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MATRIX RINGS OVER *-REGULAR RINGS AND PSEUDO-RANK FUNCTIONS

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In this paper we obtain a characterization of those *-regular rings whose matrix rings are *-regular satisfying LP $\overset{*}{\sim}$ RP. This result allows us to obtain a structure theorem for the *-regular self-injective rings of type I which satisfy LP $\overset{*}{\sim}$ RP matricially.

Also, we are concerned with pseudo-rank functions and their corresponding metric completions. We show, amongst other things, that the LP $\overset{*}{\sim}$ RP axiom extends from a unit-regular *-regular ring to its completion with respect to a pseudo-rank function. Finally, we show that the property LP $\overset{*}{\sim}$ RP holds for some large classes of *-regular self-injective rings of type II.

All rings in this paper are associative with 1.

Let R be a ring with an involution *. Recall that * is said to be *n-positive definite* if $\sum_{i=1}^{n} x_i x_i^* = 0$ implies $x_1 = \cdots = x_n = 0$. The involution * is said to be *proper* if it is 1-positive definite; and if * is n-definite positive for all n, then we say that * is *positive definite*.

Recall than an element $e \in R$ is said to be a *projection* if $e^2 = e^* = e$ and R is called a *Rickart *-ring* if for every $x \in R$ there exists a projection e in R generating the right annihilator of x, that is $r(x) = eR$. Because of the involution, we have $\ell(x) = Rf$ for some projection f. Notice that $r(x) \cap x^*R = 0$, hence the involution * is proper and R is nonsingular. The above projections e, f depend on x only, $1 - e$ $(1 - f)$ is called the right (left) projection of x and, as usual, we shall write $1 - e = \mathrm{RP}(x)$, $1 - f = \mathrm{LP}(x)$.

If R is a *-ring, we denote by $\mathrm{P}(R)$ the set of projections of R partially ordered by $e \leq f$ iff $ef = e$. Thus, if $e \leq f$ we have $eR \subseteq fR$ and $Re \subseteq Rf$. Recall [**2**, pg. 14] that if R is Rickart, then $\mathrm{P}(R)$ is a lattice.

Two idempotents e, f of a ring R are said to be *equivalent*, $e \sim f$, if there exist $x \in eRf$, $y \in fRe$ such that $xy = e$, $yx = f$. If e, f are projections in a ring with involution and we can choose $y = x^*$ then e, f are said to be **-equivalent*, $e \overset{*}{\sim} f$. A ring is *directly finite* if $e \sim 1$ implies $e = 1$. A ring with involution is said to be *finite* if $e \overset{*}{\sim} 1$ implies $e = 1$.

A ring R is *regular* if for every $a \in R$ there exists an element $b \in R$ such that $a = aba$. If R, in addition, possesses a proper involution, then R is called a **-regular* ring. By a theorem of von Neumann [**14**, Exercise 5, pg. 38] a regular ring with involution is *-regular iff it is a Rickart *-ring and in fact, if R is *-regular, then $xR = \mathrm{LP}(x)R$ and $Rx = R(\mathrm{RP}(x))$ for every $x \in R$.

If R is a *-regular ring and $r \in R$ with $e = \mathrm{RP}(r)$, $f = \mathrm{LP}(r)$, then it is well-known [**13**] that $e \sim f$, in fact there exists a unique $s \in eRf$ (the *relative inverse* of r) such that $sr = e$ and $rs = f$.

1. The property LP $\overset{*}{\sim}$ RP **for *-regular rings.** We say that a Rickart *-ring R satisfies the property LP $\overset{*}{\sim}$ RP if $\mathrm{LP}(x) \overset{*}{\sim} \mathrm{RP}(x)$ for every x in R. Also, we say that R has *partial comparability* (PC) if for every e, $f \in \mathrm{P}(R)$ such that $eRf \neq 0$ there exist nonzero subprojections $e' \leq e$ and $f' \leq f$ such that $e' \overset{*}{\sim} f'$. Clearly, in any Rickart *-ring, we have LP $\overset{*}{\sim}$ RP $\Rightarrow$ (PC).

LEMMA 1.1. *For a *-regular ring R, the following conditions are equivalent*:

(a) *R satisfies* LP $\overset{*}{\sim}$ RP.

(b) *Any two equivalent projections are *-equivalent.*

(c) *If $xx^* \in eRe$ with $e \in \mathrm{P}(R)$, then there exists $z \in eRe$ such that $xx^* = zz^*$.*

Proof. (a) $\Leftrightarrow$ (b). Since $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ for every $x \in R$.

(a) $\Rightarrow$ (c). See [**16**, Theorem 1].

(c) $\Rightarrow$ (a). First we show that R is directly finite. If $xy = 1$, then we can assume that $yx = e \in P(R)$ and $y \in eR$, $x \in Re$. We have $yy^* \in eRe$, so there exists $z \in eRe$ such that $yy^* = zz^*$. Now, we have $1 = xyy^*x^* = xzz^*x^*$. By [**1**, Theorem 3.1, (ii)], R is finite so $z^*x^*xz = 1$. This implies $e = 1$. Now, by [**16**, Theorem 1], the result follows. □

Let R be a *-ring. We say that R is a *Baer* *-ring if for every subset $S \subseteq R$ there exists a projection e in R such that $r(S) = eR$ (and so $\ell(S) = Rf$ for some projection f in R). Obviously, a Baer *-ring is Rickart and the partially ordered set $\mathrm{P}(R)$ is in fact a complete lattice.

An element $w \in R$ is said to be a *partial isometry* if $ww^*w = w$. In this case $ww^* = e$ and $w^*w = f$ are projections with $wR = eR$ and $w^*R = fR$. An element u is called *unitary* if $uu^* = u^*u = 1$.

It follows easily from Lemma 1.1 that the elements of a *-regular ring with LP $\overset{*}{\sim}$ RP have *weak polar decomposition*, that is, if $x \in R$ then

$x = wz$ where w is a partial isometry and $\mathrm{LP}(z) = \mathrm{RP}(z) = \mathrm{RP}(x)$. If, in addition, R is unit-regular (that is, for every x in R there exists a unit u in R such that $x = xux$), then w can be chosen to be a unitary.

Let R be a Baer *-ring. We say that the *-equivalence is *additive* in R if for any families $(e_i)_{i\in I}$, $(f_i)_{i\in I}$ of orthogonal projections of R such that $e_i \overset{*}{\sim} f_i$, for all $i \in I$, we have $\bigvee_{i\in I} e_i \overset{*}{\sim} \bigvee_{i\in I} f_i$ (where $\bigvee$ denotes supremum). The partial isometries are *addable* in R if for any family $(w_i)_{i\in I}$ of partial isometries such that $(w_i w_i^*)_{i\in I}$ and $(w_i^* w_i)_{i\in I}$ are families of orthogonal projections, there exists a partial isometry w in R such that $ww_i^*w_i = w_iw_i^*w = w_i$ for all $i \in I$, and $ww^* = \bigvee_{i\in I}(w_iw_i^*)$ and $w^*w = \bigvee_{i\in I}(w_i^*w_i)$.

LEMMA 1.2. (i) *If R is a self-injective *-regular ring, then the partial isometries are addable in R.*

(ii) *If R is a Baer *-regular ring, then the *-equivalence is additive in R.*

Proof. (i) Set $e_i = w_iw_i^*$, $f_i = w_i^*w_i$, with $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$ families of orthogonal projections. Consider the R-homomorphism $\varphi\colon \bigoplus_{i\in I} f_iR \to \bigoplus_{i\in I} e_iR$ for which $\varphi(f_i) = w_i$, all $i \in I$. Since R is self-injective, φ is given by left multiplication by some element, say x. Set $e = \bigvee_{i\in I} e_i$ and $f = \bigvee_{i\in I} f_i$. If $w = exf$ then it is easily seen that $e_iw = wf_i = w_i$ and $ww^* = e$, $w^*w = f$.

(ii) Since any Baer *-regular ring R is complete, it follows from [**13**, Thm. 3, p. 535] that R is a continuous ring. By [**5**, Thm. 13.17] $R = R_1 \times R_2$, where R_1 is self-injective and R_2 is an abelian continuous ring. Since a central idempotent of a Rickart *-ring is a projection, we have that R_1 and R_2 are *-regular. Moreover two *-equivalent projections in R_2 are equal so the *-equivalence is obviously additive in R_2. Since R_1 is self-injective and *-regular the partial isometries are addable in R_1. In particular the *-equivalence is additive in R_1. Therefore the *-equivalence is additive in R. □

For a ring R, we denote by $Q_r(R)$ ($Q_l(R)$) the maximal ring of right (left) quotients of R. Recall that if R is right nonsingular then $Q_r(R)$ is a regular right self-injective ring.

LEMMA 1.3. *Let R be a nonsingular *-ring. Then, the involution * extends to $Q_r(R)$ if and if $Q_r(R) = Q_l(R)$. In case * extends to $Q_r(R)$, this extension is unique and if * is n-positive definite on R, then the extended involution is also n-positive definite.*

Proof. The proof is contained in [**17**, Thm. 3.2], except the n-positive definite part.

It is well-known that if $x_1, \dots, x_m$ are nonzero elements in $Q_r(R)$, then there exist $1 \leq k \leq m$ and $r \in R$ such that $x_i r \in R$ for $i = 1, \dots, m$ and $x_k r \neq 0$. Assume that * is n-positive definite on R and let $x_1, \dots, x_m$ be nonzero elements in $Q = Q_r(R) = Q_l(R)$, with $m \leq n$. If k and r are as above, then we have $(x_1 r)^*(x_1 r) + \cdots + (x_m r)^*(x_m r) \neq 0$, and so $r^*(x_1^* x_1 + \cdots + x_m^* x_m) r \neq 0$ (we also denote by * the extended involution). Hence * is n-positive definite on Q. □

REMARKS. (1) In particular, if R is a nonsingular *-ring with proper involution and $Q = Q_r(R) = Q_l(R)$, then Q is a self-injective *-regular ring.

(2) Recall that for a nonsingular ring R the condition $Q_r(R) = Q_l(R)$ is equivalent to the Utumi's conditions:

(a) For every right ideal I, $\ell(I) = 0$ implies $I \leq_e R$.

(b) For every left ideal I, $r(I) = 0$ implies $I \leq_e R$.

Obviously, (a) $\Leftrightarrow$ (b) in any *-ring.

Let R be any *-ring. We say that R satisfies general comparability for *-equivalence (GC) if for every $e, f \in \mathrm{P}(R)$ there exists a central projection h in R such that $he \overset{*}{\lesssim} hf$ and $(1-h)f \overset{*}{\lesssim} (1-h)e$, cf. [**2**, p. 77].

THEOREM 1.4. *Let R be a *-regular ring such that $Q = Q_r(R) = Q_l(R)$. Then R satisfies (PC) if and only if Q satisfies* LP $\overset{*}{\sim}$ RP.

Proof. By Lemma 1.3, Q is a self-injective *-regular ring.

Assume that R satisfies (PC). Let e, f be two projections in Q such that $eQf \neq 0$. Since Q is regular, there exist nonzero subprojections $e_1 \leq e$ and $f_1 \leq f$ in Q such that $e_1 Q \cong f_1 Q$. Hence there exist $x \in e_1 Q f_1$ and $y \in f_1 Q e_1$ such that $e_1 = xy$ and $f_1 = yx$. Let I be a right ideal of R such that $I \leq_e R$ and $yI \leq R$. We have $yI = (ye_1)I = y(e_1 I)$ and $e_1 I \leq_e e_1 Q$. Choose a nonzero projection e_0 in R such that $e_0 \in e_1 I$. We note that $ye_0 \neq 0$, $ye_0 R \leq fQ$ and $(ye_0)R \leq R$. Set $f_0 = \mathrm{LP}(ye_0)$, and note that $f_0 \in \mathrm{P}(R)$ and $f_0 \leq f$. We observe that left multiplication by y induces an isomorphism from $e_0 R$ onto $f_0 R$ (since it is the restriction of an isomorphism from $e_1 Q$ onto $f_1 Q$), and so $e_0 R \cong f_0 R$. Since R satisfies (PC), there exist nonzero projections e_0', f_0' in R such that $e_0' \leq e_0 \leq e$, $f_0' \leq f_0 \leq f$ and $e_0' \overset{*}{\sim} f_0'$. It follows that Q satisfies (PC). By Lemma 1.2 and [**2**, Prop. 4, p. 79], we have that Q satisfies (GC). Now it follows from [**9**, Prop. 3.2] that Q satisfies LP $\overset{*}{\sim}$ RP.

Conversely, assume that Q satisfies LP $\overset{*}{\sim}$ RP. Let e, f be projections in R such that $eRf \neq 0$. Then there exist nonzero projections e_0, f_0 in R such that $e_0 \leq e$, $f_0 \leq f$ and $e_0 \sim f_0$. Thus, $e_0 \overset{*}{\sim} f_0$ in Q, and so there exists x in Q such that $xx^* = e_0$, $x^*x = f_0$. Let I be a right ideal in R such that $I \leq_e R$ and $x^*I \leq R$. Choose a nonzero projection e' in R such that $e' \in e_0R \cap I$ and note that $f' = (x^*e')(e'x)$ is a projection in R such that $e' \overset{*}{\sim} f'$. Inasmuch, $e' \leq e_0 \leq e$ and $f' \leq f_0 \leq f$. So, R satisfies (PC). □

PROPOSITION 1.5. *Let R be a Rickart *-ring. Consider the following axioms for R.*

(a) *R has* LP $\overset{*}{\sim}$ RP.

(b) *R has* (PC).

(c) *R satisfies general comparability for *-equivalence*, (GC).

(d) *The parallelogram law* (P) ($e - e \wedge f \overset{*}{\sim} e \vee f - f$, *for* $e, f \in \mathrm{P}(R)$).

(e) *If $e \sim f$, then there exists a unitary u in R such that $f = ueu^*$.*

*If R is a unit-regular *-regular ring, then* (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) *and* (c) $\Rightarrow$ (a) $\Rightarrow$ (b). *If R is a Baer *-regular ring, then all these conditions are equivalent.*

Proof. Assume that R is a unit-regular *-regular ring.

(a) $\Rightarrow$ (d). Since R is regular we have $e - e \wedge f \sim e \vee f - f$ for all projections e, f in R [**13**, Lemma 1]. The result is immediate.

(d) $\Rightarrow$ (a). This is a standard argument, cf. [**10**, Proof of Corollary 1.1, (g)].

(a) $\Leftrightarrow$ (e). This is routine.

(c) $\Rightarrow$ (a). For this, note that we can adapt the proof of [**9**, Prop. 3.2].

(a) $\Rightarrow$ (b). Obvious.

If R is a Baer *-regular ring, then R is unit-regular. By Lemma 1.2 and [**2**, Prop. 4, p. 79], (b) $\Rightarrow$ (c). This completes the proof. □

If R is *-regular and I is a two-sided ideal of R, then it is well-known that I is a *-ideal and the factor ring R/I is also *-regular with the natural involution. It is easy to see that if the involution on R is n-positive definite, then that on R/I is also n-positive definite.

LEMMA 1.6. *Let R be a *-regular ring and let I be a two-sided ideal of R. Every projection in R/I has the form $\bar{e}$, where $e \in \mathrm{P}(R)$. If v is any partial isometry in R/I and $e, f \in \mathrm{P}(R)$ are such that $\bar{e} = vv^*$ and $\bar{f} = v^*v$,* *then there exists a partial isometry* w *in* R *such that* $\overline{w} = v$, $ww^* = e_1 \leq e$ *and* $w^*w = f_1 \leq f$. *In particular, there exist orthogonal decompositions* $e = e_1 + e_2$, $f = f_1 + f_2$ *with* $e_1 \overset{*}{\sim} f_1$ *and* $e_2, f_2 \in I$.

Proof. Set $\overline{R} = R/I$. From $\overline{\mathrm{LP}(x)R} = \overline{x}\overline{R} = \mathrm{LP}(\overline{x})\overline{R}$ we deduce that $\mathrm{LP}(\overline{x}) = \overline{\mathrm{LP}(x)}$ and similarly $\mathrm{RP}(\overline{x}) = \overline{\mathrm{RP}(x)}$. So, any projection in R/I has the form $\bar{e}$, where $e \in \mathrm{P}(R)$. If v is a partial isometry in $\overline{R}$ and e, $f \in \mathrm{P}(R)$ are such that $\bar{e} = vv^*$, $\bar{f} = v^*v$ then we observe that we can choose $w' \in eRf$ such that $\overline{w'} = v$. We have

$$w'w'^* = e + y \quad \text{with } y \in I. \tag{1}$$

Put $h = \mathrm{LP}(y)$, and note that $h \leq e$. By multiplying the relation (1) on right and left by $e - h$, we obtain

$$(e-h)w'w'^*(e-h) = e - h. \tag{2}$$

Set $w = (e-h)w'$. Since $h \in I$, we have $\overline{w} = v$. Also, by (2), we have $ww^* = e - h \leq e$. Putting $e_1 = e - h$, $f_1 = w^*w = w'^*(e-h)w'$, we have $e_1 \leq e$, $f_1 \leq f$ and $e_1 \overset{*}{\sim} f_1$. Moreover, $\bar{e}_1 = \bar{e}$ and $\bar{f}_1 = \bar{f}$ and so, if we put $e_2 = h = e - e_1$, $f_2 = f - f_1$, then we have $e_2, f_2 \in I$. □

It is obvious from the relations $\mathrm{LP}(\overline{x}) = \overline{\mathrm{LP}(x)}$ and $\mathrm{RP}(\overline{x}) = \overline{\mathrm{RP}(x)}$ that if R satisfies LP $\overset{*}{\sim}$ RP, then $\overline{R} = R/I$ also satisfies LP $\overset{*}{\sim}$ RP. However, it is not true that property (PC) is preserved in factor rings, as the following example shows.

EXAMPLE 1.7. *There exists a* $*$*-regular ring* R *such that*

(a) R *is* $\aleph_0$*-continuous and* $\aleph_0$*-injective* (*see* [**5**] *for definitions*) *and* $Q_r(R) = Q_l(R)$.

(b) R *has* (PC) *but* R *does not have* LP $\overset{*}{\sim}$ RP.

(c) *There exists a maximal two-sided ideal* M *such that the factor ring* R/M *does not satisfy* (PC).

Proof. Let X be any uncountable infinite set. For $i \in X$, set $R_i = M_2(\mathbf{R})$. Consider $R = \{x \in \prod_{i\in X} R_i \mid x_i \in M_2(\mathbf{Q})$ for all but countably many $i \in X\}$. Obviously, R is a $*$-regular ring.

(a) If $(e_n)_{n\in\mathbf{N}}$ is any sequence of projections of R, then clearly $\bigvee_{n\in\mathbf{N}} e_n$ exists in $\prod_{i\in X} R_i$ and $\bigvee_{n\in\mathbf{N}} e_n \in R$. So, since $\prod_{i\in X} R_i$ is continuous, R is $\aleph_0$-continuous. Since $R \cong M_2(S)$, where $S = \{x \in \prod_{i\in X} K_i \mid K_i = \mathbf{R}$ for all $i \in X$, and $x_i \in \mathbf{Q}$ for all but countably many $i \in X\}$, it follows from [**5**, Corollary 14.13] that R is $\aleph_0$-injective. Clearly, $Q_r(R) = Q_l(R) = \prod_{i\in X} R_i$.

(b) If $eRf \neq 0$, with $e, f \in \mathrm{P}(R)$, then there exist nonzero subprojections $e_1 \leq e$, $f_1 \leq f$ such that $e_1 \sim f_1$. There exist some $i \in X$ such that e_{1i} is nonzero, and we observe that $e_{1i} \stackrel{*}{\sim} f_{1i}$ in $M_2(\mathbf{R})$. Define nonzero projections e_2, f_2 in R by $e_{2j} = f_{2j} = 0$ if $j \in X$ and $j \neq i$; $e_{2i} = e_{1i}$, $f_{2i} = f_{1i}$. Clearly, $e_2 \leq e_1$, $f_2 \leq f_1$ and $e_2 \stackrel{*}{\sim} f_2$.

To show that R does not satisfy LP $\stackrel{*}{\sim}$ RP, note first that the projections $\left(\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are equivalent but not *-equivalent in $M_2(\mathbf{Q})$. Set $p_i = \left(\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right)$ for all $i \in X$; $q_i = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for all $i \in X$, and put $p = (p_i)_{i \in X}$, $q = (q_i)_{i \in X}$. Then, p and q are equivalent but not *-equivalent projections in R.

(c) Let $J = \{x \in R \mid x_i = 0$ for all but countable many $i \in X\}$. Clearly, J is a proper two-sided ideal of R. Let M be a maximal two-sided ideal of R such that J is contained in M. It follows from [**5**, Thm. 14.33] that R/M is a simple self-injective *-regular ring. So, by Theorem 1.4, R/M has LP $\stackrel{*}{\sim}$ RP if and only if it has (PC). Consider the projections p, q constructed in (b). We note that neither p nor q belong to M. We have $p \sim q$ in R and so $\bar{p} \sim \bar{q}$ in $\overline{R} = R/M$. If $\overline{R}$ satisfies (PC), then $\bar{p} \stackrel{*}{\sim} \bar{q}$, and by applying Lemma 1.6, we see that there exist orthogonal decompositions $p = p' + p''$, $q = q' + q''$ with $p' \stackrel{*}{\sim} q'$ and p'', $q'' \in M$. Since all p_i, q_i have rank one, we deduce that each p_i' is either 0 or p_i. It follows that p', $q' \in J$ and so p, $q \in M$. This is a contradiction. So, R/M does not satisfy (PC). □

PROPOSITION 1.8. *Let R be a *-regular ring such that the intersection of the maximal two-sided ideals of R is zero. If R/M satisfies* (PC) *for all maximal two-sided ideals M of R, then R satisfies* (PC).

Proof. It suffices to see that given two nonzero equivalent projections e, f in R, there exist nonzero subprojections $e_1 \leq e$, $f_1 \leq f$ such that $e_1 \stackrel{*}{\sim} f_1$. Let M be a maximal two-sided ideal of R such that $e, f \notin M$. Then, $\bar{e}$ and $\bar{f}$ are nonzero projections in $\overline{R} = R/M$. By hypothesis, $\overline{R}$ satisfies (PC) so there exist nonzero subprojections $\bar{e}' \leq \bar{e}$, $\bar{f}' \leq \bar{f}$ such that $\bar{e}' \stackrel{*}{\sim} \bar{f}'$ in $\overline{R}$. Set $e'' = \mathrm{LP}(ee')$, $f'' = \mathrm{LP}(ff')$ and observe that $\bar{e}'' = \bar{e}'$, $\bar{f}'' = \bar{f}'$, $e'' \leq e$, $f'' \leq f$. Thus, there exist orthogonal decompositions $e'' = e_1 + e_2$, $f'' = f_1 + f_2$ with $e_1 \stackrel{*}{\sim} f_1$ and $e_2, f_2 \in M$. Clearly, e_1 and f_1 are nonzero *-equivalent projections and $e_1 \leq e$, $f_1 \leq f$. □

Proposition 1.8 and Example 1.7 suggest that maybe any *-regular ring such that the intersection of the maximal two-sided ideals is zero and the simple homomorphic images satisfy LP $\stackrel{*}{\sim}$ RP has LP $\stackrel{*}{\sim}$ RP. However, this is not true and we offer a counterexample in §3.

Now, we examine property LP $\overset{*}{\sim}$ RP in matrix rings. Recall that if R is a $*$-regular ring with n-positive definite involution, then the ring $M_n(R)$ of $n \times n$ matrices over R is also $*$-regular with involution $A^{\#} = (a_{ji}^*)$, where $A = (a_{ij})$ (the $*$-transpose involution). We shall assume in the rest of this section that $M_n(R)$ is endowed with this involution.

LEMMA 1.9. *Let R be a $*$-regular ring with 2-positive definite involution. Set $S = M_2(R)$. If E is a projection in S, then there exists an orthogonal decomposition $E = E_1 + E_2$, where $E_1 = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, with p, $q \in P(R)$ and $E_2 = \begin{pmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{pmatrix}$, with $a_1R = a_2R$ and $a_2^*R = a_3R$.*

Proof. Set $E = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$. We have

$$a^2 + bb^* = a, \tag{1}$$

$$c^2 + b^*b = c, \tag{2}$$

$$ab + bc = b, \tag{3}$$

and $a = a^*$, $c = c^*$.

Set $e = \mathrm{LP}(a) = \mathrm{RP}(a)$; $f = \mathrm{LP}(c) = \mathrm{RP}(c)$; $g = \mathrm{LP}(b)$; $h = \mathrm{LP}(b^*)$. From (1) and (2) we have $bb^* = a(1-a)$ and $b^*b = c(1-c)$ and so, $g \leq e$, $h \leq f$.

We claim that $ag = ga$. Set $d = bb^*$, and note that $ad = da$. We have $g = \mathrm{LP}(d) = \mathrm{RP}(d)$, and so $gad = da = ad$. Right multiplying this relation by $\bar{d}$, the relative inverse of d, we obtain $gag = ag$. Analogously, $ga = gag$, and we conclude that $ag = ga$.

Similarly, we can show $hc = ch$.

Now, we have

$$(e-g)a = a(e-g) = ((e-g)a)^*, \tag{4}$$

$$(e-g)a^2(e-g) = (e-g)a(e-g). \tag{5}$$

It follows that $(e-g)a$ is a projection. Note that $(e-g)aR = (e-g)eR = (e-g)R$. Hence,

$$e - g = (e-g)a \tag{6}$$

and, similarly

$$f - h = (f-h)c. \tag{7}$$

It follows from (1)–(7) that we have an orthogonal decomposition

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} = \begin{pmatrix} e-g & 0 \\ 0 & f-h \end{pmatrix} + \begin{pmatrix} ga & b \\ b^* & hc \end{pmatrix}.$$

Now, $(ga)R = geR = gR = bR$ and $(hc)R = hfR = hR = b^*R$. Putting

$$E_1 = \begin{pmatrix} e-g & 0 \\ 0 & f-h \end{pmatrix}, \qquad E_2 = \begin{pmatrix} ga & b \\ b^* & hc \end{pmatrix}$$

we have the desired projections. □

We note that the decomposition given in Lemma 1.9 is unique. Set $S = M_2(R)$. We say that a projection E of S is of *type* A if $E = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$ with p, $q \in P(R)$. We say that E is of *type* B if $E = \left(\begin{smallmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{smallmatrix}\right)$ with $a_1R = a_2R$, $a_2^*R = a_3R$. By Lemma 1.9, every projection of S is, in a unique way, an orthogonal sum of a projection of type A and a projection of type B.

We now construct some projections of type B. If $e \in \mathrm{P}(R)$ and w_1, $w_2 \in R$, we say that (w_1, w_2) is an *isometric pair* for e if $w_1R = w_1^*R = w_2R = eR$ and $w_1w_1^* + w_2w_2^* = e$. It is routine to verify that if (w_1, w_2) is an isometric pair for e, then

$$E = \begin{pmatrix} w_1^*w_1 & w_1^*w_2 \\ w_2^*w_1 & w_2^*w_2 \end{pmatrix}$$

is a projection of S of type B which is *-equivalent to $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (implemented by $\left(\begin{smallmatrix} w_1 & w_2 \\ 0 & 0 \end{smallmatrix}\right)$).

PROPOSITION 1.10. *Let R be a *-regular ring with 2-positive definite involution such that $S = M_2(R)$ satisfies* LP $\overset{*}{\sim}$ RP. *If E is a projection in S, then there exists an orthogonal decomposition $E = E_1 + E_2$, where E_1 is a projection of type* A *and there exist a projection e in R and an isometric pair for e, (w_1, w_2), such that*

$$E_2 = \begin{pmatrix} w_1^*w_1 & w_1^*w_2 \\ w_2^*w_1 & w_2^*w_2 \end{pmatrix}.$$

Proof. By Lemma 1.9, $E = E_1 + E_2$, where E_1 is type A and E_2 is type B. Set $E_2 = \left(\begin{smallmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{smallmatrix}\right)$, and put $e = \mathrm{LP}(a_1) = \mathrm{RP}(a_1) = \mathrm{LP}(a_2)$; $f = \mathrm{LP}(a_3) = \mathrm{RP}(a_3) = \mathrm{LP}(a_2^*)$. Set $G = \left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)$; $G_1 = \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$; $G_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & f \end{smallmatrix}\right)$. It is not difficult to see that

$$G \cdot S = G_1 \cdot S \oplus G_2 \cdot S = G_1 \cdot S \oplus E_2 \cdot S = G_2 \cdot S \oplus E_2 \cdot S.$$

We conclude that $G_1 \cdot S \cong G_2 \cdot S \cong E_2 \cdot S$. Since, by hypothesis, S satisfies LP $\overset{*}{\sim}$ RP, we have $E_2 \overset{*}{\sim} G_1$. Let W be a partial isometry of S implementing this *-equivalence. It is easy to see that W has the form $\left(\begin{smallmatrix} w_1 & w_2 \\ 0 & 0 \end{smallmatrix}\right)$ for w_1, $w_2 \in R$. An easy computation shows that (w_1, w_2) is an isometric pair for e. □

PROPOSITION 1.11. *Let R be a *-regular ring with 2-positive definite involution and satisfying* LP $\overset{*}{\sim}$ RP. *Set $S = M_2(R)$. Then, S satisfies* LP $\overset{*}{\sim}$ RP *if and only if for every projection $\left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ of S of type* B *with $e = \mathrm{LP}(a) = \mathrm{LP}(b)$, we have $\left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right) \overset{*}{\sim} \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$.*

Proof. We first observe that every subprojection of a projection of type B is itself of type B. This follows from Lemma 1.9 by observing that a projection of type B cannot contain a nonzero projection of type A. For, if $\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) \leq \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$, where $\left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ is of type B, then $pa = p$, $pb = 0$, $qb^* = 0$, $qc = q$. But $aR = bR$ implies $\ell(a) = \ell(b)$, so $pa = 0 = p$, and similarly $qc = 0 = q$.

If $E = \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then we say E is type A_1 and if $E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & q \end{smallmatrix}\right)$, then we say that E is type A_2. Note that every projection in S is an orthogonal sum of projections of types A_1, A_2 and B. Also, note that any subprojection of a projection E of type A_1, A_2 or B is itself of the same type as E.

Suppose that E, F are two equivalent projections in S. We will show that $E \overset{*}{\sim} F$ provided S satisfies the stated condition. Let $E = E_1 + E_2 + E_3$ be the decomposition of E into projections E_1, E_2 and E_3 of types A_1, A_2 and B respectively. Since $E \sim F$, there exists an orthogonal decomposition $F = F_1 + F_2 + F_3$, with $E_1 \sim F_1$, $E_2 \sim F_2$ and $E_3 \sim F_3$. For $i = 1, 2, 3$, we have orthogonal decompositions $F_i = F_{i1} + F_{i2} + F_{i3}$ of F_i into projections of types A_1, A_2 and B respectively. Returning to E, we obtain $E_i = E_{i1} + E_{i2} + E_{i3}$ with $E_{ij} \sim F_{ij}$ for $i, j = 1, 2, 3$. So, we have decomposed E and F into nine orthogonal projections, each one of pure type. It follows that it suffices to consider the following cases:

(a) E is type A_1 and F is type A_1.
(b) E is type A_1 and F is type A_2.
(c) E is type A_1 and F is type B.
(d) E is type B and F is type B.

Case (a). If $E = \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $F = \left(\begin{smallmatrix} p' & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $p, p' \in \mathrm{P}(R)$, then it follows that $p \sim p'$ in R. Since R satisfies LP $\overset{*}{\sim}$ RP, we have $p \overset{*}{\sim} p'$, and so $E \overset{*}{\sim} F$.

Case (b). Similar to case (a).

Case (c). By hypothesis, $F = \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right) \overset{*}{\sim} \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $eR = aR = bR$. So, $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \sim E$. By case (a), $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \overset{*}{\sim} E$, and so, $E \overset{*}{\sim} F$.

Case (d). Each one of E, F is *-equivalent, by hypothesis, to a projection of type A_1 and so, case (a) applies.

If S satisfies LP $\overset{*}{\sim}$ RP, then it follows as in the proof of Proposition 1.10 that for a projection $E = \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ of S of type B, with $e = \mathrm{LP}(a)$, we have $E \overset{*}{\sim} \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$. □

Recall that a *-ring is said to be **-Pythagorean* if for every x, y in R there exists $z \in R$ such that $xx^* + yy^* = zz^*$. Following [**11**], we say than an element a in R is a *norm* in R if it has the form $a = xx^*$, with $x \in R$. Clearly, in a *-Pythagorean ring any sum of norms is a norm.

The following theorem is an extension of some results of Handelman, cf. [**9**, Theorem 4.5] and [**11**; Theorem 4.9, Corollary 4.10].

THEOREM 1.12. *Let R be a *-regular ring with 2-positive definite involution and satisfying* LP $\overset{*}{\sim}$ RP. *Then, $M_2(R)$ satisfies* LP $\overset{*}{\sim}$ RP *if and only if R is *-Pythagorean. In this case, * is positive definite and $M_n(R)$ satisfies* LP $\overset{*}{\sim}$ RP *for all $n \geq 1$.*

Proof. The "only if" part follows from [**16**, Lemma 1].

Assume now that R is *-Pythagorean. By Proposition 1.11, it suffices to see that for any projection $E = \left(\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right)$ in $M_2(R)$ with $aR = bR$, $b^*R = cR$, $e = \mathrm{LP}(a)$, we have $E \overset{*}{\sim} \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We have $a = a^2 + bb^* = aa^* + bb^*$, so there exists w in R such that $a = ww^*$. Since R has LP $\overset{*}{\sim}$ RP, we see from Lemma 1.1 that we can choose $w \in eRe$. Let $\overline{w}$ be the relative inverse of w and note that

$$w\overline{w} = \overline{w}w = e. \tag{1}$$

Consider the relation

$$ww^*ww^* + bb^* = ww^*. \tag{2}$$

By multiplying the relation (2) on the left by $\overline{w}$ and on the right by $\overline{w}^* = \overline{w^*}$ and using (1), we get

$$w^*w + \overline{w}bb^*\overline{w}^* = e. \tag{3}$$

Hence,

$$\begin{pmatrix} w^* & \overline{w}b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w & 0 \\ b^*\overline{w}^* & 0 \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$$

and so $\left(\begin{smallmatrix} w^* & \overline{w}b \\ 0 & 0 \end{smallmatrix}\right)$ is a partial isometry. It follows that

$$F = \begin{pmatrix} w & 0 \\ b^*\overline{w}^* & 0 \end{pmatrix}\begin{pmatrix} w^* & \overline{w}b \\ 0 & 0 \end{pmatrix}$$

is a projection in S and we compute that

$$F = \begin{pmatrix} a & b \\ b^* & b^*\overline{w}^*\overline{w}b \end{pmatrix}.$$

Note that $b^*\overline{w}^*\overline{w}bR = b^*R = cR$, so F is of type B. To see that $E = F$, we observe that for any projection $\left(\begin{smallmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{smallmatrix}\right)$ of type B, a_3 is uniquely determined by a_1 and a_2. For, note that $a_2 = a_1a_2 + a_2a_3$. Let $\bar{a}_2$ be the relative inverse of a_2. Multiplying the above relation on the left by $\bar{a}_2$, and observing that $f = \bar{a}_2a_2 = \mathrm{RP}(a_2) = \mathrm{LP}(a_2^*) = \mathrm{LP}(a_3)$, we get $f = \bar{a}_2a_1a_2 + a_3$, so $a_3 = \bar{a}_2(1 - a_1)a_2$.

Clearly, if R is *-Pythagorean, then * is positive definite. By applying [**16**, Theorem 3], we see that $M_{2^n}(R)$ is *-Pythagorean for all $n \geq 0$, and so, $M_{2^n}(R)$ satisfies LP $\overset{*}{\sim}$ RP for all $n \geq 0$. Since any ring $M_m(R)$ is a corner in some ring $M_{2^n}(R)$, it follows that $M_m(R)$ satisfies LP $\overset{*}{\sim}$ RP for all $m \geq 1$. □

Let R be a *-ring such that $M_n(R)$ is Rickart for all $n \geq 1$. We say that R satisfies LP $\overset{*}{\sim}$ RP matricially if $M_n(R)$ satisfy LP $\overset{*}{\sim}$ RP for all $n \geq 1$.

COROLLARY 1.13. *Let R be a *-regular ring with 2-positive definite involution. Then, R is a *-regular ring satisfying* LP $\overset{*}{\sim}$ RP *matricially if and only if R satisfies the following condition*

If $aa^* + bb^* \in eRe$, where $a, b \in R$, $e \in \mathrm{P}(R)$, then there exists $z \in eRe$ such that $aa^* + bb^* = zz^*$. □

If R is a self-injective *-regular ring, we see from Propositions 1.5 and 1.8 that R satisfies LP $\overset{*}{\sim}$ RP if and only if all simple homomorphic images of R satisfy LP $\overset{*}{\sim}$ RP. Now we obtain a characterization of the self-injective *-regular rings of type I which satisfy LP $\overset{*}{\sim}$ RP matricially. The background of the structure theory for regular, right self-injective rings can be found in [**5**, Chapter 10].

COROLLARY 1.14. *Let R be a *-regular self-injective ring of type* I. *Then, $M_m(R)$ is a *-regular self-injective ring of type* I *satisfying* LP $\overset{*}{\sim}$ RP, *for all $m \geq 1$, if and only if R is *-isomorphic to a direct product $\prod_{n=1}^{\infty} M_n(A_n)$, where each A_n is an abelian self-injective *-regular ring and all its simple homomorphic images are *-Pythagorean division rings with positive definite involution.*

Proof. If $R \stackrel{*}{\cong} \prod_{n=1}^{\infty} M_n(A_n)$, where each A_n is an abelian self-injective *-regular ring with all division ring images *-Pythagorean and with positive definite involution, we see from 1.5, 1.8 and 1.12 that R satisfies LP $\stackrel{*}{\sim}$ RP matricially. Also, it is well-known that $M_m(R)$ is a regular self-injective ring of type I, for all $m \geq 1$.

For the converse, note that by [**5**, Thm. 10.24] there exist regular, self-injective rings $R_1, R_2, \ldots$ such that $R \cong \prod_{n=1}^{\infty} R_n$ and each R_n is of type I_n. It follows that there exist orthogonal central projections $e_1, e_2, \ldots$ in R with $\bigvee_n e_n = 1$, and orthogonal projections $f_{i1}, f_{i2}, \ldots, f_{ii}$ for $i = 1, 2, \ldots$ such that $f_{i1} \sim f_{i2} \sim \cdots \sim f_{ii}$ and $e_i = f_{i1} + f_{i2} + \cdots + f_{ii}$ for $i = 1, 2, \ldots$. Since R satisfies LP $\stackrel{*}{\sim}$ RP, also $e_i R$ satisfies LP $\stackrel{*}{\sim}$ RP and so $f_{i1} \stackrel{*}{\sim} f_{i2} \stackrel{*}{\sim} \cdots \stackrel{*}{\sim} f_{ii}$. Set $A_n = f_{n1} R f_{n1}$, and observe that $e_n R \stackrel{*}{\cong} M_n(A_n)$. We deduce that $R \stackrel{*}{\cong} \prod_{n=1}^{\infty} M_n(A_n)$ and A_n are abelian self-injective *-regular rings with positive definite involution and satisfying LP $\stackrel{*}{\sim}$ RP matricially. Since all simple homomorphic images of an abelian regular ring are division rings, the result follows. □

2. Pseudo-rank functions on *-regular rings. In this section, we study property LP $\stackrel{*}{\sim}$ RP for completions of *-regular rings with respect to pseudo-rank functions. In particular, we show that if R is a *-regular unit-regular ring satisfying LP $\stackrel{*}{\sim}$ RP and N is a pseudo-rank function on R, then its N-completion also satisfies LP $\stackrel{*}{\sim}$ RP. In [**3**], Burke showed this holds for an irreducible *-regular rank ring with order k, with $k \geq 4$, in which comparability holds, which turns out to be a very special case of the result here. Our result follows from Theorem 2.8, which is also used in §3.

A pseudo-rank function on a regular ring R is a map N: $R \to [0, 1]$ such that

(a) $N(1) = 1$

(b) $N(xy) \leq N(x)$ and $N(xy) \leq N(y)$

(c) $N(e + f) = N(e) + N(f)$ for all orthogonal idempotents $e, f \in R$.

A rank function on R is a pseudo-rank function with the additional property

(d) $N(x) = 0$ implies $x = 0$.

If N is a pseudo-rank function on R, then the rule $\delta(x, y) = N(x - y)$ defines a pseudo-metric on R. Clearly, δ is a metric iff N is a rank function. The Hausdorff completion of R with respect to δ, $\overline{R}$, is showed [**5**, Chapter 19] to be a right and left self-injective regular ring which is complete with respect to the $\overline{N}$-metric, where $\overline{N}$ is the unique extension of N to $\overline{R}$.

If R is *-regular, it follows as in [8, Prop. 1] that we can extend * in a natural way to the N-completion of R, $\overline{R}$, so that $\overline{R}$ becomes a *-regular ring.

We now show the analogue of [5, Lemma 19.5] for projections in *-regular rings.

LEMMA 2.1. *Let R be a *-regular ring with pseudo-rank function N, let $\overline{R}$ be its N-completion and let φ: $R \to \overline{R}$ be the natural map. If p, $q \in P(\overline{R})$ are orthogonal, then there exists a sequence $\{(p_n, q_n)\} \subseteq R \times R$ such that*

(a) $\varphi(p_n) \to p$, $\varphi(q_n) \to q$.

(b) *For all n, p_n and q_n are orthogonal projections.*

Proof. By [5, Lemma 19.5], there exists a sequence $\{(e_n, f_n)\} \subseteq R \times R$ such that $\varphi(e_n) \to p$, $\varphi(f_n) \to q$ and for all n, e_n and f_n are orthogonal idempotents. Set $p_n = \mathrm{LP}(e_n)$, $q_n = \mathrm{RP}(f_n)$, and note that $p_n e_n = e_n$, $e_n p_n = p_n$, $q_n f_n = q_n$, $f_n q_n = f_n$. We have $q_n p_n = q_n f_n e_n p_n = 0$, so, for all n, p_n and q_n are orthogonal projections in R.

Given $\varepsilon > 0$, we can choose M such that $\overline{N}(p - \varphi(e_n)) < \varepsilon/2$ and $\overline{N}(p - \varphi(e_n^*)) < \varepsilon/2$ for $n > M$. Now, we have

$$N(p_n - e_n) = N\big(p_n e_n^* - p_n e_n\big) \le N\big(e_n^* - e_n\big)$$
$$\le \overline{N}\big(\varphi\big(e_n^*\big) - p\big) + \overline{N}(p - \varphi(e_n)) < \varepsilon \quad \text{if } n > M.$$

It follows that $\varphi(p_n) \to p$, and similarly $\varphi(q_n) \to q$. □

PROPOSITION 2.2. (a) *Let R be a regular ring and let N be a pseudo-rank function on R. Let φ: $R \to \overline{R}$ be the natural map from R to its N-completion, $\overline{R}$. If e, f are equivalent idempotents in $\overline{R}$, then there exist sequences $\{e_n\}$, $\{f_n\}$ such that, for all n, e_n and f_n are equivalent idempotents in R and $\varphi(e_n) \to e$, $\varphi(f_n) \to f$.*

(b) *In* (a), *if e and f are orthogonal, then we can choose $\{e_n\}$, $\{f_n\}$ such that e_n and f_n are equivalent orthogonal idempotents for all n.*

(c) *If R is *-regular and p, q are (orthogonal) equivalent projections in $\overline{R}$, then there exist $\{p_n\}$, $\{q_n\}$ such that, for all n, p_n and q_n are (orthogonal) equivalent projections in R and $\varphi(p_n) \to p$, $\varphi(q_n) \to q$.*

Proof. (a) It suffices to see that given $\varepsilon > 0$, there exist equivalent idempotents h, g in R such that $\overline{N}(e - \varphi(h)) < \varepsilon$ and $\overline{N}(f - \varphi(g)) < \varepsilon$. We observe that we can get idempotents e', f' in R, and elements

$x \in e'Rf'$ and $y \in f'Re'$ such that $\overline{N}(e - \varphi(e')) < \varepsilon/2$, $\overline{N}(f - \varphi(f')) < \varepsilon/2$ while $N(e' - xy) < \varepsilon/6$ and $N(f' - yx) < \varepsilon/6$. Note that $xy \in e'Re'$. Clearly, $xyR + (e' - xy)R = e'R$ and so there exists an idempotent h in R such that $e'h = he' = h$, $hR = xyR$ and $(e' - h)R \leq (e' - xy)R$. Thus, we have $N(e' - h) < \varepsilon/6$.

Let $\lambda \in Rh$ with $xy\lambda = h$. We have

$$\begin{aligned} N(e'\lambda - e') &\leq N(e'\lambda - h) + N(h - e') \\ &= N((e' - xy)\lambda) + N(h - e') < \varepsilon/6 + \varepsilon/6 = \varepsilon/3. \end{aligned}$$

Set $g = y\lambda x$. Clearly, g is idempotent, g is equivalent to h and $g \leq f'$. We have

$$\begin{aligned} N(f' - g) = N(f' - y\lambda x) &\leq N(f' - yx) + N(yx - y\lambda x) \\ &< \varepsilon/6 + N(y(e' - e'\lambda)x) < \varepsilon/6 + \varepsilon/3 = \varepsilon/2. \end{aligned}$$

So, g and h are equivalent idempotents and

$$\begin{aligned} \overline{N}(e - \varphi(h)) &\leq \overline{N}(e - \varphi(e')) + N(e' - h) < \varepsilon/2 + \varepsilon/6 < \varepsilon, \\ \overline{N}(f - \varphi(g)) &\leq \overline{N}(f - \varphi(f')) + N(f' - g) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

(b) We note that, by [**5**, Lemma 19.5] we can choose the idempotents e', f' in the proof of (a) to be orthogonal. Since $h \in e'Re'$, $g \in f'Rf'$, h and g are orthogonal and so the result follows.

(c) If p, q are (orthogonal) equivalent projections in $\overline{R}$, then by ((b)) (a) there exist $\{e_n\}$, $\{f_n\}$ with $\varphi(e_n) \to p$, $\varphi(f_n) \to q$, and for all n, e_n and f_n (orthogonal) equivalent idempotents in R. Set $p_n = \mathrm{LP}(e_n)$, $q_n = \mathrm{RP}(f_n)$. As in the proof of Lemma 2.1, we obtain $\varphi(p_n) \to p$ and $\varphi(q_n) \to q$. Also, it is easily shown that, for all n, p_n and q_n are (orthogonal) equivalent projections in R. □

Let R be any *-ring. We say that R satisfies the **-cancellation law for projections* (briefly, R has *-cancellation) if whenever $e \overset{*}{\sim} f$ with e, $f \in P(R)$, we have $1 - e \overset{*}{\sim} 1 - f$. This is equivalent to saying that two *-equivalent projections in R are unitarily equivalent. Also, it is easy to see that if R has *-cancellation and $e, f, g, h \in \mathrm{P}(R)$ are such that e and f are orthogonal, g and h are orthogonal, $e + f \overset{*}{\sim} g + h$ and $f \overset{*}{\sim} h$, then $e \overset{*}{\sim} g$.

Examples of *-regular rings with *-cancellation are the *-regular rings with general comparability for *-equivalence. Also, the *-regular rings with primitive factors artinian and the *-regular self-injective rings of type I satisfy the *-cancellation law. The key to prove this is the following lemma.

LEMMA 2.3. *Let R be any simple artinian ring with proper involution $*$. Then, R satisfies the $*$-cancellation law.*

Proof. We note that R is $*$-regular. Since R is simple artinian, there exist orthogonal equivalent idempotents $e_1, e_2, \ldots, e_n$ such that $e_1 + \cdots + e_n = 1$ and each $e_i R$ is a simple R-module. Since R is $*$-regular, we can assume that $e_1, e_2, \ldots, e_n$ are projections, so that $e_1 R e_1 = D$ is a division ring with involution. Choose $x_i \in e_1 R e_i$, $y_i \in e_i R e_1$, $i = 1, \ldots, n$, such that $x_i y_i = e_1$, $y_i x_i = e_i$ for $i = 1, \ldots, n$. Endow $M_n(D)$ with an involution $\#$ given by $(a_{ij})^{\#} = (b_{ij})$, where $b_{ij} = (x_i x_i^*) a_{ji}^* (y_j^* y_j)$, $i, j = 1, \ldots, n$. The map $R \to M_n(D)$ given by $a \mapsto (x_i a y_j)$ is a $*$-isomorphism from R onto $M_n(D)$ with inverse map $(a_{ij}) \mapsto \sum_{i,\ j=1}^{n} y_i a_{ij} x_j$. Note that $x_i x_i^*$, $y_j^* y_j \in e_1 R e_1 = D$ are such that $(x_i x_i^*)(y_i^* y_i) = (y_i^* y_i)(x_i x_i^*) = e_1 = 1_D$. So, $x_i x_i^* = (y_i^* y_i)^{-1}$ in D. Thus, if we put $t_i = y_i^* y_i$ for $i = i, \ldots, n$ we have $t_i = t_i^*$ and $b_{ij} = t_i^{-1} a_{ji}^* t_j$, where $(a_{ij})^{\#} = (b_{ij})$.

If $x_1, \ldots, x_n$ are in D, and some x_i is nonzero, then, since $\#$ is a proper involution on $M_n(D)$, we have $x_1^* t_1 x_1 + \cdots + x_n^* t_n x_n \neq 0$. Define $\langle\ ,\ \rangle \colon D^n \times D^n \to D$ by

$$\langle a, b \rangle = \langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = a_1^* t_1 b_1 + \cdots + a_n^* t_n b_n.$$

$\langle\ ,\ \rangle$ has the following properties:

(1) $\langle a, b + c \rangle = \langle a, b \rangle + \langle a, c \rangle$,

(2) $\langle a, b \rangle = \langle b, a \rangle^*$,

(3) $\langle a, b\lambda \rangle = \langle a, b \rangle \lambda$,

(4) $\langle a, a \rangle = 0$ iff $a = 0$

for $a, b, c \in D^n$, $\lambda \in D$.

So, $\langle\ ,\ \rangle$ is a nonsingular hermitian form over D^n. It is easy to verify that $\langle Tx, y \rangle = \langle x, T^{\#} y \rangle$ for $T \in M_n(D)$, x, $y \in D^n$, and so isometric spaces in D^n correspond to $*$-equivalent projections in $M_n(D)$. So, the result follows from Witt's theorem for division rings with involution [**12**, pg. 162]. □

PROPOSITION 2.4. *Let R be a $*$-regular ring and assume that either R has all primitive factor rings artinian or R is self-injective of type* I. *Then, R satisfies the $*$-cancellation law.*

Proof. Let R be a $*$-regular ring with all primitive factor rings artinian. By [**5**, Corollary 6.7], all indecomposable factor rings of R are simple artinian. Thus, by Lemma 2.3, they satisfy the $*$-cancellation law. Also, note that we can write the $*$-cancellation law in equational terms. So, we can proceed as in [**5**, Thm. 6.10].

If R is a *-regular, self-injective ring of type I, then $R \cong \prod_{n=1}^{\infty} R_n$, where each R_n is of type I_n and so, R_n has all primitive factor rings artinian. Thus, each R_n satisfies the *-cancellation law and so, also R satisfies the *-cancellation law. □

We note that the *-cancellation law is preserved in direct products and direct limits of *-rings. If R is *-regular and R satisfies the *-cancellation law, then, by Lemma 1.6, R/I has *-cancellation and unitaries in R/I lift to unitaries in R, for every two-sided ideal I of R.

LEMMA 2.5 (*cf.* [3, *Lemma* 6.5]). *Let R be a *-regular ring with *-cancellation and let N be a pseudo-rank function on R. Let e_1, e_2, f_1, $f_2 \in \mathrm{P}(R)$ such that $e_1 \overset{*}{\sim} f_1$, $e_2 \overset{*}{\sim} f_2$ and let u_1 be a unitary such that $f_1 = u_1 e_1 u_1^*$. Then, there exists a unitary u_2 such that $u_2 e_2 u_2^* = f_2$ and $N(u_2 - u_1) \leq 2(N(e_2 - e_1) + N(f_2 - f_1))$.*

Proof. We first observe that if $e, f \in \mathrm{P}(R)$ are such that $eR \cap fR = 0$, then $eR \lesssim (e-f)R$, $fR \lesssim (e-f)R$ and so $N(e) + N(f) \leq 2N(e-f)$. Set $f_3 = u_1 e_2 u_1^*$, and note that $f_3 \overset{*}{\sim} f_2$ and

$$N(f_3 - f_1) = N\big(u_1(e_2 - e_1)u_1^*\big) = N(e_2 - e_1).$$

So,

$$\text{(1)} \quad N(f_3 - f_2) \leq N(f_3 - f_1) + N(f_2 - f_1) = N(e_2 - e_1) + N(f_2 - f_1).$$

We have orthogonal decompositions $f_2 = f_2 \wedge f_3 + f_2'$, $f_3 = f_2 \wedge f_3 + f_3'$, where $f_2', f_3' \in \mathrm{P}(R)$. Note that $f_2'R \cap f_3'R = 0$.

Since R has *-cancellation, $f_2' \overset{*}{\sim} f_3'$. Set $g = f_2' \vee f_3'$. Then, there exists $u_3' \in gRg$ such that $u_3' u_3'^* = u_3'^* u_3' = g$ and $u_3' f_2' u_3'^* = f_3'$. Set $u_3 = u_3' + 1 - g$ and note that $u_3 f_2 u_3^* = f_3$ and $1 - u_3 = (1 - u_3)g = g(1 - u_3)$.

Finally, define $u_2 = u_3^* u_1$. We have $u_2 e_2 u_2^* = u_3^* u_1 e_2 u_1^* u_3 = u_3^* f_3 u_3 = f_2$, and

$$\begin{aligned} N(u_2 - u_1) &= N\big(u_3^* u_1 - u_1\big) = N(1 - u_3) = N((1 - u_3)g) \\ &\leq N(g) = N\big(f_2'\big) + N\big(f_3'\big) \leq 2N\big(f_2' - f_3'\big) \\ &= 2N(f_2 - f_3) \leq 2\big(N(e_2 - e_1) + N(f_2 - f_1)\big). \end{aligned}$$

So, the result follows. □

LEMMA 2.6. *Let R be a *-regular ring with pseudo-rank function N. Let $\overline{R}$ be the N-completion of R and let φ: $R \to \overline{R}$ denote the natural map. If w is a partial isometry in $\overline{R}$, then there exists a sequence $\{w_n\} \subseteq R$ such that*

*$\varphi(w_n) \to w$ and, for all n, w_n is a partial isometry in R. If, in addition, R satisfies the *-cancellation law, then the group of unitaries of R is dense in that of $\overline{R}$. (These groups are endowed with the relative pseudo-rank-metric topology and they are topological groups.)*

Proof. Set $e = ww^* \in P(\overline{R})$. Choose sequences $\{e_n\}$, $\{\alpha_n\}$ such that $e_n \in \mathrm{P}(R)$, $\alpha_n \in R$, for all n and $\varphi(e_n) \to e$, $\varphi(\alpha_n) \to w$. Note that we can assume that $\alpha_n \in e_n R$ for all n. Set $\gamma_n = e_n - \alpha_n\alpha_n^*$. Then, $\varphi(\gamma_n) \to e - ww^* = 0$. Put $e_n' = \mathrm{RP}(\gamma_n) = \mathrm{LP}(\gamma_n)$, all n. Clearly, $\varphi(e_n') \to 0$. Consequently, $e_n'' = e_n - e_n'$ are projections in R and $\varphi(e_n'') \to e$. Now, we note that $0 = e_n''\gamma_n e_n'' = e_n'' - e_n''\alpha_n\alpha_n^* e_n''$. So, $e_n'' = (e_n''\alpha_n)(e_n''\alpha_n)^*$. We deduce that $w_n = e_n''\alpha_n$ are partial isometries such that $\varphi(w_n) \to ew = w$.

Clearly, the group of unitaries of R and that of $\overline{R}$ are topological groups (see [**8**, Prop. 8]). If u is a unitary in $\overline{R}$, then there exists a sequence $\{w_n\}$ such that each w_n is a partial isometry and $\varphi(w_n) \to u$. If R has *-cancellation, then there exist unitaries u_n such that $w_n w_n^* u_n = w_n$ for all n. Since $\varphi(w_n w_n^*) \to 1$, we obtain $\varphi(u_n) \to u$. □

In the next theorem, we show that the *-cancellation law extends from R to $\overline{R}$. This is not new in case $\overline{R}$ is type I, by Proposition 2.4.

THEOREM 2.7. *Let R be a *-regular ring with pseudo-rank function N. Let $\overline{R}$ be the N-completion of R. If R satisfies the *-cancellation law, then so does $\overline{R}$.*

Proof. Let $\varphi\colon R \mapsto \overline{R}$ denote the natural map.

Let e, f be two *-equivalent projections in $\overline{R}$, and let w be a partial isometry in $\overline{R}$ such that $ww^* = e$ and $w^*w = f$. By Lemma 2.6, there exists a sequence $\{w_n\}$ of partial isometries in R such that $\varphi(w_n) \to w$. Set $e_n = w_n w_n^*$ and $f_n = w_n^* w_n$ and note that $e_n, f_n \in \mathrm{P}(R)$ and $\varphi(e_n) \to e$, $\varphi(f_n) \to f$. By passing to subsequences of $\{e_n\}$ and $\{f_n\}$, we can assume that $N(e_{n+1} - e_n) < 2^{-n}$ and $N(f_{n+1} - f_n) < 2^{-n}$. Let u_1 be a unitary in R with $u_1 e_1 u_1^* = f_1$. We construct, by using Lemma 2.5, a sequence of unitaries $\{u_n\}$ in R such that $u_n e_n u_n^* = f_n$ and

$$\begin{aligned} N(u_{n+1} - u_n) &\le 2\big(N(e_{n+1} - e_n) + N(f_{n+1} - f_n)\big) \\ &< 2(2^{-n} + 2^{-n}) = 2^{-n+2}. \end{aligned}$$

It follows that $\{u_n\}$ is a Cauchy sequence. Let $u = \lim_{n\to\infty}\varphi(u_n) \in \overline{R}$. Clearly, $ueu^* = f$ and so, e and f are unitarily equivalent in $\overline{R}$. □

Next, we show the following technical, but useful, result.

THEOREM 2.8. *Let R be a *-regular ring with *-cancellation and let N be a pseudo-rank function on R. Let $\overline{R}$ be its N-completion. Then, $\overline{R}$ satisfies* LP $\overset{*}{\sim}$ RP *if and only if given $\varepsilon > 0$ and equivalent projections e, f in R, there exist subprojections $e' \leq e$, $f' \leq f$ such that $e' \overset{*}{\sim} f'$ and $N(e - e') < \varepsilon$, $N(f - f') < \varepsilon$.*

Proof. Let $\varphi \colon R \mapsto \overline{R}$ denote the natural map.

Assume that $\overline{R}$ satisfies LP $\overset{*}{\sim}$ RP. If e, f are equivalent projections in R, then $\varphi(e) \sim \varphi(f)$ and, since $\overline{R}$ satisfies LP $\overset{*}{\sim}$ RP, we have $\varphi(e) \overset{*}{\sim} \varphi(f)$. Let w be a partial isometry in $\overline{R}$ such that $ww^* = \varphi(e)$ and $w^*w = \varphi(f)$. We observe that, in this situation, we can choose the partial isometries $\{w_n\}$ constructed in the proof of Lemma 2.6 in such a way that $w_n \in eRf$. Set $e_n = w_n w_n^*$, $f_n = w_n^* w_n$. Clearly, $\varphi(e_n) \to \varphi(e)$ and $\varphi(f_n) \to \varphi(f)$, and $e_n \overset{*}{\sim} f_n$ for all n. It follows that $N(e - e_n) \to 0$ and $N(f - f_n) \to 0$. So, given $\varepsilon > 0$, there exist e', f' such that $e' \leq e$, $f' \leq f$, $e' \overset{*}{\sim} f'$ and $N(e - e') < \varepsilon$, $N(f - f') < \varepsilon$.

Conversely, assume that e and f are equivalent projections in $\overline{R}$. By Proposition 2.2, (c), there exist sequences $\{e_n\}$, $\{f_n\}$, with $e_n, f_n \in \mathrm{P}(R)$, $\varphi(e_n) \to e$, $\varphi(f_n) \to f$, and $e_n \sim f_n$ for all n. Thus, by application of our hypothesis with $\varepsilon_n = 2^{-n}$, we have that there exist, for each n, subprojections $e_n' \leq e_n$, $f_n' \leq f_n$ such that $e_n' \overset{*}{\sim} f_n'$, $N(e_n - e_n') < 2^{-n}$ and $N(f_n - f_n') < 2^{-n}$. It follows that $\varphi(e_n') \to e$ and $\varphi(f_n') \to f$. Now, as in the proof of Theorem 2.7, we get a unitary u in $\overline{R}$ such that $ueu^* = f$. In particular, we obtain that $e \overset{*}{\sim} f$. □

So, if R has *-cancellation, then $\overline{R}$ satisfies LP $\overset{*}{\sim}$ RP iff any two equivalent projections e, f in R can be "well approximated" with respect to N by *-equivalent subprojections in R. Since any *-regular unit-regular ring with LP $\overset{*}{\sim}$ RP obviously satisfies the *-cancellation law, we have

THEOREM 2.9. *Let R be a *-regular unit-regular ring with pseudo-rank function N, and let $\overline{R}$ be its N-completion. If R satisfies* LP $\overset{*}{\sim}$ RP, *then so does $\overline{R}$.* □

REMARK. Let R be any regular ring. Denote by $\mathbf{P}(R)$ the set of pseudo-rank functions of R. Define ([**6**]), if $\mathbf{P}(R) \neq \varnothing$, $N^*(x) = \sup\{P(x) \mid P \in \mathbf{P}(R)\}$ and $N^*(x) = 0$ if $\mathbf{P}(R) = \varnothing$. Then, N^* induces a pseudo-metric $\delta(x, y) = N^*(x - y)$ on R and the completion of R with respect to δ, S, is a regular ring, called the N^*-completion of R. If R is *-regular, then S is also *-regular in a natural way. It can be seen that the results of this section also hold for the N^*-completion of a *-regular ring. In particular, the *-cancellation law and, if R is unit-regular, the LP $\overset{*}{\sim}$ RP axiom, extends from R to S.

3. Applications to the study of property LP $\overset{*}{\sim}$ RP **for certain *-regular self-injective rings.** Let R be a *-regular ring with positive definite involution. We assume throughout in this section that $M_n(R)$ is endowed with the *-transpose involution (see §1). We proceed to construct a Grothendieck group for R which is attached to the *-equivalence of projections in the rings $M_n(R)$. We shall call this group $K_0^*(R)$. For to construct it, we follow the construction in [7] for C^*-algebras. Set $P_\infty(R) = \bigcup_{n=1}^{\infty} P(M_n(R))$. For e, $f \in P_\infty(R)$, set $e \oplus f = \left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right) \in P_\infty(R)$. If e, $f \in P_\infty(R)$, then we say that e and f are *-equivalent, $e \overset{*}{\sim} f$, if $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \overset{*}{\sim} \left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in some ring $M_m(R)$, for some suitably-sized zero matrices. Also, define e, $f \in P_\infty(R)$ to be *stably *-equivalent*, written $e \overset{*}{\approx} f$, provided $e \oplus g \overset{*}{\sim} f \oplus g$ for some $g \in P_\infty(R)$. Let $P_\infty(R)/\overset{*}{\approx}$ denote the family of all the equivalence classes defined by $\overset{*}{\approx}$ (which is clearly an equivalence relation). For $e \in P_\infty(R)$, we use $[e]_*$ to denote the equivalence class of e with respect to $\overset{*}{\approx}$. It follows easily that $P_\infty(R)/\overset{*}{\approx}$, with the operation $[e]_* + [f]_* = [e \oplus f]_*$, is an abelian semigroup with cancellation. So, we may formally adjoin inverses to $P_\infty(R)/\overset{*}{\approx}$, obtaining an abelian group, denoted by $K_0^*(R)$.

Recall that, if we use in the above construction equivalence instead of *-equivalence, we obtain the group $K_0(R)$, which can also be defined by using finitely generated projective modules over R (see [**5**, Chapter 15]).

We have a map Φ: $K_0^*(R) \to K_0(R)$ given by $\Phi([e]_*) = [e]$ where $[e]$ denotes the corresponding equivalence class of e in $K_0(R)$. This map is clearly a group homomorphism from $K_0^*(R)$ onto $K_0(R)$.

Define a cone C in $K_0^*(R)$ by $C = K_0^*(R)^+ = \{[e]_* \mid e \in P_\infty(R)\}$. It follows from [**1**, Thm. 3.1, (b)] that $(K_0^*(R), [1]_*)$ is a partially ordered group with order unit ([5, pg. 203]) for any *-regular ring R with positive definite involution. Also, we may view Φ: $(K_0^*(R), [1]_*) \to (K_0(R), [1])$ as a morphism in the category $\mathscr{P}$ defined in [**5**, pg. 203].

Now, we study $K_0^*(F)$, where F is any *-field with positive definite involution. In this case, $K_0^*(F)$ and $K_0(F)$ admit in a natural way a structure of ring, where the product is induced by the tensor product. Recall that $M_n(F) \otimes M_m(F) \cong M_{nm}(F)$ and the usual isomorphism is in fact a *-isomorphism of *-algebras, if we define $(x \otimes y)^* = x^* \otimes y^*$ for $x \in M_n(F)$ and $y \in M_m(F)$. Also, note that $K_0(F) \cong \mathbf{Z}$, and so Φ: $K_0^*(F) \mapsto K_0(F)$ induces a ring map r: $K_0^*(F) \to \mathbf{Z}$ given by $r([e]_* - [f]_*) = \operatorname{rank}(e) - \operatorname{rank}(f)$. If we set $K = \operatorname{Ker}(r)$, we have an exact sequence of groups

$$0 \to K \to K_0^*(F) \to \mathbf{Z} \to 0$$

Hence, $K_0^*(F) \cong \mathbf{Z} \oplus K$ as abelian groups. In fact, $K_0^*(F)$ is the ring generated by $[1]_*$ and K. Since K is an ideal of $K_0^*(F)$, this is the unitification of the (non unital) ring K.

We now relate $K_0^*(F)$ with the Witt ring of F, $W(F)$. The construction of $W(F)$ can be found in [**15**]. There are no extra difficulties in constructing $W(F)$ using hermitian forms instead of symmetric bilinear forms. We now fix some notation.

For any *-field F, an hermitian form over F is a map Φ: $V \times V \to F$, where V is a finite-dimensional vector space over F, such that

(1) $\Phi(e_1 + e_2, v) = \Phi(e_1, v) + \Phi(e_2, v)$,

(2) $\Phi(\lambda e, v) = \lambda\Phi(e, v)$ for $\lambda \in F$,

(3) $\Phi(e, v) = \Phi(v, e)^*$.

Let F_s denote the fixed field of F, that is $F_s = \{x \in F \mid x = x^*\}$. For $a \in V$, we note that $\Phi(a, a) \in F_s$. We define $D_F(\Phi) = \{\lambda \in \dot{F} \mid \lambda = \Phi(a, a)$ for some $a \in V\} \subseteq \dot{F}_s$.

Each hermitian form Φ is isometric to a form $\langle a_1, \dots, a_n\rangle$, with $a_1, \dots, a_n \in D_F(\Phi)$, where $\langle a_1, \dots, a_n\rangle$ denotes the hermitian form ψ: $F^n \times F^n \to F$ defined by $\psi((x_1, \dots, x_n), (y_1, \dots, y_n)) = a_1x_1y_1^* + \cdots + a_nx_ny_n^*$.

If $\operatorname{ch}(F) \neq 2$, then we construct $W(F)$ as in [**15**, Chapter 2] using hermitian forms instead of symmetric bilinear forms. Recall [**15**, Prop. II.1.4] that

(1) The elements of $W(F)$ are in one-one correspondence with the isometry classes of all anisotropic hermitian forms.

(2) Two nonsingular hermitian forms Φ, Φ' represent the same element in $W(F)$ iff the anisotropic part of Φ, Φ_a, is isometric to the anisotropic part of Φ', Φ'_a; in symbols, $\Phi_a \simeq \Phi'_a$.

(3) If $\dim\Phi = \dim\Phi'$ (where Φ, Φ' are nonsingular) then Φ and Φ' represent the same element in $W(F)$ iff $\Phi \simeq \Phi'$.

We now return to the case where * is positive definite. For $e \in P(M_n(F))$, we have an hermitian form associated $H(e) = (e(F^n), h_e)$, where h_e is the restriction to $e(F^n)$ of the hermitian form $\langle x, y\rangle = x_1y_1^* + \cdots + x_ny_n^*$ over F^n. Set $-H(e) = (e(F^n), -h_e)$; and note that $\{-H(e)\} = -\{H(e)\}$, where $\{\Phi\}$ denotes the class of Φ in $W(F)$.

PROPOSITION 3.1. (a) *There exists an injective ring map* $\varphi\colon K_0^*(F) \mapsto W(F)$ *such that* $\varphi([e]_* - [f]_*) = \{H(e) \oplus (-H(f))\}$, *for* $e, f \in P_\infty(F)$.

(b) *The hermitian form* $H(e) \oplus (-H(f))$ *is isotropic if and only if there exist nonzero subprojections* $e' \le e$, $f' \le f$ *such that* $e' \overset{*}{\sim} f'$ *in* $P_\infty(F)$.

Proof. Define $\varphi'\colon K_0^*(F)^+ \to W(F)$ by $\varphi'([e]_*) = \{H(e)\}$. We show that φ' is well-defined, $\varphi'([e]_* + [f]_*) = \varphi'([e]_*) + \varphi'([f]_*)$ and $\varphi'([e]_* \cdot [f]_*) = \varphi'([e]_*) \cdot \varphi'([f]_*)$, for $e, f \in P_\infty(F)$. For, assume that $[e]_* = [f]_*$, with $e \in M_n(F)$, $f \in M_m(F)$. There exist $g \in P_\infty(F)$ and suitably-sized zero matrices such that

$$\begin{pmatrix} e & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{*}{\sim} \begin{pmatrix} f & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in some ring $M_k(F)$. By Lemma 2.3, $M_k(F)$ has *-cancellation, so $\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \overset{*}{\sim} \left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $M_k(F)$. It follows easily that $(e(F^n), h_e)$ is isometric to $(f(F^m), h_f)$. So, $\{H(e)\} = \{H(f)\}$ and φ' is well-defined. If e, $f \in P_\infty(F)$, then

$$\begin{aligned} \varphi'([e]_* + [f]_*) &= \varphi'([e \oplus f]_*) = \{H(e \oplus f)\} \\ &= \left\{\left((e \oplus f)(F^{n+m}), h_{e\oplus f}\right)\right\} = \left\{\left(e(F^n), h_e\right)\right\} + \left\{\left(f(F^m), h_f\right)\right\} \\ &= \{H(e)\} + \{H(f)\} = \varphi'([e]_*) + \varphi'([f]_*). \end{aligned}$$

Since the products in $K_0(F)$ and in $W(F)$ are both induced by the tensor product, we obtain similarly $\varphi'([e]_* \cdot [f]_*) = \varphi'([e]_*) \cdot \varphi'([f]_*)$.

From this, we deduce that we can define $\varphi\colon K_0^*(F) \to W(F)$ such that $\varphi([e]_* - [f]_*) = \varphi([e]_*) - \varphi([f]_*)$. So,

$$\begin{aligned} \varphi([e]_* - [f]_*) &= \{H(e)\} - \{H(f)\} = \{H(e)\} + \{-H(f)\} \\ &= \{H(e) \oplus (-H(f))\}. \end{aligned}$$

We note that, since the involution on F is positive definite, $H(e)$ is anisotropic for every $e \in P_\infty(F)$.

Suppose that $\varphi([e]_* - [f]_*) = 0$. Then, $\{H(e)\} = \{H(f)\}$ and so, $H(e) = H(e)_a \simeq H(f)_a = H(f)$. It follows that $e \overset{*}{\sim} f$ in $P_\infty(F)$ and so, $[e]_* = [f]_*$.

(b) Assume that $H(e) \oplus (-H(f))$ is isotropic. Then, there exist nonzero vectors $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_m)$ such that $u \in e(F^n)$, $v \in f(F^m)$ and $u_1 u_1^* + \cdots + u_n u_n^* = v_1 v_1^* + \cdots + v_m v_m^*$. We infer that there exist (nonzero) subprojections $e' \le e$ and $f' \le f$ with $e'(F^n) = uF$ and $f'(F^m) = vF$. It follows that $e' \overset{*}{\sim} f'$.

Conversely, assume that $e' \leq e$, $f' \leq f$ are nonzero *-equivalent projections. Then, $H(e')$ and $H(f')$ are nonzero isometric subspaces of $H(e)$ and $H(f)$ respectively. So, $H(e) \oplus (-H(f))$ is isotropic. □

We define $D_F(m) = D(m\langle 1\rangle)$ and $D_F(\infty) = \bigcup_{m=1}^{\infty} D_F(m)$. Let $W_t(F)$ denote the subgroup of additive torsion of $W(F)$. Clearly, $W_t(F)$ is an ideal and by [**15**, Corollary XI.3.2], $W_t(F)$ is a 2-primary group. If $w \in D_F(\infty)$, let 2^n be the smallest power of 2 for which $w \in D_F(2^n)$. Then, by [**15**, Prop. XI.1.3], the additive order of the form $\langle 1, -w\rangle$ is precisely 2^n. So, $\langle 1, -w\rangle \in W_t(F)$ if $w \in D_F(\infty)$ and, by [**15**, Prop. XI.3.3 and supplement], $W_t(F)$ coincides with the ideal generated by these elements.

PROPOSITION 3.2. *Let K be the kernel of the map r: $K_0^*(F) \mapsto \mathbf{Z}$ given by $r([e]_* - [f]_*) = \operatorname{rank}(e) - \operatorname{rank}(f)$ and let φ: $K_0^*(F) \mapsto W(F)$ be the map defined in Proposition* 3.1. *Then, $\varphi(K) \subseteq W_t(F)$ and so, K is a 2-primary group. Moreover, $\varphi(K) = \tilde{W}_t(F)$, where $\tilde{W}_t(F)$ is the* (*non unital*) *subring of $W(F)$ generated by $\{\langle 1, -w\rangle \mid w \in D_F(\infty)\}$ and $K_0^*(F)$ is ring isomorphic, via φ, to the unitification of $\tilde{W}_t(F)$.*

Proof. We first observe that K is generated by the elements $[1]_* - [e]_*$, where $e \in P_\infty(F)$ is of rank 1. If $e \in M_n(F)$, then we deduce that $\varphi([1]_* - [e]_*) = \{\langle 1, -w\rangle\}$, where $w \in D_F(n)$. Thus, clearly $\varphi(K) = \tilde{W}_t(F)$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, hook] & K_0^*(F) \arrow[r, "r"] \arrow[d, hook, "\varphi"] & \mathbf{Z} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & W_t(F) \arrow[r] & W(F) \arrow[r] & W(F)/W_t(F) \arrow[r] & 0
\end{tikzcd}$$

So, $K_0^*(F) = \mathbf{Z} \oplus K \xrightarrow{\sim} \mathbf{Z} \oplus \tilde{W}_t(F) \subseteq W(F)$ and clearly $K_0^*(F)$ is ring isomorphic to the unitification of $\tilde{W}_t(F)$. □

If $D_F(\infty)$ induces a total ordering on F, that is, if $F = D_F(\infty) \cup \{0\} \cup (-D_F(\infty))$, then $K_0^*(F) \cong W(F)$. On the other hand, if F is *-Pythagorean, then $W_t(F) = \tilde{W}_t(F) = 0$ and $K_0^*(F) \cong \mathbf{Z}$.

DEFINITIONS. Let $(F, *)$ be a field with positive definite involution. A *-algebra A over F is said to be *matricial* if A is isomorphic as *-algebra to $M_{n(1)}(F) \times \cdots \times M_{n(r)}(F)$ for some positive integers $n(1), \ldots, n(r)$. The *-algebra is *ultramatricial* if A contains a sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of matricial *-algebras such that $\bigcup_{n=1}^{\infty} A_n = A$.

In [7, Prop. 16.1], it is shown that a *-algebra A is ultramatricial iff A is isomorphic as *-algebra to a direct limit (in the category of *-algebras) of a sequence of matricial *-algebras and *-algebra maps.

The *-algebra A is *standard matricial* if $A = M_{n(1)}(F) \times \cdots \times M_{n(r)}(F)$ for some positive integers $n(1), \dots, n(r)$; (see [7, Chapter 17]).

If $A = M_{n(1)}(F) \times \cdots \times M_{n(k)}(F)$ and $B = M_{m(1)}(F) \times \cdots \times M_{m(l)}(F)$ are standard matricial *-algebras, then a *standard map* from A to B is any map which sends the element $(a_1, \dots, a_k)$ of A to

$$\left(\begin{bmatrix} a_1 & & & & & & \\ & \overset{s_{11}}{\ddots} & & & & & \\ & & a_1 & & & & \\ & & & \ddots & & & \\ & & & & a_k & & \\ & & & & & \overset{s_{1k}}{\ddots} & \\ & & & & & & a_k \end{bmatrix}, \dots, \begin{bmatrix} a_1 & & & & & & \\ & \overset{s_{l1}}{\ddots} & & & & & \\ & & a_1 & & & & \\ & & & \ddots & & & \\ & & & & a_k & & \\ & & & & & \overset{s_{lk}}{\ddots} & \\ & & & & & & a_k \end{bmatrix}\right)$$

where s_{ij} are nonnegative integers such that $s_{i1}n(1) + \cdots + s_{ik}n(k) = m(i)$ for all i. Clearly any standard map is a *-algebra map. We observe that the maps we obtain by iterated composition of standard ones are precisely the "block diagonal" maps.

A *standard ultramatricial* *-algebra is a direct limit of a sequence $A_1 \xrightarrow{\Phi_1} A_2 \xrightarrow{\Phi_2} A_3 \xrightarrow{\Phi_3} \cdots$ of standard matricial *-algebras A_n and standard maps $\Phi_n \colon A_n \to A_{n+1}$.

PROPOSITION 3.3. *If F is *-Pythagorean then every ultramatricial *-algebra over F is isomorphic as *-algebra to a standard ultramatricial *-algebra. Moreover, if A and B are ultramatricial *-algebras over F, then A and B are isomorphic as rings if and only if they are isomorphic as *-algebras.*

Proof. We know that property LP $\overset{*}{\sim}$ RP holds in $M_n(F)$ for all n. So we can adapt the proofs of [7, Prop. 17.2] and [7, Thm. 20.6]. □

We do not know if Proposition 3.3 remains true for arbitrary fields with positive definite involution. By using [5, Thm. 15.26] one can show that any ultramatricial algebra over a field F is isomorphic as F-algebra to a standard ultramatricial algebra.

Now we proceed to study completions of direct limits of direct systems of standard matricial *-algebras and standard maps with respect to a pseudo-rank function. We need a lemma which gives a characterization of those pseudo-rank functions N on a regular ring R such that the N-completion of R is type II.

LEMMA 3.4. *Let R be a regular ring with pseudo-rank function N and let $\overline{R}$ be its N-completion. Then, $\overline{R}$ is type* II *if and only if for each idempotent e in R, for each $\varepsilon > 0$, and for each $m \geq 1$ there exist equivalent orthogonal idempotents $e_1, e_2, \ldots, e_m \in R$ such that $e_i e = e e_i = e_i$ for all i, and $N(e - (e_1 + \cdots + e_m)) < \varepsilon$.*

Proof. Let $\varphi\colon R \to \overline{R}$ denote the natural map.

Assume that for each idempotent $e \in R$, $\varepsilon > 0$, and $m \geq 1$, there exist equivalent orthogonal idempotents $e_1, \ldots, e_m$ such that $ee_i = e_i e = e_i$ for all i, and $N(e - (e_1 + \cdots + e_m)) < \varepsilon$. If $\overline{R}$ is not type II then there exists a central idempotent $h \in \overline{R}$ such that $h \neq 0$ and $h\overline{R}$ is type I_n for some $n \geq 1$. Set $\varepsilon = \overline{N}(h)$, where $\overline{N}$ denotes the natural extension of N to $\overline{R}$. There exist equivalent orthogonal idempotents $e_1, e_2, \ldots, e_{n+1} \in R$ such that $N(1 - (e_1 + \cdots + e_{n+1})) < \varepsilon$. We observe that $h\varphi(e_1), \ldots, h\varphi(e_{n+1})$ are equivalent orthogonal idempotents of $\overline{R}$. We have

$$\begin{aligned} &\overline{N}(h(1 - (\varphi(e_1) + \cdots + \varphi(e_{n+1})))) \\ &\qquad \leq N(1 - (e_1 + \cdots + e_{n+1})) < \varepsilon = \overline{N}(h). \end{aligned}$$

In particular $h(\varphi(e_1) + \cdots + \varphi(e_{n+1})) \neq 0$. So $h\varphi(e_1), \ldots, h\varphi(e_{n+1})$ are nonzero equivalent orthogonal idempotents in $h\overline{R}$. This contradicts [**5**, Thm. 7.2] and consequently we deduce that $\overline{R}$ is type II.

Conversely, assume that $\overline{R}$ is type II. First we show that for each $e \in R$, for each $\varepsilon > 0$, and for each $n \geq 1$, there exist 2^n equivalent orthogonal idempotents $e_1, e_2, \ldots, e_{2^n} \in R$ such that $ee_i = e_i e = e_i$ for all i, and $N(e - (e_1 + \cdots + e_{2^n})) < \varepsilon$. We proceed by induction on n. Set $n = 1$. If $N(e) = 0$ then the result is trivial. So assume that $N(e) \neq 0$ and consider the pseudo-rank function N' on eRe defined by $N'(z) = N(z)/N(e)$ for $z \in eRe$. Then the N'-completion of eRe is precisely $\varphi(e)\overline{R}\varphi(e)$ which is also type II. So we can assume without loss of generality that $e = 1$. Since $\overline{R}$ is type II it follows from [**5**, Prop. 10.28] that there exist equivalent orthogonal idempotents $g_1, g_2 \in \overline{R}$ such that $1 = g_1 + g_2$. By Proposition 2.2, (b) we can choose sequences $\{g_{1r}\}$, $\{g_{2r}\}$ such that, for each r, g_{1r} and g_{2r} are equivalent orthogonal idempotents in R and $\varphi(g_{1r}) \to g_1$, $\varphi(g_{2r}) \to g_2$. Consequently there exist equivalent orthogonal idempotents $e_1, e_2 \in R$ such that $\overline{N}(g_1 - \varphi(e_1)) < \varepsilon/2$ and $\overline{N}(g_2 - \varphi(e_2) < \varepsilon/2$. Hence

$$N(1 - (e_1 + e_2)) \leq \overline{N}(g_1 - \varphi(e_1)) + \overline{N}(g_2 - \varphi(e_2)) < \varepsilon.$$

Now assume that the result is true for $1 \leq k < n$ with $n \geq 2$. Taking $k = 1$ we see that there exist equivalent orthogonal idempotents e_1', $e_2' \in R$ such that $e_1' + e_2' \leq e$ and $N(e - (e_1' + e_2')) < \varepsilon/3$. Taking now $k = n - 1$ we obtain 2^{n-1} equivalent orthogonal idempotents $e_1, \dots, e_{2^{n-1}} \in R$ such that $e_1 + \cdots + e_{2^{n-1}} \leq e_1'$ and $N(e_1' - (e_1 + \cdots + e_{2^{n-1}})) < \varepsilon/3$. Since $e_1' \sim e_2'$ there exist equivalent orthogonal idempotents $e_{2^{n-1}+1}, \dots, e_{2^n} \in R$ such that $e_{2^{n-1}+1} + \cdots + e_{2^n} \leq e_2'$ and $e_1 \sim e_{2^{n-1}+1} \sim \cdots \sim e_{2^n}$. We have

$$
\begin{aligned}
N&\big(e_2' - (e_{2^{n-1}+1} + \cdots + e_{2^n})\big) \\
&= N(e_2') - N(e_{2^{n-1}+1}) - \cdots - N(e_{2^n}) \\
&= N(e_1') - N(e_1) - \cdots - N(e_{2^{n-1}}) < \varepsilon/3.
\end{aligned}
$$

So, $e_1, \dots, e_{2^n}$ are 2^n equivalent orthogonal idempotents such that $e_1 + \cdots + e_{2^n} \leq e$ and

$$
\begin{aligned}
N(e - (e_1 + \cdots + e_{2^n})) \leq\, & N\big(e - (e_1' + e_2')\big) \\
& + N\big(e_1' - (e_1 + \cdots + e_{2^{n-1}})\big) \\
& + N\big(e_2' - (e_{2^{n-1}+1} + \cdots + e_{2^n})\big) < \varepsilon.
\end{aligned}
$$

Now let $e \in R$ be an idempotent and let $\varepsilon > 0$, $m \geq 1$. Choose $n \geq 1$ such that $m/2^n < \varepsilon/2$ and put $2^n = mr + k$ where $r \geq 0$ and $0 \leq k < m$. As we have seen there exist equivalent orthogonal idempotents $e_1', \dots, e_{2^n}' \in R$ such that $e_i'e = ee_i' = e_i'$ for all i, and $N(e - (e_1' + \cdots + e_{2^n}')) < \varepsilon/2$. Observe that $N(e_i') \leq 2^{-n}$ for all i. Define $e_i = e_{(i-1)r+1}' + \cdots + e_{ir}'$ for $i = 1, \dots, m$. Then $e_1, \dots, e_m$ are equivalent orthogonal idempotents of R such that $e_i e = ee_i = e_i$ all i. Moreover we have

$$
\begin{aligned}
N(e - (e_1 + \cdots + e_m)) &= N\big(e - (e_1' + \cdots + e_{mr}')\big) \\
&\leq N\big(e - (e_1' + \cdots + e_{2^n}')\big) + N\big(e_{mr+1}' + \cdots + e_{2^n}'\big) \\
&< \varepsilon/2 + kN(e_{2^n}') \\
&\leq \varepsilon/2 + m/2^n < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

Hence $N(e - (e_1 + \cdots + e_{2^n})) < \varepsilon$ as desired. □

THEOREM 3.5. *Let F be a $*$-field with positive definite involution. Let $\{R_i, \Phi_{ji}\}_{i,j \in I}$ be a direct system such that, for every $i \in I$, R_i is a standard matricial $*$-algebra over F and, if $i \leq j$, Φ_{ji}: $R_i \to R_j$ is a composition of standard maps. Let R be the direct limit of $\{R_i, \Phi_{ji}\}$ and let N be a pseudo-rank function on R. Then the type* II *part of the N-completion of R satisfies* LP $\stackrel{*}{\sim}$ RP *matricially.*

Proof. It suffices to see that the type II part of the N-completion of R satisfies LP $\overset{*}{\sim}$ RP.

Let $\overline{R}$ be the N-completion of R and let φ: $R \to \overline{R}$ denote the natural map. There exists a unique decomposition $\overline{R} = R_1 \times R_2$ where R_1 is type I and R_2 is type II. Let $\overline{N}$ be the natural extension of N to $\overline{R}$, and note that $\overline{N}$ is a rank function on $\overline{R}$. If R_1 and R_2 are nonzero, then there exists a central projection $h \neq 0, 1$ such that $h\overline{R} = R_1$ and $(1-h)\overline{R} = R_2$. By [5, Prop. 16.4] there exist unique rank functions N_1', N_2' on R_1, R_2 such that

$$\overline{N}(x) = \overline{N}(h)N_1'(hx) + \overline{N}(1-h)N_2'((1-h)x)$$

for all $x \in \overline{R}$. For $y \in R$, define $N_2(y) = N_2'((1-h)\varphi(y))$. Then, it is easily seen that N_2 is a pseudo-rank function on R. Also, one can see that the map ψ: $R \to R_2$ defined by $\psi(y) = (1-h)\varphi(y)$ is the natural map from R to its N_2-completion, so that the completion of (R, N_2) is precisely (R_2, N_2').

If $R_2 = 0$, there is nothing to prove. If $R_2 \neq 0$, then we see from the above discussion that R_2 is the completion of R with respect to a certain pseudo-rank function on R. So, we can assume without loss of generality that $\overline{R}$ is of type II.

Since each R_i has *-cancellation, so does R. Thus, by Theorem 2.8, it suffices to prove that given $\varepsilon > 0$ and equivalent projections e, f in R, there exist subprojections $e' \leq e$, $f' \leq f$ such that $e' \overset{*}{\sim} f'$ and $N(e-e') < \varepsilon$, $N(f-f') < \varepsilon$. For $i \in I$, let θ_i: $R_i \mapsto R$ be the natural map from R_i to the direct limit. There exist $i \in I$ and projections g, h in R_i such that $\theta_i(g) = e$, $\theta_i(h) = f$ and $g \sim h$ in R_i. Since R_i is a standard matricial *-algebra, there exist some positive integers $c(1), \ldots, c(n)$ such that $R_i = M_{c(1)}(F) \times \cdots \times M_{c(n)}(F)$. Clearly, we may assume without loss of generality that $g = (0, \ldots, 0, g', 0, \ldots, 0)$ and $h = (0, \ldots, 0, h', 0, \ldots, 0)$ where g' and h' are projections of rank one in some ring $M_{c(\alpha)}(F)$ for some $1 \leq \alpha \leq n$.

Let k be the additive order of $[g']_* - [h']_*$ in $K_0^*(F)$. By Proposition 3.2, k is a power of 2. Moreover, since $M_n(F)$ has *-cancellation for all n, we have

$$\begin{bmatrix} g' & \\ & \underbrace{\begin{matrix} g' & & \\ & \ddots & \\ & & g' \end{matrix}}_{k} \end{bmatrix} \overset{*}{\sim} \begin{bmatrix} h' & \\ & \underbrace{\begin{matrix} h' & & \\ & \ddots & \\ & & h' \end{matrix}}_{k} \end{bmatrix}.$$

Let l be a positive integer with $1/l < \varepsilon/2$, and set $m = kl$. By Lemma 3.4 (and a standard argument) there exist m orthogonal equivalent projections $e_1, \dots, e_m$ in R such that $e_1 + \cdots + e_m \leq e$ and $N(e - (e_1 + \cdots + e_m)) < \varepsilon/2$. Now, there exist $j \in I$ such that $j \geq i$ and m orthogonal equivalent projections $g_1, \dots, g_m$ in R_j such that $g_p \leq \Phi_{ji}(g)$ and $\theta_j(g_p) = e_p$ for $p = 1, \dots, m$. There exist positive integers $d(1), \dots, d(r)$ such that $R_j = M_{d(1)}(F) \times \cdots \times M_{d(r)}(F)$. Set $g_p = (g_{p1}, \dots, g_{pr})$ for $p = 1, \dots, m$, and note that, for each $q = 1, \dots, r$, $g_{1q}, \dots, g_{mq}$ are m orthogonal equivalent projections in $M_{d(q)}(F)$. Without loss of generality, we can assume that $g_{11}, \dots, g_{1r'} \neq 0$ and $g_{1r'+1} = \cdots = g_{1r} = 0$. Set $\Phi_{ji}(g) = (e'_1, \dots, e'_r)$. We note that

$$\begin{aligned} N\big(\theta_j\big((0, \dots, 0, e'_{r'+1}, \dots, e'_r)\big)\big) &\leq N\big(\theta_j\big(\Phi_{ji}(g) - (g_1 + \cdots + g_m)\big)\big) \\ &= N\big(e - (e_1 + \cdots + e_m)\big) < \varepsilon/2. \end{aligned}$$

Since Φ_{ji} is a composition of standard maps, each e'_q has the form

$$\begin{bmatrix} 0 & & & & \\ & g' & & & \\ & & 0 & & \\ & & & g' & \\ & & & & \ddots \end{bmatrix}$$

for suitably-sized zero matrices.

Since $g_{1q} + \cdots + g_{mq} \leq e'_q$ for $q = 1, \dots, r$, we have $\operatorname{rank}(e'_q) \geq m$ for $q = 1, \dots, r'$. If we put $\Phi_{ji}(h) = (f'_1, \dots, f'_r)$ we see that $\operatorname{rank}(f'_q) \geq m$ for $q = 1, \dots, r'$.

For $q = 1, \dots, r'$, set $t(q) = \operatorname{rank}(e'_q)$ and note that $t(q)$ is precisely the number of copies of g' that appear in the expression of e'_q. Put $t(q) = s(q)k + t'(q)$ with $0 \leq t'(q) < k$. We observe that $m \leq s(q)k$. For each $q = 1, \dots, r'$, let e''_q be the projection of $M_{d(q)}(F)$ which has $s(q)k$ g'-blocks in the same places as the first $s(q)k$ g'-blocks of e'_q and zeroes elsewhere, that is

$$e''_q = \begin{bmatrix} 0 & & & & & & & \\ & g' & & & & & & \\ & & 0 & & & & & \\ & & & \ddots & & & & \\ & & & & g' & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}.$$

For $q = 1, \ldots, r'$, let f''_q be the projection of $M_{d(q)}(F)$ formed in the same way as e''_q but with h' instead of g'.

Set $e' = \theta_j((e''_1, \ldots, e''_r, 0, \ldots, 0))$, $f' = \theta_j((f''_1, \ldots, f''_r, 0, \ldots, 0))$. Clearly, $e' \leq e$ and $f' \leq f$. Since $e''_q \overset{*}{\sim} f''_q$ for $q = 1, \ldots, r'$, we have $e' \overset{*}{\sim} f'$.

Set $N_j = N\theta_j$. Then, N_j is a pseudo-rank function on R_j and by [**5**, Corollary 16.6], we have that there exist nonnegative real numbers $\alpha_1, \ldots, \alpha_r$ with $\alpha_1 + \cdots + \alpha_r = 1$ such that

$$N_j((x_1, \ldots, x_r)) = \alpha_1 \operatorname{rank}(x_1)/d(1) + \cdots + \alpha_r \operatorname{rank}(x_r)/d(r).$$

For $q = 1, \ldots, r'$ we have

$$\begin{aligned}\operatorname{rank}\left(e'_q - e''_q\right)/d(q) &= t'(q)/d(q)\\ &\leq t'(q)/m < k/m = k/(kl) = 1/l < \varepsilon/2.\end{aligned}$$

Finally,

$$\begin{aligned}N(e - e') &= N\left(\theta_j\left(e'_1 - e''_1, \ldots, e'_{r'} - e''_{r'}, e'_{r'+1}, \ldots, e'_r\right)\right)\\ &\leq N_j\left(\left(e'_1 - e''_1, \ldots, e'_{r'} - e''_{r'}, 0, \ldots, 0\right)\right)\\ &\quad + N_j\left(\left(0, \ldots, 0, e'_{r'+1}, \ldots, e'_r\right)\right)\\ &< N_j\left(\left(e'_1 - e''_1, \ldots, e'_{r'} - e''_{r'}, 0, \ldots, 0\right)\right) + \varepsilon/2\\ &= \alpha_1 \operatorname{rank}\left(e'_1 - e''_1\right)/d(1)\\ &\quad + \cdots + \alpha_{r'} \operatorname{rank}\left(e'_{r'} - e''_{r'}\right)/d(r') + \varepsilon/2\\ &< (\alpha_1 + \cdots + \alpha_{r'})\varepsilon/2 + \varepsilon/2 \leq \varepsilon.\end{aligned}$$

Similarly, $N(f - f') < \varepsilon$. So, the proof is complete. □

As a consequence of Theorem 3.5, we see that if F is any *-field with positive definite involution, then there exists a simple, *-regular, self-injective ring of type II satisfying LP $\overset{*}{\sim}$ RP whose center is F. For example, let $n(1) < n(2) < \cdots$ be positive integers such that $n(k) \mid n(k+1)$ for all k, and set $S = \lim M_{n(k)}(F)$ (with respect to the obvious standard maps). Let R be the completion of S with respect to the unique rank function on S. Then, R is a simple, *-regular, self-injective ring of type II whose center is F ([**4**, Thm. 2.8]). By Theorem 3.5, R satisfies LP $\overset{*}{\sim}$ RP matricially.

Next, we shall construct a simple, *-regular, self-injective ring of type II which does not satisfy LP $\overset{*}{\sim}$ RP. In [**9**, pg. 31, Example 1] Handelman tries to offer an example of a simple, *-regular, type II self-injective ring R which does not satisfy LP $\overset{*}{\sim}$ RP and a Baer *-subring S of R which contains all the partial isometries of R and does not satisfy neither LP $\overset{*}{\sim}$ RP nor the (EP)-axiom. The ring R constructed by Handelman is the completion of $\lim M_{2^n}(\mathbf{Q}(x))$ with respect to its unique rank function. So, it follows from Theorem 3.5 that R satisfies LP $\overset{*}{\sim}$ RP and therefore, also the Baer *-subring S has LP $\overset{*}{\sim}$ RP. It is true, however, that they do not satisfy the (SR)-axiom of [**2**, pg. 66].

EXAMPLE 3.6. *There exists a simple, *-regular, self-injective ring of type* II *which does not satisfy* LP $\overset{*}{\sim}$ RP.

Proof. Let F be a formally real field such that $D_F(1) \subsetneq D_F(2) \subsetneq \cdots$ (for example we can take $F = \mathbf{R}(x_1, x_2, \dots)$, [**15**, Exercise 6, pg. 315]). Set $S = \prod_{n=1}^{\infty} M_{2^n}(F)$. Let M be a maximal two-sided ideal of S which contains the direct sum $\oplus_{n=1}^{\infty} M_{2^n}(F)$. Set $R = S/M$. By [**5**, Thm. 10.30] R is a simple, regular, right and left self-injective ring of type II. Clearly, both R and S are *-regular rings (here, the involution on F is the identity). For $n \geq 1$, choose $w_n \in D_F(2^n) - D_F(2^{n-1})$. From Propositions 3.1 and 3.2, we see that there exist rank one projections $f_{n,i} \in M_{2^n}(F)$, $i = 1, \dots, 2^n$ such that for each n, $f_{n,i}$ are 2^n orthogonal *-equivalent projections adding to the identity in $M_{2^n}(F)$, that is $f_{n,1} + \cdots + f_{n,2^n} = 1_{2^n}$, and $\varphi([f_{n,i}]_*) = \{\langle w_n \rangle\}$ for $i = 1, \dots, 2^n$. Set

$$g_{n,1} = f_{n,1} + \cdots + f_{n,2^{n-1}}; \quad g_{n,2} = f_{n,2^{n-1}+1} + \cdots + f_{n,2^n};$$

$$h_{n,1} = \operatorname{diag}\Big(\overbrace{1, \dots, 1}^{2^{n-1}}, 0, \dots, 0\Big); \quad h_{n,2} = \operatorname{diag}\Big(0, \dots, 0, \overbrace{1, \dots, 1}^{2^{n-1}}\Big).$$

From [**15**, Corollary X.1.6] and 3.1 (b) we deduce that for each n, $g_{n,1}$ and $h_{n,1}$ does not have nonzero *-equivalent subprojections. Set $g_1 = (g_{1,1}, g_{2,1}, \dots)$; $g_2 = (g_{1,2}, g_{2,2}, \dots)$; $h_1 = (h_{1,1}, h_{2,1}, \dots)$; $h_2 = (h_{1,2}, h_{2,2}, \dots)$. We have $g_1 \overset{*}{\sim} g_2$, $h_1 \overset{*}{\sim} h_2$ and $g_1 + g_2 = h_1 + h_2 = 1$. Note that $g_1 \sim h_1$ and $g_2 \sim h_2$ in S. So, in R we have $\bar{g}_1 \sim \bar{h}_1$ and $\bar{g}_2 \sim \bar{h}_2$. Clearly, $\bar{g}_1$, $\bar{h}_1 \neq 0$.

Suppose that $\bar{g}_1 \overset{*}{\sim} \bar{h}_1$. By Lemma 1.6, there exist orthogonal decompositions $g_1 = g_1' + g_1''$, $h_1 = h_1' + h_1''$ such that $g_1' \overset{*}{\sim} h_1'$ and g_1'', $h_1'' \in M$. But $g_{n,1}$ does not have any nonzero subprojection *-equivalent to a subprojection of $h_{n,1}$. We conclude that $g_1' = h_1' = 0$, and so g_1, $h_1 \in M$ which is a contradiction. So, $\bar{g}_1$ and $\bar{h}_1$ are equivalent but not *-equivalent projections in R and we conclude that R does not have LP $\overset{*}{\sim}$ RP. □

We now consider the special case in which F is chosen to be a formally real number field.

LEMMA 3.7. *Let F be a formally real number field and let e, f be two projections in $M_n(F)$. Then, if $e \sim f$, there exist subprojections $e' \leq e$, $f' \leq f$ such that $e' \overset{*}{\sim} f'$ and* $\operatorname{rank}(e - e') < 4$, $\operatorname{rank}(f - f') < 4$.

Proof. If $\operatorname{rank}(e) < 4$, then the result is trivial. If $\operatorname{rank}(e) \geq 4$, set $q = H(e)$. By [**15**, Thm. XI.1.4] we see that q represents 1 (since $\dim q \geq 4$) and so $q \simeq \langle 1 \rangle \perp q'$. Thus, we conclude that we can get a quadratic form r such that $\dim r = 3$ and

$$q \simeq \left\langle \overbrace{1, \ldots, 1}^{s} \right\rangle \perp r.$$

This implies that there exists an orthogonal decomposition

$$e = e' + e'' \quad \text{with } e' \overset{*}{\sim} \operatorname{diag}\left(\overbrace{1, \ldots, 1}^{s}, 0, \ldots, 0 \right).$$

Similarly,

$$f = f' + f'' \quad \text{with } f' \overset{*}{\sim} \operatorname{diag}\left(\overbrace{1, \ldots, 1}^{s}, 0, \ldots, 0 \right).$$

So, $e' \overset{*}{\sim} f'$ and $\operatorname{rank}(e - e') = \operatorname{rank}(e'') = \operatorname{rank}(f'') = \operatorname{rank}(f - f') = 3$. □

PROPOSITION 3.8. *Let F be a formally real number field.*

(a) *Let $\{R_i, \Phi_{ji}\}_{j,i \in I}$ be any direct system where each R_i is a matricial $*$-algebra over F (with the identity involution on F). Set $R = \lim R_i$ and let N be a pseudo-rank function on R. Then, the type* II *part of the N-completion of R satisfies* LP $\overset{*}{\sim}$ RP *matricially.*

(b) *Set $S = \prod_{i=1}^{\infty} M_{n(i)}(F)$ with $n(1) < n(2) < \cdots$, and let M be any maximal two-sided ideal of S which contains $\oplus_{i=1}^{\infty} M_{n(i)}(F)$. Then, the factor ring S/M is a simple, $*$-regular, self-injective ring of type* II *satisfying* LP $\overset{*}{\sim}$ RP *matricially.*

Proof. (a) The proof is analogous to that of Theorem 3.5, using Lemma 3.7 adequately.

(b) Set $R = S/M$. By [**5**, Thm. 10.30], R is a simple, regular, right and left self-injective ring of type II. Also, R is $*$-regular with positive definite involution. It suffices to show that R satisfies LP $\overset{*}{\sim}$ RP.

Let e, f be two nonzero equivalent projections in R. By Proposition 1.5, we only have to prove that there exist nonzero subprojections $e' \leq e$, $f' \leq f$ such that $e' \overset{*}{\sim} f'$. Let n be any integer such that $n \geq 6$. By [**5**, 10.28] (and a standard argument), there exist n orthogonal equivalent projections $e_1, \ldots, e_n$ in R such that $e = e_1 + \cdots + e_n$.

Choose equivalent projections p, $q \in S$ such that $\bar{p} = e$ and $\bar{q} = f$. By applying [**5**, Prop. 2.18] we obtain orthogonal projections $p'_1, \ldots, p'_n \in S$ such that $p'_j \leq p$ and $\bar{p}'_j = e_j$ for $j = 1, \ldots, n$. By [**5**, Prop. 2.19] there exist projections $p_j \leq p'_j$ such that $p_1 \sim \cdots \sim p_n$ and $\bar{p}_j = \bar{p}'_j = e_j$ for $j = 1, \ldots, n$. Set $g = p_1 + \cdots + p_n \leq p$. Since $p \sim q$ there exists a projection $h \leq q$ such that $g \sim h$. Note that $\bar{g} = \bar{p}_1 + \cdots + \bar{p}_n = e_1 + \cdots + e_n = e$ and $\bar{h} \sim \bar{g} = e \sim f$. Since $\bar{h} \leq f$ and R is directly finite, we obtain $\bar{h} = f$. Summarizing we have $\bar{g} = e$, $\bar{h} = f$, $g \sim h$ and $g = p_1 + \cdots + p_n$ where the p_i are equivalent orthogonal projections.

Set $g = (g_1, g_2, \ldots)$, $h = (h_1, h_2, \ldots)$ where g_i, $h_i \in \mathrm{P}(M_{n(i)}(F))$. Note that $g_i \sim h_i$ in $M_{n(i)}(F)$ and that each g_i (and so each h_i) is the sum of n equivalent orthogonal projections. By Lemma 3.7 we can choose subprojections $g'_i \leq g_i$, $h'_i \leq h_i$, for $i = 1, 2, \ldots$ such that $g'_i \overset{*}{\sim} h'_i$, $\operatorname{rank}(g_i - g'_i) < 4$ and $\operatorname{rank}(h_i - h'_i) < 4$. Set $g''_i = g_i - g'_i$, $h''_i = h_i - h'_i$. Since $n \geq 6$ we have $g''_i \lesssim g'_i$ and $h''_i \lesssim h'_i$ for $i = 1, 2, \ldots$. Set $g' = (g'_i)$, $h' = (h'_i)$, $g'' = (g''_i)$, $h'' = (h''_i)$. We have $g' \overset{*}{\sim} h'$, $g' + g'' = g$, $h' + h'' = h$, $g'' \lesssim g'$ and $h'' \lesssim h'$. Hence $\bar{g}' \overset{*}{\sim} \bar{h}'$, $\bar{g}' \leq \bar{g} = e$ and $\bar{h}' \leq \bar{h} = f$. It only remains to prove that $g' \notin M$. If $g' \in M$ then since $g'' \lesssim g'$ we have $g'' \in M$ and so $g \in M$ which is a contradiction. Therefore $\bar{g}' \neq 0$ and this completes the proof. □

EXAMPLE 3.9. *There exists a *-regular ring such that*

(a) *The intersection of the maximal two-sided ideals is zero.*

(b) *For every maximal two-sided ideal M of R, R/M satisfies* LP $\overset{*}{\sim}$ RP *matricially, but R does not satisfy* LP $\overset{*}{\sim}$ RP.

Proof. Set $R = \{x \in \prod_{n=1}^{\infty} M_n(\mathbf{R}) \mid x_n \in M_n(\mathbf{Q})$ for all but finitely many $n\}$. Clearly the intersection of the maximal two-sided ideals of R is zero. If M is a maximal two-sided ideal of R such that M does not contain the direct sum $\bigoplus_{n=1}^{\infty} M_n(\mathbf{R})$, then $R/M \overset{*}{\cong} M_m(\mathbf{R})$ for some m and so R/M satisfies LP $\overset{*}{\sim}$ RP matricially. If M contains the direct sum $\bigoplus_{n=1}^{\infty} M_n(\mathbf{R})$ then $R/M \overset{*}{\cong} \prod_{n=1}^{\infty} M_n(\mathbf{Q})/(M \cap \prod_{n=1}^{\infty} M_n(\mathbf{Q}))$ and so, by Proposition 3.8, (b), R/M satisfies LP $\overset{*}{\sim}$ RP matricially. On the other hand it is clear that R does not satisfy LP $\overset{*}{\sim}$ RP. □

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