLES. *Example* 1. A C^k monoid need have no nontrivial one parameter subsemigroups. For example from [G1] let S be the subset of the plane to which (x, y) belongs if and only if (x, y) is $(0,0)$ or $x > 0$ and y is between 0 and x^2. S forms a C^k monoid under vector addition for each positive k. There is no non-constant homomorphism from R^+ into S.

Example 2. Local C^k groups must have some neighborhood of the identity element covered with one parameter subsemigroups (actually subroups). The following example shows that some differentiability is needed for this conclusion.

Let S be the Banach space of continuous real valued functions on R with limit at $+\infty$ and $-\infty$. For each f and g in S define fg by $fg(x) = f(g(x) + x) + g(x)$. This multiplication is obtained by mapping f and g to $f + \mathrm{id}$ and $g + \mathrm{id}$, composing, then subtracting id from the result. Thus, S is isomorphic to a subsemigroup of the semigroup under composition of continuous functions from R to R. S is a Banach space when equipped with the supremum norm. The multiplication is continuous but not differentiable relative to the Banach space topology. The constantly zero function is the only idempotent and is also an identity element.

There are elements arbitrarily close to the identity element which have no square root and hence are in no one parameter subsemigroup. To see this consider for a positive number c the function f defined by $f(x) = c$ if $x \le 0$, $f(x) = -2x$ if x is in $[0, c/2]$, and $f(x) = 0$ if $x > c/2$. It is fairly easy to see that if g is in S and $gg = f$ then $g(x) = c/2$ if $x \le 0$ and

$g(x) = 0$ if $x > c/2$. But this implies that

$$c = f(0) = g(g(0) + 0) + g(0) = g(c/2) + c/2 = c/2.$$

thus there is no g so that $gg = f$. These properties of this example are explicated in [H3].

There are also elements arbitrarily close to the identity element which are in uncountably many one parameter subgroups. To see this consider the isomorphism into the semigroup of functions from R to R noted above. If a function agrees with id outside the interval $[0, c]$ ($c > 0$) then it is in the image of the isomorphism.

Suppose $c > 0$ and f is a homeomorphism from R to R which agrees with id outside $(0, c)$ and so that $f(x) > x$ if x is in $(0, c)$. We can build an increasing homeomorphism g from $(0, c)$ onto R satisfying $g(f(x)) = g(x) + 1$ in the following way. Choose a in $(0, c)$ and note that the sequence

$$\ldots, f^{-2}(a), f^{-1}(a), a, f(a), f^2(a), \ldots$$

is increasing and converges to 0 on the left and c on the right. Define g on $[a, f(a)]$ to be an arbitrary increasing homeomorphism onto $[0, 1]$. There is exactly one way to define g on $[f(a), f^2(a)]$ to make $g(f(x)) = g(x) + 1$ for x in $[a, f(a)]$ since each y in $[f(a), f^2(a)]$ is $f(x)$ for exactly one x in $[a, f(a)]$. This extension of g is continuous, increasing, and maps $[a, f^2(a)]$ onto $[0, 2]$. Continuing in this way we obtain a g with the desired property. If T is defined from R into $S + $ id by $T(s)(x) = x$ if x is not in $(0, c)$ and $T(s)(x) = g^{-1}(g(x) + s)$ if x is in $(0, c)$ then the function Q defined from R into S by $Q(s) = T(s) -$ id is a continuous isomorphism. The norm of each $Q(s)$ is no more than c and

$$Q(1) = f - \text{id}.$$

The arbitrariness of g on $[a, f(a)]$ accounts for the existence of uncountably many such Q's. The author was introduced to this construction by Gordon Johnson.

Locally complete semigroups. We now turn our attention to C^k monoids in which k is at least 2 and which have a neighborhood, U, of 1 so that $g_1(U)$ is a closed subset of E. We say that such a monoid is locally complete. The main result is that if such a monoid contains a C^2 arc with one end at 1 then it has a C^k one parameter subsemigroup. The examples show that some completeness near 1 is needed and indicate that some differentiability hypothesis may be needed. On the other hand we are not aware of a locally complete connected C^1 monoid with no one parameter subsemigroups. Our argument implies that if a closed submonoid of a Lie group contains a C^2 arc with one end at 1 then it contains a one parameter subsemigroup.

THEOREM 1. *Suppose $k \geq 2$, S is a C^k locally complete monoid, and there is a C^2 function, h, from $[0,1]$ into S with $h(0) = 1$. Then there is a C^k function, T, from R^+ to S so that $T(s)T(t) = T(s + t)$ whenever each of s and t is in R^+ and $T'(0) = h'(0)$.*

In [N1] Neuberger showed for C^2 local groups that if y is the solution of the initial value problem $y' = L(y)$ and $y(0) = 1$, for L a left invariant vector field, then $y(t)$ can be approximated by products of scalar multiples of $L(1)$ with the coefficients adding to t. The existence of y is assured because L must be C^1. Of course this y is a local one parameter subgroup of the local group.

In [M] Marsden gives sufficient conditions for the pointwise convergence of $K(t/n)^n$ where, for each t, $K(t)$ is a self map of the ordinary Banach manifold M and composition is the operation used in forming $K(t/n)^n$. If we interpreted $K(t)$ as left multiplication by $h(t)$ (which is a one-to-one correspondence for t near 0) we could obtain Theorem 1 as a consequence of a translation of Marsden's argument for [M, Theorem 2.1] to the present setting of generalized Banach manfolds. Our Lemmas 1.1 and 1.3 would serve as the non-trivial part of verifying that this version of K satisfies a translated version of the hypotheses of [M, Theorem 2.1]. The argument we present here is more straightforward in the present setting and shows that left multiplication by $T(t)$ is approximated, for t near 0, by products of the form $K(t_n)K(t_{n-1}) \cdots K(t_0)$ where the t_i's are non-negative and sum to t.

In our setting, solutions to the initial value problem $y' = L(y)$, $y(0) = 1$ for L a left invariant vector field are not guaranteed by standard existence and uniqueness theorems because such an L might point outside of S. Thus, we resort to product integrals which stay in S because S is a semigroup.

The author appreciates the help of the referee in pointing out reference [M].

To formulate an argument choose a closed (in E) neighborhood, D, of 0 in $g_1(S)$ so that

$$V(x, y) = g_1\big(g_1^{-1}(x)g_1^{-1}(y)\big)$$

exists whenever each of x and y is in D. We may slow h down if necessary so that it maps into $g_1^{-1}(D)$ and let f be defined by $f(t) = g_1(h(t))$ for each t in $[0,1]$. Since h is C^2 so is f. The first lemma is a sort of Taylor's theorem for V.

LEMMA 1.1. *There are positive numbers $M, r,$ and d so that if s is in $[0, r]$, q is in D, and $|q| \le d$ then*
$$V(f(s), q) = q + V'(0, q)(f(s), 0) + A(s, q)$$
where $|A(s, q)| \le Ms^2$ and
$$|V(f(s), q)| \le |q| + M(s + s^2).$$

Proof. Choose, by continuity of V' and V'' at $(0, 0)$, positive numbers d and N so that if each of p and q is in D and each has norm not exceeding d then $|V'(p, q)| \le N$ and $|V''(p, q)| \le N$. Choose N' so that $|f'(t)| \le N'$ for each t in $[0, 1]$. Note that
$$|f(x) - f(t)| \le N'|s - t|$$
whenever each of s and t is in $[0, 1]$. Choose r in $(0, 1)$ so that $rN' \le d$ and let $M = NN'^2 + NN'$.

Suppose s is in $[0, r]$, q is in D, an $|q| \le d$. Then
$$
\begin{aligned}
V(f(s), q) - q &= V(f(s), q) - V(0, q) \\
&= \int_0^1 da\, [V'(f(0), q)(f'(as)(s), 0)] \\
&\quad + \int_0^1 da\, [V'(f(as\cdot), q) - V'(f(0), q)](f'(as)(s), 0) \\
&= V'(0, q)(f(s) - f(0), 0) \\
&\quad + \int_0^1 da \int_0^1 db\, [V''(f(bas), q)((f'(as)(s), 0), (f'(bas)(as), 0))] \\
&= V'(0, q)(f(s), 0) + A(s, q).
\end{aligned}
$$

By choice of r, $|f(bas)| \le N'bas \le d$ if each of a and b is in $[0, 1]$ and moreover each of $|f'(as)(s)|$ and $|f'(bas)(s)|$ is bounded by $N's$. Hence the integrand in $A(s, q)$ is bounded in norm by N'^2s^2. It follows that $|A(s, q)| \le Ms^2$. Finally, since $|q| \le d$ and $|f(s)| \le sN'$, we have
$$|V(f(s), q)| \le |q| + M(s + s^2).$$

COROLLARY 1.2. *There are positive numbers B and e so that if each of $s_1, \ldots, s_n$ is a positive number and $a = \sum_1^n s_i \le e$ then*
$$V\big(f(s_n), V(f(s_{n-1}), \ldots, V(f(s_2), f(s_1)) \ldots)\big)$$
exists and has norm not exceeding aB.

Proof. Choose $M, r,$ and d as in Lemma 1.1, let $B = 2M$, and choose e positive so that $e \le r$ and $eB \le d$. Suppose each of s_1 and s_2 is positive and $s_1 + s_2 \le e$. Each of $s_2 \le r$ and $f(s_1)$ has norm not exceeding d so by Lemma 1.1 we have
$$|V(f(s_2), f(s_1))| \le s_2(M + s_2 M) + |f(s_1)| \le B(s_2 + s_1).$$

Assume that n is an integer which is at least 2 and suppose inductively that if and each of $s_1, \ldots, s_n$ is positive and $b = \sum_1^n s_i \le e$ then $q = V(f(s_n), V(f(s_{n-1}, \ldots, V(f(s_2), f(s_1)) \ldots)))$ exists and has norm not exceeding bB.

Suppose that each of $s_1, \ldots, s_{n+1}$ is positive and $a = \sum_1^{n+1} s_i \le e$. By the inductive hypothesis we have the existence of q as above with norm not exceeding $Bb \le Be \le d$. Moreover, s_{n+1} is in $[0, r]$ so Lemma 1.1 implies that

$$\left| V(f(s_{n+1}), q) \right| \le s_{n+1}(M + s_{n+1}M) + |q| \le aB.$$

LEMMA 1.3. *There are positive numbers a, b, and C so that if p is in D and $|p| \le b$ and each of u, s, and $t = u + s$ is in $[0, a]$ then*

$$\left| V(f(t), p) - V(f(s), V(f(u), p)) \right| \le Cst.$$

Proof. Choose M, r, and d as in Lemma 1.1. Choose N'' to be an upper bound for $|f''(x)|$ for each x in $[0, 1]$. Choose a' positive so that if each of x, y, p, and q is in D and each has norm not exceeding a' then

$$\left| V'(x, y) - V'(p, q) - V''(0, 0)(x - p, y - q) \right| \le \left| (x - p, y - q) \right|.$$

Choose b positive and less than each of $a'/2$ and $d/2$. Choose the number a positive and less than r so that $2aM$ is less than each of b and r and so that aN' is less than a'. Choose C so that $C/3$ exceeds each of $[|V''(0, 0)| + 1]2[N' + M]N'$, MN'', and M.

If p is in D, $|p| \le b$, and each of u, s, an $t = u + s$ is in $[0, a]$ then from Lemma 1.1 we have $|V(f(u), p)| \le |p| + 2Mu$ which by choice of a and b is smaller than each of d and a'.

Suppose p is in D, $|p| \le b$, and each of u, s, and $t = u + s$ is in $[0, a]$. Then

$$\left| V(f(t), p) - V(f(s), V(f(u), p)) \right|$$
$$= \left| V(f(t), p) - V(f(u), p) + V(f(u), p) - V(f(s), V(f(u), p)) \right|.$$

Since u is in $[0, r]$ and $|V(f(u), p)| \le d$ we have by Lemma 1.1 that the last term is

$$= \left| \int_0^1 dv \left[V'(f'(u + vs), p)(f'(u + vs)(s), 0) \right] \right.$$
$$\left. - \int_0^1 dv \left[V'(0, V(f(u), p))(f'(vs)(s), 0) \right] - A(s, V(f(u), p)) \right|$$
$$\le \left| \int_0^1 dv \left[V'(f(u + vs), p) - V'(0, V(f(u), p)) \right](f'(u + vs)(s), 0) \right|$$
$$+ \left| \int_0^1 dv \left[V'(0, V(f(u), p)(f'(u + vs) - f'(vs))(s), 0) \right] \right|$$
$$+ \left| A(s, V(f(u), p)) \right|.$$

We consider the three summands separately. For each v in $[0, 1]$ each of $V(f(u), p)$, p, and $f(u + vs)$ has norm not exceeding a' so

$$\left| [V'(f(u + vs), p) - V'(0, V(f(u), p))](f'(u + vs)(s), 0) \right|$$
$$\leq \left[|V''(0,0)| + 1 \right]\left[|f(u + vs)| + |p - V(f(u), p)| \right] N's$$
$$\leq \left[|V''(0,0)| + 1 \right]\left[N't + N'u + 2Mu \right] N's$$
$$\leq \left[|V''(0,0)| + 1 \right] 2[N' + M] N'st \leq (C/3)st.$$

Thus the first summand does not exceed $(C/3)st$. The second summand is, since $|V(f(u), p)| \leq d$,

$$\left| V'(0, V(f(u), p))\left(\int_0^1 dv \int_0^1 dw [f''(wu + vs)(u, s)], 0 \right) \right|$$
$$\leq MN''us \leq (C/3)st.$$

Finally, since s is in $[0, r]$ and $|V(f(u), p)| \leq d$,

$$\left| A(s, V(f(u), p)) \right| \leq Mss \leq (C/3)st.$$

By a subdivision, H, of the interval $[0, x]$ we mean a finite non-decreasing sequence $0 = t_1 \leq t_2 \leq \cdots \leq t_n = x$. By mesh$(H)$ we mean the length of the largest interval formed by adjacent t's. Finally, by $P(H)$ we mean the product

$$V(f(t_n - t_{n-1}), \ldots, V(f(t_3 - t_2), f(t_2 - t_1)) \ldots)$$

provided it exists. The next lemma provides for convergence of the $P(H)$'s.

COROLLARY 1.4. *There are positive numbers F and c so that if x is in $[0, c]$, H is a subdivision of $[0, x]$, and H' refines H then each of $P(H)$ and $P(H')$ exist and $|P(H) - P(H')| \leq F \operatorname{mesh}(H)$.*

Proof. Let B and e be chosen as in Corollary 1.2 and a, b, and C be chosen as in Lemma 1.3. Choose $a' > 0$ as in the proof of Lemma 1.3 so that if each of x, y, p, and q is in D and has norm not exceeding a' then

$$|V(x, y) - V(p, q) - V'(0,0)(x - p, y - q)|$$
$$\leq |(x - p, y - q)|.$$

Note that for such x, y, p, an q we have

$$|V(x, y) - V(p, q)| \leq G|(x - p, y - q)|$$

where $G = |V'(0,0)| + 1$. Choose c not exceeding each of e and a so that Bc is smaller than each of b and a' and let $F = GCc$.

Suppose x is in $[0, c]$,

$$H = \{t_1, \ldots, t_n\}$$

is a subdivision of $[0, x]$, and s is between t_i and t_{i-1} for some i between 0 and n. The product, $P(H)$, exists by Corollary 1.2. Let $u_j = t_j - t_{j-1}$ for $j = 1, \ldots, n$, $v = s - t_{i-1}$, and $w = t_i - s$. Note that by choice of c each of v, w, and $u_i = v + w$ is in $[0, a]$. Let

$$p = V\big(f(t_{i-1} - t_{i-2}), \ldots, V(f(t_3 - t_2), f(t_2 - t_1)) \ldots \big)$$

and

$$q = V\big((t_n - t_{n-1}), \ldots, V(f(t_{i+2} - t_{i+1}), f(t_{i+1} - t_i)) \ldots \big).$$

By Corollary 1.2 $|p| \le Bt_{i-1} \le b$, $|V(f(u), p)| \le Bs \le a'$, $|V(f(w), V(f(v), p))| \le Bt_i \le a'$, and $|q| \le B(t_n - t_i) \le a'$.

Note that $P(H) = V(q, V(f(u_i), p))$. We wish to compare this with $Q = V(q, V(f(w), V(f(v), p)))$.

$|P(H) - Q|$ by choice of a'

$\le G|V(f(u_i), p) - V(f(w), (f(v), p))|$ by Lemma 1.3

$\le MCu_iw = (F/c)u_iw.$

Now suppose H' refines H. Let $s_1, \ldots, s_j$ be an increasing sequence of numbers so that $H' = H \cup \{s_1, \ldots, s_j\}$. Let $H' = H_j$, $H = H_0$ and

$$H_i = H_{i+1} - \{s_{i+1}\}$$

for $i = j, \ldots, 1$ and let r_i be the next larger member of H_i to s_i for $i = 1, \ldots, j$. The intervals $[s_i, r_i]$ are nonoverlapping and fill up less than $[0, c]$. The mesh of each H_i is no larger than mesh(H). Thus we have

$$|P(H') - P(H)| \le \sum_1^j |P(H_i) - P(H_{i-1})|$$

$$\le (F/c)\, \text{mesh}(H) \sum_1^j (r_i - s_i) \le F\, \text{mesh}(H).$$

Now it is clear by completeness of D that for each t in $[0, c]$ there is a unique $R(t)$ in D so that if ε is a positive number there is a positive number δ so that if H is a subdivision of $[0, t]$ with mesh less than δ then $P(H)$ is within ε of $R(t)$. From Corollary 1.2 we have $|R(t)| \le Bt$ and from construction it is clear that $V(R(t), R(s)) = R(t + s)$ whenever each of s, t, and $s + t$ is in $[0, c]$. From these observations and continuity of V we have R continuous on $[0, c]$.

We comment that associativity was used in the proof of Corollary 1.4 for convenience and the proof of the existence of the product integral $R(t)$ can be carried out for nonassociative V. Of course, in the nonassociative situation we do not obtain a local homomorphism of $[0, c]$ into D.

Let T be defined on $[0, c]$ into S by $T(t) = g_1^{-1}(R(t))$. It is easy to see that there is a unique extension of T to R^+ which is continuous. It remains to show that T is C^k and $T'(0)$ can be made to be $h'(0)$.

LEMMA 1.5. *With f and R defined above, R is C^k and $R'(0) = f'(0)$.*

Proof. Suppose c' is a positive number and choose d positive so that if each of p and q is in $D \times D$ and each of p and q is within d of $(0, 0)$ then

$$|V(p) - V(q) - V'(0, 0)(p - q)| < c'|p - q|.$$

If each of x and y is in D and each is within d of 0 then

$$|x - y - V'(0, 0)(x - y, 0)| < c'|x - y|.$$

Since 0 is a limit point of the interior of D, $x - y$ can be taken in any direction. This implies that $V'(0, 0)(x, 0) = V'(0, 0)(0, x) = x$ for each x in E.

Suppose a is a positive number, choose M an upper bound for $|f'(x)|$ for x in $[0, 1]$, choose F and c as in Corollary 1.4, and choose B and e as in Corollary 1.2.

Using differentiability of V at $(0, 0)$ and using $V'(0, 0)(x, y) = x + y$ choose d positive so that if each of p and q is in D and each has norm not exceeding d then

$$|V(p, q) - V(0, q) - p| \le (a/3M)|p|.$$

Choose t positive and less than each of c, e, and d/B and choose N so that if $n \ge N$ then $3F/n < a$. Choose such an n and for each $i = 1, \ldots, n$ let

$$q_i = V(f(t/n), \ldots, V(f(t/n), f(t/n)) \ldots)$$

where the product has i factors.

Note that $q_n = P(H)$ where H is the subdivision of $[0, t]$ with n equal length subintervals. It follows from the choice of t in $[0, e]$ that each q has norm less than Bt which is smaller than d. It follows from the choice of F and c that since t is in $[0, c]$ we have

$$|R(t) - q_n| \le Ft/n < at/3.$$

But

$$|q_n - nf(t/n)| \le |V(f(t/n), q_{n-1}) - q_{n-1} - f(t/n)|$$
$$+ |V(f(t/n), q_{n-2}) - q_{n-2} - f(t/n)|$$
$$\cdots$$
$$+ |V(f(t/n), f(t/n)) - f(t/n) - f(t/n)|$$
$$\le n(a/3M)|f(t/n)| \le at/3.$$

Thus we have for each $n \ge N$ and each t sufficiently small that

$$|R(t) - nf(t/n)| < at.$$

Hence $R'(0)$ exists and is $f'(0)$. Since $R(t + s) = V(R(t), R(s))$ for t in $[0, 1)$ and s sufficiently small the chain rule implies that $R'(t)$ exists for t in $[0, 1)$ and

$$R'(t) = V'(R(t), 0)(0, R'(0)).$$

Thus R is C^k on $[0, 1)$. This implies that T is C^k on R^+ and we are done with Theorem 1.

COROLLARY 1.6. *If S is a closed submonoid of a C^k group with k at least 2 and S contains a C^2 curve with one end at 1 then S has a C^k one parameter subsemigroup.*

This is immediate from the above argument. The product integral may be carried out in the group but the fact that S is a closed set which is closed under products implies that the one parameter subsemigroup constructed maps into S.

In [HL] Hofmann and Lawson study certain subsemigroups of Lie groups which they term Lie semigroups. They are C^k monoids in the present sense. These monoids are generated by one parameter subsemigroups and the structure of such a monoid is connected with that of its Lie wedge (an analogue of Lie algebra in this setting). In view of Theorem 1 one might hope to obtain geometric conditions on a C^k monoid which imply that it is a Lie semigroup in this sense.

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AUBURN UNIVERSITY
AUBURN, AL 36849-3501