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PARACOMPACT C-SCATTERED SPACES

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Telgársky calls a topological space C-scattered when each of its non-empty closed sets contains a compact set with non-empty relative interior. With respect to infinite products, hyperspaces, and the partially ordered set of compactifications, we study the class of paracompact C-scattered spaces and two of its subclasses, MacDonald and Willard's A'-spaces and A-spaces.

0. Introduction. All spaces are Hausdorff spaces. A space X is said to be *C-scattered* [**16**] provided that each of its non-empty closed subspaces contains a compact set with non-empty relative interior. The notion of C-scatteredness seems a simple simultaneous generalization of scattered ($\equiv$ each non-empty set has a relative isolated point) and of local compactness. However, the class of paracompact C-scattered spaces is most interesting because [**19**] it contains its perfect pre-images, it is closed under finite products, it contains all closed continuous images of paracompact locally compact spaces, and for each of its members X, $X \times Y$ is paracompact iff Y is paracompact. Presently we study this class and two of its subclasses.

Section 1 is due to the third author and §§2 and 3 are due to the first two authors.

In §1 of our paper, we show that each countable product of paracompact C-scattered spaces is paracompact. This result improves upon the same theorem, due to Rudin and Watson [**18**], for paracompact scattered spaces, and answers the question raised for A'-spaces by the first two authors of this paper. As a corollary, we find that each countable product of Lindelöf C-scattered spaces is Lindelöf, a result due to Alster [**2**].

In the second section, we investigate hyperspaces of paracompact C-scattered spaces—a situation so complex that we limit our attention to A'-spaces. An *A'-space* is a space whose set of accumulation points is compact [**10**]. Thus, an A'-space is paracompact C-scattered. It is known [**12**] that the compact-set hyperspace $\mathscr{C}(X)$ is locally compact (metrizable) iff X is locally compact (respectively, metrizable). Here we present an example of a Lindelöf scattered A'-space X such that $\mathscr{C}(X)$ is neither C-scattered or normal. Further, we prove that $\mathscr{C}(X)$ is an A'-space (contains a dense A'-space containing X) iff X is either compact or discrete (respectively, or $\operatorname{int}(\operatorname{acc}(X)) = \varnothing$).

In our final section we consider the metrizable A'-spaces termed as *A-spaces* by Willard [**21**]. A-spaces occur naturally in several ways; for example, a metrizable space is an A-space iff each closed continuous image is metrizable ([**17**] and [**21**]) iff each Hausdorff quotient space is metrizable ([**1**], [**9**], and [**18**]). A-spaces are also studied in [**3**], [**4**], [**7**], [**11**], [**13**], and [**14**]. The main result in the section shows that $K_M(X)$, the partially ordered set of metrically compactible Hausdorff compactifications, is a lattice when X is an A-space. However, we also obtain a characterization of A-space: A metrizable space is an A-space iff $K_M(X)$ has maximal element.

0.1. *Conventions.* All ordinals are von-Neumann ordinals. $\mathbf{N}$ denotes the set of positive integers and $\mathbf{R}$ denotes the set of reals. The interior, closure, and accumulation point-set operators are denoted, respectively, by int, cl, and acc.

0.2. DEFINITION. Let X be a space and $X^{(1)}$ be the set of points of X which fail to have a compact neighborhood in X. Now, letting $X^{(0)} = X$, inductively define for each ordinal α, $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)(1)}$. Then X is C-scattered iff there exists an ordinal γ such that $X^{(\gamma)} = \varnothing$ [**19**].

Suppose X is C-scattered and $Y \subset X$. For each ordinal α define $Y^{(\alpha)} = X^{(\alpha)} \cap Y$. Then the *rank* of Y (in X), denoted by $\operatorname{rk}(X)$, is the least ordinal γ such that $Y^{(\gamma)} = \varnothing$. It is easily proved that an A'-space is a paracompact C-scattered space of rank at most 2.

1. Products of C-scattered spaces. The entirety of this section is directed to proving the following result:

1.1. THEOREM. *Suppose X_n is a paracompact C-scattered space for each positive integer n. Then $\prod_n X_n$ is paracompact.*

1.2. *A Reduction.* We begin our proof of 1.1 with a reduction to an easier case. We first observe that it suffices in Theorem 1.1 to assume that all the spaces X_n are homeomorphic and 0-dimensional in the sense of small inductive dimension. To see this, let

$$Y = \Big(\bigcup_{n\in\mathbf{N}} X_n \times \{n\}\Big) \cup \{\infty\}.$$

The space Y has the topology which makes each $X_n \times \{n\}$ clopen in Y and homeomorphic to X_n. Basic neighborhoods of ∞ have the form

$$Y \setminus \bigcup_{n \leq k} X_n \times \{n\}, \quad \text{for some } k \in \mathbf{N}.$$

Then Y is paracompact and C-scattered. So $\prod^{\mathbf{N}} Y$ is paracompact implies $\prod_n X_n$ is paracompact. Now let X be Ponomarev's absolute of Y **[16]**. Then X is paracompact, extremally disconnected and C-scattered **[19]**. Since $\prod^{\mathbf{N}} X$ maps perfectly onto $\prod^{\mathbf{N}} Y$, $\prod^{\mathbf{N}} X$ is paracompact iff $\prod^{\mathbf{N}} Y$ is paracompact. □

Henceforth, X will be a paracompact C-scattered 0-dimensional space, and we will show $\prod^{\mathbf{N}} X$ paracompact. Actually we show a stronger result: Each open cover of $\prod^{\mathbf{N}} X$ has a pairwise-disjoint open refinement; i.e. $\prod^{\mathbf{N}} X$ is *ultraparacompact*. We approach this in stages.

1.3. LEMMA **[19]**. *If X is a paracompact C-scattered 0-dimensional space, then so is X^n for each $n \in \mathbf{N}$.* □

The following result was obtained (unpublished) by the third author in 1974.

1.4. LEMMA. *For a paracompact space Y the following are equivalent:*

(1) *Y is ultraparacompact.*

(2) $\operatorname{Ind}(Y) = 0$ (Ind $\equiv$ *large inductive dimension*).

(3) *Each non-empty closed subset F of Y contains an ultra-paracompact subspace with non-empty F-interior.*

Proof. The equivalence of (1) and (2) is straightforward and (1) implies (3) is obvious. We prove (3) implies (1).

For a closed subset Z of Y, define Z^* to be the set of all points of Z which do not have Y-closed Y-ultraparacompact neighborhoods. Since Z^* is closed in Y, there is a family $\mathscr{A}$ of Z-open sets such that $\bigcup\mathscr{A} = Z \setminus Z^*$ and $\operatorname{Ind}(A) = 0 \ \forall A \in \mathscr{A}$.

Claim. If $Z^* = \varnothing$, then Z is ultraparacompact.

To see the claim, suppose $\mathscr{R}$ is a Z-open cover of Z. Since Z is paracompact and $Z^* = \varnothing$, there is a Z-locally finite refinement $\mathscr{T}$ such that $\{\operatorname{cl}_Y(T)\colon T \in \mathscr{T}\}$ refines $\{R \cap A\colon R \in \mathscr{R},\ A \in \mathscr{A}\}$. Applying the normality of Z and the condition $\operatorname{Ind}(A) = 0 \ \forall A \in \mathscr{A}$, we may choose a refinement $\mathscr{U} = \{U_T: T \in \mathscr{T}\}$ of $\mathscr{T}$ such that for each $T \in \mathscr{T}$, $U_T \subset T$ and U_T is $\mathrm{cl}_Y(T)$-clopen. Note that each U_T is actually Z-clopen. Since $\mathscr{T}$ is Z-locally finite, $\mathscr{U}$ is Z-locally finite. Let $\prec$ be a well-ordering of $\mathscr{U}$. Then

$$\{U \setminus \bigcup\{V \in \mathscr{U}: V \prec U\}: U \in \mathscr{U}\}$$

is the desired pairwise-disjoint Z-open refinement of $\mathscr{R}$. The claim is now proved.

According to the claim, Y is ultraparacompact whenever $Y^* = \varnothing$. We contend the latter is true. Suppose, by way of contradiction, that $Y^* \neq \varnothing$. Then applying regularity and (3), there is a Y-closed set Z such that $Z \cap Y^*$ is ultraparacompact and $Y^* \cap \mathrm{int}_Y(Z) \neq \varnothing$. Clearly, $\varnothing \neq Z^* \subset Y^*$. Now suppose $\mathscr{R}$ is a Z-open cover of Z. Then there is a pairwise-disjoint Z^*-open cover $\mathscr{S}$ of Z refining $\mathscr{R}|Z^*$. Since Z is collectionwise normal, there is a pairwise-disjoint family $\mathscr{T} = \{T_S: S \in \mathscr{S}\}$ consisting of Z-open sets such that for each $S \in \mathscr{S}$, $T_S \cap Z^* = S$ and there exists $R_S \in \mathscr{R}$ with $T_S \subseteq R_S$. Let $K = Z \setminus \bigcup\mathscr{T}$, and, by normality, choose a Z-open set G such that

$$K \subset G \subset \mathrm{cl}_Y(G) \subset Z \setminus Z^*.$$

Clearly, $(\mathrm{cl}_Y(G))^* = \varnothing$. So the claim above shows there is a $\mathrm{cl}_Y(G)$-open pairwise-disjoint family $\mathscr{U}$ covering $\mathrm{cl}_Y(G)$ and refining the family

$$\{\mathrm{cl}_Y(G) \setminus K\} \cup \{R \cap G: R \in \mathscr{R}\}.$$

Let $\mathscr{V} = \{U \in \mathscr{U}: U \cap K \neq \varnothing\}$. Then each $V \in \mathscr{V}$ is a $\mathrm{cl}_Y(G)$-open subset of G and $\mathrm{cl}_Y(G)$-closed. Hence, each $V \in \mathscr{V}$ is Z-clopen. So $\mathscr{V}$ is Z-locally-finite and $\bigcup\mathscr{V}$ is clopen. Certainly

$$\mathscr{V} \cup \{T \setminus \bigcup\mathscr{V}: T \in \mathscr{T}\}$$

is a Z-open pairwise-disjoint refinement of $\mathscr{R}$. So Z is ultraparacompact. Hence $\mathrm{int}_Y(Z) \cap Y^* = \varnothing$—a contradiction. □

Now we know $\prod^n X$ is ultraparacompact for each $n \in \mathbf{N}$. However, we need a much stronger result.

1.5. Definition. Suppose that Y is a C-scattered space and $A \subset Y$. Define the *top* of A by

$$\mathrm{tp}(A) = \begin{cases} \varnothing & \text{if } \mathrm{rk}(A) \text{ is a limit ordinal,} \\ A^{(\alpha)} & \text{if } \mathrm{rk}(A) = \alpha + 1. \end{cases}$$

We say that A is *capped* provided there exists an α such that $A^{(\alpha)}$ is compact and non-empty. Obviously if A is open and capped, then $\operatorname{rk}(A)$ will be $\alpha + 1$ when $A^{(\alpha)}$ is compact and non-empty.

1.6. LEMMA. *A 0-dimensional C-scattered space Y has a base of clopen capped sets.*

Proof. For $y \in Y$, $\operatorname{rk}(y) = \alpha + 1$ for some α. Given a neighborhood G of Y, choose a clopen set H with $y \in H \subseteq G \setminus Y^{(\alpha+1)}$. Then $H^{(\alpha)}$ is clopen in the locally compact space $Y^{(\alpha)} \setminus Y^{(\alpha+1)}$. So there is a Y-clopen neighborhood K of y such that $K \cap H^{(\alpha)}$ is compact. □

1.7. LEMMA. *Each open covering of X is refined by a pairwise-disjoint family of clopen capped sets.*

Proof. Suppose that $\mathscr{R}$ is an open covering X. According to 1.4 we may assume $\mathscr{R}$ to consist of pairwise-disjoint clopen sets. Inductively, we construct for each $n \in \mathbf{N}$, a family $\mathscr{R}_n$ as follows: First set $\mathscr{R}_1 = \mathscr{R}$. For each $n > 1$

(i) $\mathscr{R}_n$ is a pairwise-disjoint open refinement of $\mathscr{R}_{n-1}$.
(ii) If $R \in \mathscr{R}_{n-1}$ is capped, then $R \in \mathscr{R}_n$.
(iii) If $R \in \mathscr{R}_{n-1}$ is not capped, then $\operatorname{rk}(R^*) < \operatorname{rk}(R)$ for each non-capped $R^* \in \mathscr{R}_n$ with $R' \subseteq R$.

Assume that we have $\mathscr{R}_n$ for all $n \leq m$; we will find $\mathscr{R}_{m+1}$. Let

$$\mathscr{S} = \{R \in \mathscr{R}_m \colon R \text{ is not capped}\}$$

and fix $S \in \mathscr{S}$. Note that S is clopen. For each $x \in S$ we use 1.6 to find an open capped set S_x such that $x \in \operatorname{tp}(S_x)$ and $S_x \subseteq S$.

Now suppose $\operatorname{rk}(S)$ is a limit ordinal. Then $\operatorname{rk}(S_x) < \operatorname{rk}(S)$ for each $x \in S$. From 1.6 there is a pairwise-disjoint refinement $\mathscr{T}_S$ of $\{S_x\colon x \in S\}$ (we are assuming that the union of the refinement is the union of the family that it refines).

On the other hand, suppose $\operatorname{rk}(S) = \alpha + 1$. Then $(X^{(\alpha)} \cap S) \setminus X^{(\alpha+1)}$ is a locally compact ultraparacompact space, and hence, the union of a pairwise-disjoint family $\mathscr{U}$ consisting of compact $X^{(\alpha)}$-open sets. For each $U \in \mathscr{U}$, let $V_U \subset S$ be an open set such that $(V_U)^{(\alpha)} = U$. From 1.4 there is a pairwise-disjoint X-open refinement $\mathscr{T}_S$ of $\{S \setminus X^{(\alpha)}\} \cup \{V_U\colon U \in \mathscr{U}\}$. Observe that for each $T \in \mathscr{T}_S$, $T^{(\alpha)}$ is closed in some $U \in \mathscr{U}$. Hence, $T^{(\alpha)}$ is compact. Clearly, if T is not capped, then $T^{(\alpha)} = \varnothing$ and so $\operatorname{rk}(T) < \operatorname{rk}(S)$.

We complete the construction by defining

$$\mathscr{R}_{m+1} = (\mathscr{R}\setminus\mathscr{S}) \cup \left(\bigcup\{\mathscr{T}_S: S \in \mathscr{S}\}\right).$$

Now $\mathscr{R}_n$ is defined for each $n \in \mathbf{N}$. Define

$$\mathscr{R}_\infty = \left\{R \in \bigcup_{n\in\mathbf{N}} \mathscr{R}_n: R \text{ is capped}\right\}.$$

Then (i) and (ii) imply $\mathscr{R}_\infty$ is a pairwise-disjoint open capped family. Also, (i) shows that if $\mathscr{R}_\infty$ covers X, then $\mathscr{R}_\infty$ refines $\mathscr{R}$. So suppose $\mathscr{R}_\infty$ does not cover X, i.e., $x \in X\setminus \bigcup\mathscr{R}_\infty$. Then we may find for each $n > 1$ a set $R_n \in \mathscr{R}_n$ such that $x \in R_n \subset R_{n-1}$. As R_n is not capped, $\operatorname{rk}(R_n) < \operatorname{rk}(R_{n-1})$. However, this implies there is a decreasing sequence of ordinals, an impossibility. So $\mathscr{R}_\infty$ must cover X. □

1.8. Definition. Fix $n \in \mathbf{N}$. By a *box* in $\prod^n X$, we mean a set of the form $B = \prod_{i\le n} B_i$, where each B_i is open in X. A *capped box* is a box B such that B_i is capped in X for each $i \le n$.

1.9. Lemma. *For a fixed natural number n, each open cover of $\prod^n X$ is refined by a pairwise-disjoint collection of capped box.*

Proof. The proof is by induction on n. As the Lemma 1.6 shows our result for $n = 1$, we suppose it is true for some $m \in \mathbf{N}$ and show it is true for $m + 1$. Suppose $\mathscr{R}$ is an open cover of $\prod^{m+1} X$. Let Y denote the set of all points in X such that x does not have a neighborhood 0 for which $\mathscr{R}|(0 \times \prod^m X)$ is refined by a pairwise-disjoint family of capped boxes. If $Y =$ empty, then we may, by 1.7, write X as the union of a disjoint family $\mathscr{G}$ if open capped sets G such that $\mathscr{R}|(G \times \prod^m X)$ is refined by a pairwise-disjoint capped box family $\mathscr{S}_G$. Clearly, in this case, $\bigcup\{\mathscr{S}_G: G \in \mathscr{G}\}$ is the desired refinement of $\mathscr{R}$.

Of course, it is always true that $Y = \varnothing$. To see this suppose not. Then there is a compact subset K of Y such that $\operatorname{int}_Y(K) \neq \varnothing$. For each $v \in \prod^m X$, there is a capped box $A_v = \prod_i A_{vi}$ neighborhood of v such that $K \times A_v$ is the union of a finite pairwise-disjoint $(K \times \prod^m X)$-capped box refinement $\mathscr{T}_v$ of $\mathscr{R}|K \times T$. By the induction hypothesis, there is a pairwise-disjoint capped box refinement $\mathscr{U}$ of $\{A_v: v \in \prod^m X\}$. For each $U \in \mathscr{U}$, choose a $v(U) \in \prod^m X$ such that $U \subseteq A_{v(U)}$. Then

$$\mathscr{W} = \left\{(K \times U) \cap T: U \in \mathscr{U},\ T \in \mathscr{T}_{v(U)}\right\}$$

is a pairwise-disjoint $(K \times \prod^m X)$-capped box refinement of the restriction of $\mathscr{R}$ to $K \times \prod^m X$. For each $W = \prod_{i\le m+1} W_i \in \mathscr{W}$ choose an open set H_W of X and an $R_W \in \mathscr{R}$ such that $H_W \cap K = W_{m+1}$ and $H_W \times \prod_{i \le m} W_i \subset R_W$. But, according to the definition of Y, the existence of the family

$$\{H_W \times \textstyle\prod_{i \le m} W_i \colon W \in \mathscr{W}\}$$

implies that $\operatorname{int}_X(K) \cap Y = \varnothing$, a contradiction. So $Y = \varnothing$. □

We are now ready to prove the main result of this section, from which 1.1 follows.

1.10. THEOREM. *$\prod^{\mathbf{N}} X$ is ultraparacompact.*

Proof. By a *cube* in $\prod^{\mathbf{N}} X$, we mean a set of the form $C = \prod_n C_n$, where each C_n is a clopen capped set in X, and there exists $m(C) \in \mathbf{N}$ such that $C_n \ne X$ $\forall n < m(C)$ and $C_n = X$ $\forall n \ge m(C)$ [Notice that in the reduction 1.2, X is capped. We assume, without loss of generality, that such is the case here.] Therefore, the family of cubes in $\prod^{\mathbf{N}} X$ forms a base. For a cube C let $\operatorname{tp}(C) = \prod_n \operatorname{tp}(C_n)$.

Suppose that $\mathscr{R}$ is a cube cover of $\prod^{\mathbf{N}} X$. We construct, inductively, for each $i \in \mathbf{N}$, a cube cover $\mathscr{S}_i$ of $\prod^{\mathbf{N}} X$ satisfying the conditions below for $i = j + 1$:

(1) $\mathscr{S}_i$ is a pairwise-disjoint refinement of $\mathscr{S}_j$.

(2) If $S \in \mathscr{S}_j$ is such that $\operatorname{tp}(S) \subset \bigcup \mathscr{R}'$, where $\mathscr{R}' \subset \mathscr{R}$ and $m(R) \le m(S)$ $\forall R \in \mathscr{R}'$, then
 (a) $m(S') = m(S)$ for each S' with $S \supset S' \in \mathscr{S}_i$, and
 (b) $S' \in \mathscr{S}_i$ and $S' \cap \operatorname{tp}(S) \ne \varnothing$ implies $\exists R \in \mathscr{R}$ with $S' \subset R$.

(3) If $S \in \mathscr{S}_j$ is such that $\nexists \mathscr{R}' \subset \mathscr{R}$ with $\operatorname{tp}(S) \subset \bigcup \mathscr{R}'$ and $m(R) \le m(S)$ $\forall S' \in \mathscr{S}_i$ with $S' \subset S$.

(4) If $S \in \mathscr{S}_j$ is such that $\exists R \in \mathscr{R}$ with $S \subset R$, then $S \in \mathscr{R}_i$.

The construction proceeds as follows. First let $\mathscr{S}_1 = \{\prod^{\mathbf{N}} X\}$, and suppose we have found $\mathscr{S}_i$ $\forall i \le k$. We construct $\mathscr{S}_{k+1}$. Since $\mathscr{S}_k$ is a pairwise-disjoint clopen cover of $\prod^{\mathbf{N}} X$, it is sufficient to find $\mathscr{S}_{k+1} \mid S$ for a fixed $S = \prod_n S_n \in \mathscr{S}_k$. If there is an $R \in \mathscr{R}$ with $S \subset R$, then define $(\mathscr{S}_{k+1} \mid S) = \{S\}$. If there does not exist a subfamily $\mathscr{R}'$ of $\mathscr{R}$ such that $\operatorname{tp}(S) \subset \bigcup \mathscr{R}'$ and $m(R) \le m(S)$ $\forall R \in \mathscr{R}'$, then we apply 1.9 to obtain a pairwise-disjoint capped box family $\mathscr{T}$ covering $\prod_{n \le m(S)} S_n$ such that $\forall T \in \mathscr{T}$ $\forall n \le m(S)$ $T_n \ne X$. In this case we define

$$(\mathscr{S}_{k+1} \mid S) = \left\{ \left(\prod_{n \le m(S)} T_n \right) \times \prod^{\{n \in \mathbf{N} \colon n > m(S)\}} X \colon T \in \mathscr{T} \right\}.$$

Finally, if $\exists\mathscr{R}' \subset \mathscr{R}$ such that $\operatorname{tp}(S) \subset \bigcup\mathscr{R}'$ and $m(R) \leq m(S)\ \forall R \in \mathscr{R}'$, then there is a finite subfamily of $\{\prod_{n<m(S)} R_n\colon R \in \mathscr{R}'\}$ covering the compact set $\prod_{n<m(S)} \operatorname{tp}(S_n)$. Applying 1.9 we choose a pairwise-disjoint clopen capped box cover $\mathscr{W}$ of $\prod_{n<m(S)} S_n$ refining

$$\left\{\left(\prod^{m(S)} X\right)\setminus\left(\prod_{n<m(S)} \operatorname{tp}(S_n)\right)\right\} \cup \left\{\prod_{n<m(S)} R_n\colon R \in \mathscr{R}'\right\}.$$

We define

$$(\mathscr{S}_{k+1} \mid S) = \left\{\left(\prod_{n<m(S)} W_n\right) \times \prod^{\{n\colon n\geq m(S)\}} X\colon W \in \mathscr{W}\right\}.$$

It is clear that the conditions (1) through (4) are satisfied. So we assume that we have defined the families $\mathscr{S}_i\ \forall i \in \mathbf{N}$.

Define $\mathscr{S} = \{\bigcap\mathscr{B}\colon \mathscr{B}$ is a maximal chain of $\bigcup_{i\in\mathbf{N}}\mathscr{S}_i$ and $\bigcap\mathscr{B} \neq \varnothing\}$. Now (i) implies $\mathscr{S}$ is a pairwise-disjoint cover of $\prod^{\mathbf{N}} X$. So the proof is complete once we show that $\mathscr{S}$ consists of open sets. This follows immediately from

(#) *Each maximal chain $\mathscr{B}$ of $\bigcup_{i\in\mathbf{N}}\mathscr{S}_i$ is finite.*

Suppose # is false. Then for each $i \in \mathbf{N}\ \exists S(i) \in \mathscr{S}_i$ such that, by (1), $S(i+1)$ is a proper subset of $S(i)$. Also $m(S(i+1)) \geq m(S(i))\ \forall i \in \mathbf{N}$, and $\operatorname{rk}(S(i+1)_n) \leq \operatorname{rk}(S(i)_n)\ \forall i \in \mathbf{N}$. Since any non-increasing sequence of ordinals is eventually constant, we may choose for each $n \in \mathbf{N}$, $i_n \in \mathbf{N}$ such that $\operatorname{rk}(S(i)_n) = \operatorname{rk}(S(i_n)_n)\ \forall i \geq i_n$. There is a finite family $\mathscr{R}' \subset \mathscr{R}$ covering the compact set $\prod_n \operatorname{tp}(S(i_n)_n)$. Let $m = \sup\{m(R)\colon R \in \mathscr{R}'\}$.

Suppose there is a j so large that $m(S(j)) \geq m$. Let $i \geq j$ be such that $i \geq i_n\ \forall n < m$. Then for each $n < m$, $\operatorname{tp}(S(i)_n) \subset \operatorname{tp}(S(i_n)_n)$. Since $R_n = X\ \ \forall R \in \Omega'\ \ \forall n \geq m$, $\operatorname{tp}(S(i)) \subset \bigcup\mathscr{R}'$. Since $m(S(i)) > m$, (2a) implies $m(S(i+1)) = m(S(i))$. Thus, for each $k \in \mathbf{N}$, $m(S(i+k)) \neq m(S(i))$. since $S(i+k+1) \neq S(i+k+2)$, (2b) implies that

$$\operatorname{tp}(S(i+k+1)) \cap \operatorname{tp}(S(i+k)) = \varnothing.$$

So for each $k \in \mathbf{N}$, there exists $n(k) \in \mathbf{N}$ such that $n(k) < m(S(i))$ and

$$\operatorname{rk}(S(i+k+1)_{n(k)}) < \operatorname{rk}(S(i+k)_{n(k)}).$$

But then there exists $n \in \mathbf{N}$ with $n = n(k)$ for infinitely many n. As this implies there is an infinite decreasing sequence of ordinals, we have a contradiction.

Now suppose there is $m(S(i)) < m\ \forall i \in \mathbf{N}$. Then (3) implies there is $j \in \mathbf{N}$ such that $\forall i \geq j\ \ \exists\mathscr{R}_i \subset \mathscr{R}$ with $\operatorname{tp}(S(i)) \subset \bigcup\mathscr{R}_i$ and $m(R) \leq m(S(i)) < m\ \ \forall R \in \mathscr{R}_i$. Let $i \geq j$. As in the previous paragraph, (2) implies there exists $n(i) < m$ such that

$$\mathrm{rk}\big(S(i+1)_{n(i)}\big) < \mathrm{rk}\big(S(i)_{n(i)}\big).$$

So there is an $n < m$ with $n(i) = n$ for infinitely many i, which leads to a contradiction. □

1.11. Corollary [**2**]. *If X_n is a Lindelöf C-scattered space for each natural number n, then $\prod_n X_n$ is Lindelöf.*

Proof. The absolute of a Lindelöf space is Lindelöf, as is the space X constructed in the reduction 1.2. In particular, each of the families $\mathscr{S}_i$ of the Theorem 1.10 may be taken to be countable. □

1.12. Corollary [**18**]. *If X_n is a paracompact scattered space for each natural number n, then $\prod_n X_n$ is ultraparacompact.*

Proof. The Lemma 1.4(3) $\Rightarrow$ (2) shows that each X_n is 0-dimensional. So we simply replace X, in the reduction, by Y. Then this result is a consequence 1.10. □

2. Hyperspaces of C-scattered spaces. Given a space X, we use 2^X, according to Michael [**12**], for the set of all non-empty closed subsets of X topologized by the *Vietoris topology* as follows:

First, given a finite set $\{S_1, \dots, S_n\}$ of subsets of X, define

$$\langle S_1, \dots, S_n \rangle = \Big\{ F \in 2^X \colon F \subset \bigcup_{i \leq n} S_i \text{ and } F \cap S_i \neq \varnothing \ \forall i \leq n \Big\}.$$

Then the Vietoris topology is the topology on 2^X with base the set of all sets of form $\langle V_1, \dots, V_n \rangle$, where $\{V_1, \dots, V_n\}$ is some finite (n is not fixed) family of open subsets of X.

The hyperspace 2^X has two distinguished subspaces, the *compact-set hyperspace* $\mathscr{C}(X) = \{F \in 2^X\colon F \text{ is compact in } X\}$, and the *finite-set hyperspace* $\mathscr{F}(X) = \{F \in 2^X\colon F \text{ is a finite subset of } X\}$.

It is known that X is compact iff 2^X is compact iff 2^X is normal [**20**]. Further, we know that $\mathscr{C}(X)$ is locally compact (discrete) iff X is locally compact (discrete) [**12**]. More recently, Bell [**5**] discovered that $\mathscr{F}(X)$ is paracompact iff $\prod^n X$ is paracompact $\forall n \in \mathbf{N}$; hence $\mathscr{F}(X)$ is paracompact whenever X is a paracompact C-scattered space (a result, unpublished, due to the third author). From these results one might conjecture that $\mathscr{C}(X)$ is C-scattered whenever X is C-scattered, or that $\mathscr{C}(X)$ is an A'-space whenever X is an A'-space. In this section we kill both conjectures and prove the right theorems in their stead.

2.1. LEMMA. *For a space X, $\mathscr{C}(X)^{(1)} \subset \langle X, X^{(1)} \rangle$.*

Proof. If $F \notin \langle X, X^{(1)} \rangle$, then $F \cap X^{(1)} = \varnothing$. So F is covered by open sets with compact closures. If $F \in \mathscr{C}(X)$, then F has a compact neighborhood K in X. Since $\langle K \rangle = 2^K$ is compact, $\langle K \rangle \cap \mathscr{C}(X)^{(1)} = \varnothing$. So $F \notin \mathscr{C}(X)^{(1)}$. □

2.2. LEMMA. *For an A'-space X,* $\operatorname{acc}(\mathscr{C}(X)) = \mathscr{C}(X) \cap \langle X, \operatorname{acc}(X) \rangle$.

Proof. If $F \in \mathscr{C}(X)$ and $F \cap \operatorname{acc}(X) = \varnothing$, then F is finite and clopen in X, say $F = \{x_1, \dots, x_n\}$. But then $\{F\} = \langle x_1, \dots, x_n \rangle$ is open in $\mathscr{C}(X)$. So $f \notin \operatorname{acc}(\mathscr{C}(X))$.

Conversely, if $F \in \mathscr{C}(X) \cap \langle X, \operatorname{acc}(X) \rangle$, let $x \in F \cap \operatorname{acc}(X)$ and suppose $\langle V_1, \dots, V_n \rangle$ is an arbitrary basic neighborhood of F in $\mathscr{C}(X)$. Since x is an accumulation point of X, we may choose, for each $i \leq n$, an $x_i \in V_i \setminus \{x\}$. Clearly,

$$F \neq \{x_1, \dots, x_n\} \in \langle V_1, \dots, V_n \rangle.$$

So $F \in \operatorname{acc}(\mathscr{C}(X))$. □

2.3. EXAMPLE. *There is a Lindelöf scattered A'-space X such that $\mathscr{C}(X)$ is neither C-scattered nor normal.*

Proof. Let $X = (\mathbf{N} \times \omega_1) \cup \{\infty\}$ have as a base the set

$$\{\{(n, \alpha)\} : (n, \alpha) \in \mathbf{N} \times \omega_1\}$$
$$\cup \{X \setminus (\mathbf{N} \times W) : W \text{ is a finite subset of } \omega_1\}.$$

Then the only non-isolated point of X is ∞, and the complement of a neighborhood of ∞ is countable. So X is a Lindelöf scattered A'-space. Since $\prod^{\omega_1} \mathbf{N}$ is not normal (see 2.7.16 in [6]), our proof will be complete once we show the following assertions:

(1) $(\mathscr{C}(X))^{(1)} \neq \varnothing$.

(2) If $\mathscr{V} = \langle V_1, \dots, V_n \rangle$ is a basic open set of $\mathscr{C}(X)$ and if $\mathscr{V} \cap (\mathscr{C}(X))^{(1)} \neq \varnothing$, then there is a closed set $\mathscr{K}$ of $(\mathscr{C}(X))^{(1)}$ such that $\mathscr{K} \subset \mathscr{V}$ and $\mathscr{K}$ is homeomorphic to $\prod^{\omega_1} \mathbf{N}$ [i.e., every compact subset of $(\mathscr{C}(X))^{(1)}$ has empty interior].

For simplicity, let $\mathscr{C}^1$ denote $(\mathscr{C}(X))^{(1)}$ and Π denote $\Pi^{\omega_1}\mathbf{N}$. Of course (1) follows since X is not locally compact; however, using the Lemmas 2.1 and 2.2, we can easily establish more:

(3) $\mathscr{C}^1 = \mathscr{C}(X) \cap \langle X, \{\infty\}\rangle$.

In order to see (2), we use (3) to assume, without loss of generality, $V_i = \{(k_i, \alpha_i)\} \subset \mathbf{N} \times \omega_1\ \forall i < n$, and $V_n = X \setminus (\mathbf{N} \times W)$, where W is a finite subset of ω_1. Let $\{\bar{\alpha}: \alpha \in \omega_1\}$ be a listing of $\omega_1 | (W \cup \{\alpha_i: i < n\})$. Define a function $\Phi: \Pi \to \mathscr{C}^1$ by

$$\Phi(g) = \{\infty\} \cup \{(k_i, \alpha_i): i < n\} \cup \{(g(\alpha), \bar{\alpha}): \alpha \in \omega_1\} \quad \forall g \in \Pi.$$

Now Φ is a function because each $\Phi(g)$ is (homeomorphic to) the one-point compactification of the discrete subspace $\Phi(g) \setminus \{\infty\}$. Clearly Φ is an injection into $\mathscr{V}$. Further, for each finite subset S of ω_1 and each $g \in \Pi$, we have

$$\Phi\left(\bigcap_{\alpha \in S} \overset{-1}{\prod_{\alpha}}\{g(\alpha)\}\right) = \Phi(\Pi) \cap \langle (k_1, \alpha_1), \ldots, (k_n, \alpha_n), \{x_1\}, \ldots, \{x_m\}, \\ X \setminus (\mathbf{N} \times (S \cup W))\rangle,$$

where $\{x_1, \ldots, x_m\} = g(S)$. Therefore, Φ is an embedding.

In order to see that $\Phi(\Pi)$ is closed in $\mathscr{C}^1$, suppose that $F \in \mathscr{C}^1 \setminus \Phi(\Pi)$. Then at least one of the following hold:

(4) $\exists \beta \in \omega_1$ such that $F \cap (N \times \{\bar{\beta}\})$ has more than one element, or

(5) $\exists \beta \in \omega_1$ such that $F \cap (N \times \{\bar{\beta}\}) = \varnothing$, or

(6) $\exists x \in F \cap (N \times \{\alpha_i: i < n\}) \setminus \{(k_i, \alpha_i): i < n\}$, or

(7) $\exists i < n$ such that $(k_i, \alpha_i) \notin F$.

In case (4), suppose $\{(j_1, \bar{\beta}), (j_2, \bar{\beta})\} \subset F$ and $j_1 \neq j_2$. Then $\langle X, \{(j_1, \bar{\beta})\}, \{(j_2, \bar{\beta})\}\rangle$ is a neighborhood of F missing $\Phi(\Pi)$. In case (5)

$$\langle X \setminus (\mathbf{N} \times \{\bar{\beta}\})\rangle$$

is a neighborhood of F missing $\Phi(\Pi)$. The cases (6) and (7) are similar to (4) and (5), respectively. So $\Phi(\Pi)$ is a closed subset of $\mathscr{C}^1$. □

An example similar to our 2.3 was discovered independently by S. Mrowka.

2.4. THEOREM. *The hyperspace $\mathscr{C}(X)$ is an A'-space iff X is either compact or discrete.*

Proof. If X is compact, then $\mathscr{C}(X) = 2^X$. If X is discrete, then $\mathscr{C}(X) = \mathscr{F}(X)$, which is discrete. In either case, $\mathscr{C}(X)$ is an A'-space.

Conversely, suppose $\mathscr{C}(X)$ is an A'-space. Since $x \to \{x\}$ gives an embedding of X onto a closed subspace of $\mathscr{C}(X)$, X is an A'-space. Now suppose X is neither compact nor discrete. Since X is not discrete, there exists $y \in \operatorname{acc}(X)$. Since X is not compact and $\operatorname{acc}(X)$ is compact, there exists a non-compact, closed, discrete subspace $D \subset X \setminus \operatorname{acc}(X)$. Let $\mathscr{D} = \{\{y, d\} \colon d \in D\}$. Obviously $\mathscr{D}$ is a closed, non-compact subset of $\mathscr{C}(X)$. According to 2.2, $\mathscr{D} \subset \operatorname{acc}(\mathscr{C}(X))$. Thus, $\operatorname{acc}(\mathscr{C}(X))$ is not compact—a contradiction. □

Since $\mathscr{C}(X)$ is metrizable iff X is metrizable **[12]**, the following is an immediate consequence of Theorem 2.4.

2.5. COROLLARY. *The hyperspace $\mathscr{C}(X)$ is an A-space iff A is compact metrizable or discrete.* □

We do not, at this time, have a reasonable characterization for "$\mathscr{C}(X)$ is paracompact C-scattered". However, according to **[22]** and to the inverse limit characterization of absolute, the absolute of $\mathscr{C}(X)$ is $\mathscr{C}$ (the absolute of X). Thus, one might follow the path we used in 1.2 to reduce the situation to the extremally disconnected X case.

For the remaining part of this section, identify X with its image in $\mathscr{C}(X)$ under the map $x \to \{x\}$. We wish to examine the truth of the statement "When X is an A'-space, there is an A'-space X' such that $X \subset X'$ and X' is dense in $\mathscr{C}(X)$". Since $X \subset \mathscr{F}(X)$ and since $\mathscr{F}(X)$ will be paracompact (see the first paragraph of this section), $\mathscr{F}(X)$ is a natural candidate for X' in the statement in question. However, when X is the space of example 2.3, $\mathscr{F}(X)$ is not even C-scattered. Before we present the last result of this section, we state a lemma whose proof is straight-forward and easy.

2.6. LEMMA. *Suppose Y is dense in a space X. Then* $\operatorname{acc}(Y) = Y \cap \operatorname{acc}(X)$. □

2.7. THEOREM. *Suppose X is an A'-space. Then there is an A'-space X' such that $X \subset X'$ and X' is dense in $\mathscr{C}(X)$ iff X is compact or* $\operatorname{int}_X(\operatorname{acc}(X)) = \varnothing$.

Proof. Suppose X is compact. Then $X' = \mathscr{C}(X)$ works. So we suppose that $\operatorname{int}_X(\operatorname{acc}(X)) = \varnothing$. Define

$$X' = \mathscr{C}(X) \cap (\langle \operatorname{acc}(X) \rangle \cup \langle X \setminus \operatorname{acc}(X) \rangle).$$

Clearly $X \subset X'$. To see that X' is dense in $\mathscr{C}(X)$, suppose $\langle V_1, \ldots, V_n \rangle$ is a basic open set in 2^X. Since $\operatorname{int}_X(\operatorname{acc}(X)) = \varnothing$, we may choose an isolated point $x_i \in V_i$ $\forall i \leq n$. Then

$$\{x_i\colon i \leq n\} \in X' \cap \langle V_1, \ldots, V_n\rangle.$$

So X' is dense in $\mathscr{C}(X)$. To see that X' is an A'-space, first observe that 2.6 shows that $\operatorname{acc}(X') = X' \cap \operatorname{acc}(\mathscr{C}(X))$. Applying 2.2, we find that

$$\begin{aligned}\operatorname{acc}(X') &= X' \cap \langle X, \operatorname{acc}(X')\rangle \\ &= X' \cap \langle \operatorname{acc}(X)\rangle = \mathscr{C}(X) \cap \langle \operatorname{acc}(X)\rangle.\end{aligned}$$

Since $\operatorname{acc}(X)$ is compact, $\operatorname{acc}(X') = \mathscr{C}(X) \cap \langle \operatorname{acc}(X)\rangle$ is compact [**12**].

For the converse, suppose X is a non-compact space such that $U = \operatorname{int}_X(\operatorname{acc}(X)) = \varnothing$, and assume $X \subset \mathscr{Z}$ and $\mathscr{Z}$ is dense in $\mathscr{C}(X)$. We shall show that $\mathscr{Z}$ is not an A'-space. Since X is an A'-space, there is an infinite closed set $D \subset X \setminus \operatorname{acc}(X)$. For each $d \in D$, $\langle U, \{d\}\rangle \cap \mathscr{C}(X) = \varnothing$. Since $\mathscr{Z}$ is dense in $\mathscr{C}(X)$, there is, for each $d \in D$, $K_d \in \mathscr{Z} \cap \langle U, \{d\}\rangle$. Clearly, $K_d \setminus \{d\} \subset \operatorname{acc}(X)$. Hence, Lemma 2.2 shows that each $K_d \in \operatorname{acc}(\mathscr{C}(X))$. According to 2.6, each $K_d \in \operatorname{acc}(\mathscr{Z})$. Now $\mathscr{K} = \{K_d\colon d \in D\}$ is certainly discrete in $\mathscr{C}(X)$, and hence, in $\mathscr{Z}$. Suppose $F \in \mathscr{Z} \setminus \mathscr{K}$. Since F is compact, $F \cap D$ is finite, say $F \cap D = \{d(1), \ldots, d(n)\}$, where $n = 0$ is the $F \cap D = \varnothing$ case. Then

$$\langle \{x_1\}, \ldots, \{x_n\}, X \setminus D\rangle \setminus \{K_{d(1)}, \ldots, K_{d(n)}\}$$

is a neighborhood of F missing $\mathscr{K}$. Thus, $\mathscr{K}$ is closed and discrete in $\operatorname{acc}(\mathscr{Z})$. So $\operatorname{acc}(\mathscr{Z})$ is not compact. $\square$

3. Compactifications of metrizable C-scattered spaces. For a metrizable space X, let $K(X)$ denote the collection of all Hausdorff compactifications of X. $BX \in K(X)$ is said to be *metrically compatible* provided that BX is the Smirnov compactification [**15**] induced by a metrizable proximity. Let $K_M(X)$ denote the set of all metrically compatible members of $K(X)$. We will consider $K_M(X)$ to be partially ordered, inheriting the natural partial order of $K(X)$.

In this section we will present a characterization of A-spaces in terms of $K_M(X)$, and a study of $K_M(X)$ when X is an A-space. The principal result of our study is that $K_M(X)$ is a lattice when X is an A-space. In order to facilitate our study we first develop some machinery.

3.1. Definition. Suppose that X is a metrizable space. Let $M(X)$ denote the set of all metrics (on X) compatible with X. Define a partial order $\ll$ on $M(X)$ as follows: if $d_1, d_2 \in M(X)$, then $d_1 \ll d_2$ holds provided that for each pair $\{a_n\}$ and $\{b_n\}$ of sequences in X, $d_2(a_n, b_n) \to 0$ implies $d_1(a_n, b_n) \to 0$.

Given $d_1, d_2 \in M(X)$, we say that d_1 and d_2 are *coherently equivalent*, and write $d_1 \equiv d_2$, provided that both $d_1 \ll d_2$ and $d_2 \ll d_1$ hold. It is easily determined that coherent equivalence is an equivalence relation on $M(X)$. Let $[d]$ designate the equivalence class of $d \in M(X)$ under $\equiv$. Let $(E(X), \ll)$ denote the quotient partially ordered set $M(X)/\equiv$ ordered by $[d_1] \ll [d_2]$ iff $d_1 \ll d_2$. That $(E(X), \ll)$ is an upper semi-lattice follows from defining $[d_1] \vee [d_2] = [d_1 \vee d_2]$, where

$$(d_1 \vee d_2)(x, y) = \max\{d_1(x, y), d_2(x, y)\} \qquad \forall x, y \in X.$$

The following lemma is the principal reason why we introduced $E(X)$, and is a consequence of the theorem: Suppose that $d_1, d_2 \in M(X)$. Then $d_1 \equiv d_2$ iff d_1 and d_2 induce the same proximity [**8**].

3.2. LEMMA. *Suppose that X is a metrizable space. Then $K_M(X)$ and $(E(X), \ll)$ are order-isomorphic.* □

When X is an A-space, we can simplify our study by considering a less complicated space.

3.3. DEFINITION. Suppose that X is an A'-space. We define a space X^* as follows: First let ∞ be an object not in X and define

$$X^* = \begin{cases} (X \setminus \operatorname{acc}(X)) \cup \{\infty\}, & \text{if } \operatorname{acc}(X) \neq \varnothing, \\ X, & \text{otherwise.} \end{cases}$$

X^* is topologized by $U \subset X^*$ is open iff $U \cap X$ is open.

Obviously $\operatorname{acc}(X^*) = \{\infty\}$, and X^* is the perfect image of X under the map $x \to x$ if $x \notin \operatorname{acc}(X)$, and $x \to \infty$ otherwise. Further, X^* is an A-space whenever X is an A-space.

3.4. LEMMA. *Suppose that X is an A-space. Then $(E(X), \ll)$ and $(E(X^*), \ll)$ are order isomorphic.*

Proof. We may assume, without loss of generality, that $\operatorname{acc}(X) \neq \varnothing$. Suppose $d \in M(X)$. We may define $d^* \in M(X^*)$ by allowing

(i) $d^*(\infty, \infty) = 0$,

(ii) $d^*(x, \infty) = d(x, \operatorname{acc}(X))$, if $x \neq \infty$, and

(iii) $d^*(x, y) = \min\{d(x, y),\ d(x, \operatorname{acc}(X)) + d(y, \operatorname{acc}(X))\}$, if $\infty \notin \{x, y\}$.

Define $\Phi = \{([d], [d^*]): d \in M(X)\}$. We will show that Φ is an order-preserving bijection from $(E(X), \ll)$ onto $(E(X^*), \ll)$.

Claim 1. If $d_1, d_2 \in M(X)$ and if $d_1 \ll d_2$, then $d_1^* \ll d_2^*$. To establish this claim, let (x_n) and (y_n) be sequences in X^* such that $d_2^*(x_n, y_n) \to 0$. We show that $d_1^*(x_n, y_n) \to 0$. First, let $I_1 = \{n \in \mathbf{N}\colon \infty \in \{x_n, y_n\}\}$.

If I_1 is finite, then go to the next paragraph. Let (a_n) and (b_n) be the subsequences, respectively, of (x_n) and (y_n) such that $\infty \in \{u_n, v_n\}$ $\forall n \in \mathbf{N}$. Since $d_2^*(x_n, y_n) \to 0$, we must have $a_n \to \infty$ and $b_n \to \infty$. Therefore, the following holds:

$$d_1^*(a_n, b_n) \to 0. \tag{1}$$

If $\mathbf{N} \setminus I_1$ is finite, then (1) shows $d_1^*(x_n, y_n) \to 0$. So we assume $\mathbf{N} \setminus I_1$ is infinite, and proceed to the next paragraph.

Let $I_2 = \{n \in \mathbf{N}\colon \{x_n, y_n\} \subset X \text{ and } d_2^*(x_n, y_n) = d_2(x_n, y_n)\}$. If I_2 is finite, then go to the next paragraph. Let (p_n) and (q_n) be the subsequences, respectively, of (x_n) and (y_n) such that $\{p_n, q_n\} \subset X$ and $d_2^*(p_n, q_n) = d_2(p_n, q_n)$. Since $d_1 \ll d_2$ and $d_2^*(p_n, q_n) \to 0$, we have $d_1(p_n, q_n) \to 0$. Since $d_1^*(p_n, q_n) \le d_1(p_n, q_n)$, we have

$$d_1^*(p_n, q_n) \to 0. \tag{2}$$

If $\mathbf{N} \setminus (I_1 \cup I_2)$ is finite, then (1) and (2) show $d_1^*(x_n, y_n) \to 0$. So we assume $\mathbf{N} \setminus (I_1 \cup I_2)$ is infinite, and proceed to the next paragraph.

Let $I_3 = \mathbf{N} \setminus (I_1 \cup I_2)$. Let (s_n) and (t_n) be the subsequences, respectively, of (x_n) and (y_n) whose indices come from I_3. Since $\operatorname{acc}(X)$ is compact, we may choose for each $n \in \mathbf{N}$, $u_n, v_n \in \operatorname{acc}(X)$ such that

$$d_2(s_n, u_n) = d_2(s_n, \operatorname{acc}(X)) \quad \text{and} \quad d_2(t_n, v_n) = d_2(t_n, \operatorname{acc}(X)).$$

Since $d_2(s_n, \operatorname{acc}(X)) + d_2(t_n, \operatorname{acc}(X)) = d_2^*(s_n, t_n) \to 0$, we have that $d_2(s_n, u_n) \to 0$ and $d_2(t_n, v_n) \to 0$. Since $d_1 \ll d_2$, we find that $d_1(s_n, u_n) \to 0$ and $d_1(t_n, v_n) \to 0$. So $d_1^*(s_n, \infty) \to 0$ and $d_1^*(t_n, \infty) \to 0$. From the triangular inequality, we have

$$d_1^*(s_n, t_n) \to 0. \tag{3}$$

Certainly (1), (2), and (3) together imply $d_1^*(x_n, y_n) \to 0$. Thus, claim 1 is established.

Claim 2. Φ is an order-preserving function.

This is obvious from claim 1 which shows that $d_1^* \equiv d_2^*$ whenever $d_1 \equiv d_2$.

Claim 3. Φ is an injection.

Let $d_1, d_2 \in M_2(X)$ be such that $d_1 \not\equiv d_2$. Without loss of generality we may assume that there exist sequences (x_n) and (y_n) in X and an $\varepsilon > 0$ such that the following hold:

$$d_2(x_n, y_n) \to 0, \quad \text{and} \tag{4}$$

$$d_1(x_n, y_n) > \varepsilon \quad \forall n \in \mathbf{N}. \tag{5}$$

If there is a subsequence (a_i) of (x_n) such that $d_2(a_i, \operatorname{acc}(X)) \to 0$, then (a_i) has a subsequence (b_j) converging to some $z \in \operatorname{acc}(X)$. But then (4) implies (y_n) has a subsequence converging to z, contradicting (5). Thus, without loss of generality, we may assume that $\varepsilon > 0$ is chosen so that

$$d_1(x_n, \operatorname{acc}(X)) > \varepsilon \quad \forall n \in \mathbf{N} \tag{6}$$

holds. In a similar manner, we may additionally assume that the $\varepsilon > 0$ and the sequences (x_n) and (y_n) satisfy the following:

$$d_i(z_n, \operatorname{acc}(X)) > 0 \quad \forall i \in \{1,2\}\ \forall z_n \in \{x_n, y_n\}\ \forall n \in \mathbf{N}. \tag{7}$$

Now combining (4), (7), and the definition of d_1^* and d_2^*, we conclude that sufficiently large n, we have $d_i^*(x_n, y_n) = d_i(x_n, y_n)\ \forall i \in \{1,2\}$. Therefore, $d_2^*(x_n, y_n) \to 0$ while $d_1^*(x_n, y_n) > \varepsilon$ for sufficiently large n. Thus, $d_1^* \not\equiv d_2^*$.

Claim 4. Φ is a surjection.

Suppose $\delta \in M(X^*)$ and $d \in M(X)$. Given $x \in X \setminus \operatorname{acc}(X)$, we use the compactness of $\operatorname{acc}(X)$ to choose $\bar{x} \in \operatorname{acc}(X)$ such that $d(x, \bar{x}) = d(x, \operatorname{acc}(X))$. For each pair $x, y \in X$, define $\rho(x, y) = 0$, if $x = y$; otherwise define

$$\rho(x,y) = \begin{cases} d(x,y), & \text{if } x, y \in \operatorname{acc}(X) \\ d(\bar{x}, y) + \delta(x, \infty), & \text{if } x \in X \setminus \operatorname{acc}(X) \text{ and } y \in \operatorname{acc}(X) \\ d(\bar{x}, \bar{y}) + \delta(x, \infty) + \delta(y, \infty), & \text{if } x, y \in X \setminus \operatorname{acc}(X). \end{cases}$$

It is easy to see that ρ is a metric for X. Since $X \setminus \operatorname{acc}(X)$ is discrete, $\rho \in M(X)$. Clearly, $\rho^* = \delta$. Thus, Φ is surjective. □

It is interesting to note that much of 3.4 did not require the full force of "A-space". For example, if in 3.3 we merely assume that X is metrizable with $X^{(2)} = \varnothing$, and replace $\operatorname{acc}(X)$ with $X^{(3)}$ in the definitions of X^* and d^*, then Φ is still an order-preserving function. Requiring $X^{(1)}$ to be compact seems necessary for showing Φ is injective. However, it is unclear how to prove Φ is surjective in this context.

3.5. LEMMA [**13**]. *Suppose that* (X, d) *is a metric space such that for each pair* F_0 *and* F_1 *of non-empty disjoint closed subsets of* X, $d(F_0, F_1) > 0$, *then* X *is an* A*-space.* □

Nagata [**14**] has shown that the A-spaces are precisely those spaces whose finest compatible uniformities are metric. Here is a similar characterization.

3.6. THEOREM. *A metrizable space* X *is an* A*-space iff* $K_M(X)$ *has a maximum.*

Proof. According to 3.2, we may use $(E(X), \ll)$ as a representation for $K_M(X)$.

Assume that X is not an A-space. Let $d \in M(X)$. From 3.5 there exist non-empty disjoint closed sets F_0 and F_1 such that $d(F_0, F_1) = 0$. By Urysohn's lemma there exists a continuous map $f\colon X \to [0,1]$ such that $f(F_i) = \{i\}$ for each $i \in \{0,1\}$. Define a metric $\rho \in M(X)$ by $\rho(x,y) = d(x,y) + |f(x) - f(y)|$ (this is standard, see [7]). Since $d \leq \rho$, $[d] \ll [\rho]$. Since $d(F_0, F_1) = 0$, there exist sequences (x_n) and (y_n) in, respectively, F_0 and F_1 such that $d(x_n, y_n) \to 0$. However, $\rho(x_n, y_n) > 1$ $\forall n \in \mathbf{N}$. Thus $d \not\equiv \rho$. So $[d]$ is not a maximum.

Now suppose that X is an A-space. According to 3.4, it suffices to show $(E(X^*), \ll)$ has a maximum element. Let $\delta \in M(X^*)$ be arbitrary. If $\operatorname{acc}(X) \neq \varnothing$, we define

$$\mu(x,y) = \begin{cases} 0, & \text{if } x = y \\ \delta(x,\infty) + \delta(y,\infty), & \text{if } x \neq y. \end{cases}$$

If $\operatorname{acc}(X) = \varnothing$, we define

$$\mu(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

It is easy to verify that $d \ll \mu$ $\forall d \in M(X)$. □

It is known that a metrizable space X is locally compact iff $K(X)$ is a lattice. The main result of this section is similar in nature.

3.7. THEOREM. *If X is an A-space, then $K_M(X)$ is a lattice.*

Proof. From 3.2, we need only show $(E(X), \ll)$ is a lattice. From 3.4, it suffices to prove that $(E(X^*), \ll)$ is a lattice. So we assume X has at most one accumulation point which will be denoted by ∞. As we have already established $(E(X), \ll)$ to be an upper semi-lattice under the operation $\vee$, we only need to define $\wedge$.

Claim. If $d_1, d_2 \in M(X)$, then there is $d \in M(X)$ such that $d(x, y) \leq d_i(x, y)$ for each $i \in \{1,2\}$.

To establish the claim, first define a continuous semi-metric ρ compatible with X by $\rho(x, y) = \min\{d_1(x, y), d_2(x, y)\}$. The semi-metric ρ generates a shortest path semi-metric d in the following standard way. Let $x, y \in X$. Define

$$d(x, y) = \inf\left\{ \sum_{i=0}^{n} \rho(x_{i-1}, x_i): \{x_0, \dots, x_n\} \subset X,\ n \in \mathbf{N},\ x_0 = x,\ x_n = y \right\}.$$

Suppose that $x \neq y$. Not both x and y are ∞, so suppose $x \neq \infty$. Since x is an isolated point, each of $d_1(x, X \setminus \{x\}) > 0$ and $d_2(x, X \setminus \{x\}) > 0$. So

$$0 < \rho(x, X \setminus \{x\}) \leq d(x, y) \leq \min\{d_1(x, y), d_2(x, y)\}.$$

It is easy to verify that $d \in M(X)$. Thus, our claim is proved.

Now define, for each pair $x, y \in X$,

$$(d_1 \wedge d_2)(x, y) = \sup\{\delta(x, y): \delta \in M(X), \delta(u, v) \leq \rho(u, v)\ \forall u, v \in X\}.$$

It is easy to verify that $d_1 \wedge d_2 \in M(X)$, that $d_1 \wedge d_2 \ll d_1$, and that $d_1 \wedge d_2 \ll d_2$. Define $[d_1] \wedge [d_2] = [d_1 \wedge d_2]$. □

Question Suppose X is a metrizable space with $X^{(1)}$ compact. Is $K_M(X)$ a lattice?

We complete this section with a result on the size of $K_M(X)$ when X is an A-space. First observe that $|K_M(X)| = 1$ whenever X is compact.

3.8. THEOREM. *Suppose that X is a non-compact A-space. Then there is $K \subset K_M(X)$, $|K| = 2^{\aleph_0}$, such that each distinct pair of members of K are pairwise incomparable. Further, if X is separable, then $|K_M(X)| = 2^{\aleph_0}$.*

Proof. We show the result for $(E(X), \ll)$, and we assume, without loss of generality, that X has at most one accumulation point to be denoted by ∞. Since X is non-compact, it has a countably infinite closed discrete subset $\{x_i\colon i \in \mathbf{N}\}$. Let $\mathscr{I}$ be an independent set in $\mathbf{N}$ (i.e., for each disjoint pair $\mathscr{I}_1$ and $\mathscr{I}_2$ of non-empty finite subsets of $\mathscr{I}$ we have $\bigcap\mathscr{I}_1 \setminus \bigcup\mathscr{I}_2$ is infinite) of cardinality $2^{\aleph_0}$ (see 3.6F in [**6**]). Let $\mu \in M(X)$ be as defined in 3.6, above, such that $[\mu]$ is the maximum of $(E(X), \ll)$. For each $I \in \mathscr{I}$, define $\mu_I\colon X \times X \to \mathbf{R}$ by

$$\mu_I(x, y) = \begin{cases} \left|\frac{1}{i} - \frac{1}{j}\right|\mu(x, y), & \text{if } x = x_i,\ y = x_j, \text{ and } i, j \in I, \\ \mu(x, y), & \text{otherwise.} \end{cases}$$

It is easy to verify that $\mu_I \in M(X)\ \forall I \in \mathscr{I}$.

Suppose $I \in \mathscr{I}$. Then $\mu_I(a_n, b_n) \to 0$ iff either $a_n \to \infty$ and $b_n \to \infty$, or $a_n = b_n$ for sufficiently large n, or $\{a_n, b_n\} \subset \{x_i\colon i \in I\}$ for all but finitely many n. So if $J \in \mathscr{I} \setminus \{I\}$ and if $j(n)$ is the nth element of J, then $\mu_J(x_{j(n)}, x_{j(n+1)}) \to 0$. While

$$0 < \mu(\infty, D) \leq \mu_I(x_{j(n)}, x_{j(n+1)})$$

for infinitely many $n \in J$. Therefore, $\mu_J \ll \mu_I$ is false. Let $K = \{[\mu_I]\colon I \in \mathscr{I}\}$.

Further, suppose $Y \subset X$ is countable and dense. Then there are at most $2^{\aleph_0}$ many continuous functions from $Y \times Y$ into $\mathbf{R}$. Hence $|M(X)| \leq 2^{\aleph_0}$. □

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