

INTRINSIC MEASURES AND HOLOMORPHIC RETRACTS

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In this paper we examine the consequences of the equality of the Eisenman and Carathéodory norms on k -vectors, $2 \leq k \leq n - 1$, at a point in an n -dimensional complex manifold M . We also investigate the consequence of the existence of a large number of two-dimensional holomorphic retracts of a complex manifold—one tangent to each 2-vector at a given point.

1. Introduction. The equality of the Kobayashi and Carathéodory metrics on strictly convex domains in \mathbf{C}^n is one consequence of the work of Lempert [12, 13, 14, 19]. This remarkable fact is deduced from the existence of a large supply of one-dimensional holomorphic retracts of such domains—one through each point tangent to each direction. In this paper we examine the consequences of the equality of the Eisenman and Carathéodory norms on k -vectors, $2 \leq k \leq n - 1$, at a point in an n -dimensional complex manifold M . (The Eisenman norm is the analog of the infinitesimal Kobayashi metric and is the object of a recent study by H. Wu and the author [9]; see also the papers of Eisenman [7, 15].) H. Wu and the author considered the top-dimensional case in [10] (for earlier results see [6, 17, 22]) obtaining a criterion for biholomorphic equivalence with the unit ball in \mathbf{C}^n . We also investigate the consequences of the existence of a large number of two-dimensional holomorphic retracts of a complex manifold—one tangent to each 2-vector at a given point. Both assumptions lead to the conclusion, if M is Carathéodory-hyperbolic (see §4 for precise statements of the theorems) that the indicatrix of the Carathéodory metric at the point in question is an analytic ellipsoid; the first assumption also implies that the infinitesimal Kobayashi and Carathéodory metrics coincide at the point in question. Later in the paper we consider intrinsic norms on k -vectors at the origin of a circular domain in \mathbf{C}^n , exhibiting some differences between the $k = 1$ case and the $k > 1$ case.

Lempert [13] has already observed that there do not in general exist 2-dimensional holomorphic retracts containing three given points of a strictly convex domain; his argument shows that there does not in general exist a 2-dimensional retract tangent to a given 2-vector at a point. It is

worth pointing out that some facts from the theory of Banach and Hilbert spaces shed some light on this situation and on the theorems proved here. Specifically, if X is a real or complex Banach space, there always exists a projection of norm 1 onto any one-dimensional subspace by the Hahn-Banach theorem. However, if there exists a projection of norm 1 onto all two-dimensional subspaces, then the given Banach space X is actually a Hilbert space. (The real case of this result was proved by Kakutani [11] and Phillips [16], the complex case by Bohnenblust [3].) Of course a projection of norm 1 is just a linear retraction of the unit ball of X , and, in the finite-dimensional case, X is a Hilbert space iff the unit ball is an ellipsoid.

The plan of the paper is as follows: In §2 we recall the definitions of the Eisenman and Carathéodory norms on decomposable k -vectors. In §3 we give estimates in the form of two lemmas for these norms which are used in the proof of the theorems as well as in some further results in §6. The main theorems are proved in §4. In §5 we give a variant on the known versions of the Schwarz lemma on circular domains; this result as well as the lemmas of §3 are used in §6 to give further formulas and estimates for the Eisenman and Carathéodory norms at the origin of circular domains.

It is a pleasure to record my thanks to Man-Duen Choi and Peter Rosenthal for discussions about projections in Banach spaces and for locating the characterizations of Hilbert space by Kakutani [11] and Bohnenblust [3].

2. Definitions. In this section we recall some basic definitions, most of which appear in [9]. Let M be an n -dimensional complex manifold and let $p \in M$. Let $T_p M$ (respectively TM) denote the holomorphic tangent space to M at p (respectively the holomorphic tangent bundle). Let $\Lambda^k T_p M$ (respectively $\Lambda^k TM$) be the k th exterior power of $T_p M$ (respectively TM). Of special interest are the decomposable elements of $\Lambda^k T_p M$ (respectively $\Lambda^k TM$); these we denote by $D_p^k M$ (respectively $D^k M$). There is a natural way of extending a hermitian metric $\langle \cdot, \cdot \rangle$ on TM to a hermitian metric on $\Lambda^k TM$: first, for $\alpha, \beta \in D_p^k M$ with $\alpha = v_1 \wedge \cdots \wedge v_k$ and $\beta = w_1 \wedge \cdots \wedge w_k$ we define

$$(2.1) \quad \langle \alpha, \beta \rangle = \det(\langle v_i, w_j \rangle)_{i,j=1, \dots, k}$$

and then extend the definition to arbitrary elements of $\Lambda^k T_p M$ by linearity.

The unit ball in \mathbf{C}^n is denoted by B_n . B_n carries the Euclidean metric which may be extended to k -vectors as above. We orient \mathbf{C}^n so that the (n, n) -form $(i/2)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$ is positive.

For decomposable k -vectors, we define the Eisenman and Carathéodory norms by constructions modelled on the definitions of the differential Kobayashi and Carathéodory pseudo-metrics. Let $1 \leq k \leq n$ and let $\alpha \in D_p^k M$.

DEFINITION 1. The Eisenman norm of α is

$$(2.2) \quad E_k(p; \alpha) = \inf\{\|\gamma\|^2 \mid \gamma \in D_0^k B_k \text{ and there exists a holomorphic mapping } f: B_k \rightarrow M \text{ such that } f(0) = p \text{ and } f_*(\gamma) = \alpha\}.$$

DEFINITION 2. The Carathéodory norm of α is

$$(2.3) \quad C_k(p; \alpha) = \sup\{\|\gamma\|^2: \gamma \in D_0^k B_k \text{ and there exists a holomorphic mapping } g: M \rightarrow B_k \text{ such that } g(p) = 0 \text{ and } g_*(\alpha) = \gamma\}.$$

By $\|\gamma\|$ we mean the Euclidean norm of γ ; the reason for squaring (as in [2, 9, 10]) arises from the computation of volumes in local holomorphic coordinates. It suffices to define E_k and C_k for decomposable k -vectors in order to compute volumes. If A is a k -dimensional complex submanifold of an open subset U of M we introduce intrinsic volume forms τ_A^M and η_A^M as follows: let $p \in A$ and let z_1, \dots, z_n be local holomorphic coordinates in M near p so that $\partial/\partial z_1, \dots, \partial/\partial z_k$ are tangent to A at p . Then

$$(2.4) \quad \tau_A^M(p) = E_k\left(p; \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_k}(p)\right) \left(\frac{i}{2}\right)^k dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_k \wedge d\bar{z}_k$$

is the Eisenman volume form of A (as a submanifold of M) while

$$(2.5) \quad \eta_A^M = C_k\left(p; \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_k}(p)\right) \left(\frac{i}{2}\right)^k dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_k \wedge d\bar{z}_k$$

is the Carathéodory volume form of A (as a submanifold of M). If S is a Borel subset of A then its Eisenman volume is

$$(2.6) \quad \mathcal{E}_k(S) = \int_S \tau_A^M$$

and its Carathéodory volume is

$$(2.7) \quad \mathcal{C}_k(S) = \int_S \eta_A^M.$$

When necessary we shall indicate the manifold on which the intrinsic k -norms are defined by a superscript, as in $E_k^M(\ ; \)$, $C_k^M(\ ; \)$. It is elementary to verify that if A is a complex submanifold of M which is a

holomorphic retract of M then $E_k^M = E_k^A$ and $C_k^M = C_k^A$ on decomposable k -vectors of A . (There is a general theorem of Rossi to the effect that any holomorphic retract of a complex manifold must be a complex submanifold [18, Theorem 5.2].) Also the inequality $C_k \leq E_k$ is elementary [7].

The indicatrix of the Kobayashi pseudo-metric at $p \in M$ is the set $I_K(p) = \{v \in T_p M \mid E_1(p; v) < 1\}$. E_1 as defined above is actually the square of the Kobayashi pseudo-metric. Similarly C_1 is the square of the Carathéodory pseudo-metric so that the indicatrix of the Carathéodory pseudo-metric is the set $I_C(p) = \{v \in T_p M \mid C_1(p; v) < 1\}$. Both indicatrices are complete circular domains; $I_C(p)$ is convex.

By an analytic ellipsoid we shall mean a domain of the form

$$\left\{ z \in \mathbf{C}^n \mid \sum_{i,j=1}^n a_{ij} z_i \bar{z}_j < 1 \right\}$$

where $(a_{ij})_{i,j=1}^n$ is a positive definite hermitian matrix. Note that all such domains are centered at the origin.

Finally we recall that a complex manifold M is said to be taut if for any other complex manifold N the set of holomorphic mappings from N into M form a normal family [23].

3. Estimates for the Eisenman and Carathéodory norms. Let p be a point in an n -dimensional complex manifold M which carries a hermitian metric $\langle \cdot, \cdot \rangle$. The unit ball B_k in \mathbf{C}^k is equipped with the Euclidean metric. A preliminary estimate is given by the following:

LEMMA 1 *Let $\alpha \in D_p^k M$. Let L be the k -dimensional (complex) linear subspace of $T_p M$ which is tangent to α . Then*

$$(3.1) \quad E_k(p; \alpha) \geq \frac{\|\alpha\|^2 \text{vol}(B_k)}{\text{vol}(L \cap I_K(p))},$$

$$(3.2) \quad C_k(p, \alpha) \leq \frac{\|\alpha\|^2 \text{vol}(B_k)}{\text{vol}(L \cap I_C(p))}.$$

($\|\alpha\|^2$, $\text{vol}(L \cap I_K(p))$, and $\text{vol}(L \cap I_C(p))$ are computed with respect to the hermitian metric $\langle \cdot, \cdot \rangle$; $\text{vol}(B_k)$ denotes the Euclidean volume of the unit ball in \mathbf{C}^k . The right-hand sides of (3.1) and (3.2) are of course independent of the choice of hermitian metric on M .)

Proof of (3.1) Suppose that $f: B_k \rightarrow M$ is a holomorphic map and $\gamma \in D_0^k B_k$ are such that $f(0) = p$ and $f_*(\gamma) = \alpha$. Then

$$(3.3) \quad \frac{\|f_*(\gamma)\|^2}{\|\gamma\|^2} = \frac{\text{vol}(f_*(B_k))}{\text{vol}(B_k)} \leq \frac{\text{vol}(L \cap I_K(p))}{\text{vol}(B_k)}$$

where the last inequality is a consequence of the fact that f_* maps the indicatrix of the Kobayashi metric of B_k at 0 into $L \cap I_K(p)$. (For the terms in the numerator of (3.3) the metric $\langle \cdot, \cdot \rangle$ is used, while the Euclidean metric is used for the terms in the denominator. Also by abuse of notation we are denoting the unit ball in $T_0 B_k$ by B_k .) Thus

$$(3.4) \quad \|\gamma\|^2 \geq \frac{\|\alpha\|^2 \text{vol}(B_k)}{\text{vol}(L \cap I_K(p))}$$

from which (3.1) follows.

Proof of (3.2) Suppose that $g: M \rightarrow B_k$ is a holomorphic map and $\gamma \in D_0^k B_k$ are such that $g(p) = 0$ and $g_*(\alpha) = \gamma$. Then

$$(3.5) \quad \frac{\|g_*(\alpha)\|^2}{\|\alpha\|^2} = \frac{\text{vol}(g_*(L \cap I_C(p)))}{\text{vol}(L \cap I_C(p))} \leq \frac{\text{vol}(B_k)}{\text{vol}(L \cap I_C(p))}$$

(Euclidean metric in the numerator, $\langle \cdot, \cdot \rangle$ in the denominator) from which (3.2) follows. As before we have used the fact that $g_*(I_C(p)) \subset B_k$.

The estimates in Lemma 1 can be sharpened. In fact we actually have

LEMMA 2. *Let $\alpha \in D_p^k M$. Let L be the k -dimensional linear subspace of $T_p M$ which is tangent to α .*

(a) *Let s be the supremum of the volumes (w.r.t. $\langle \cdot, \cdot \rangle$) of the analytic ellipsoids which can be inscribed in $L \cap I_K(p)$. (There is an ellipsoid of maximal volume if $L \cap I_K(p)$ is bounded.) Then*

$$(3.6) \quad E_k(p; \alpha) \geq \|\alpha\|^2 \text{vol}(B_k) s^{-1}.$$

(b) *Let t be the infimum of the volumes (w.r.t. $\langle \cdot, \cdot \rangle$) of the analytic ellipsoids which can be circumscribed about $L \cap I_C(p)$. (t is finite iff $L \cap I_C(p)$ is bounded, in which case there is a circumscribing ellipsoid of minimal volume.) Then*

$$(3.7) \quad C_k(p; \alpha) \leq \|\alpha\|^2 \text{vol}(B_k) t^{-1}.$$

(The estimates (3.6) and (3.7) are of course independent of the choice of $\langle \cdot, \cdot \rangle$.)

Proof (a) Referring to the proof of (3.1) in the previous lemma, we note that $f_*(B_k)$ must be an analytic ellipsoid and hence $\text{vol}(f_*(B_k)) \leq s$. This gives the sharpened estimate (3.6). To see that the supremum is attained if $L \cap I_K(p)$ is bounded, let $T_j: \mathbf{C}^k \rightarrow L$ be a sequence of complex-linear maps such that $T_j(B_k) \subset I_K(p)$ and $\text{vol} T_j(B_k) \rightarrow s$. The maps T_j are uniformly bounded, hence a subsequence converges in the norm topology to a linear map $T: \mathbf{C}^k \rightarrow T_p M$. Evidently $\text{vol}(T(B_k)) = s$ and $T(B_k) \subset \overline{I_K(p)}$. But since T is open we must have $T(B_k) \subset I_K(p)$.

(b) Referring to the proof of (3.2) in the previous lemma, we note that if there exist a holomorphic map $g: M \rightarrow B_k$ and $\gamma \in D_0^k B_k$ such that $g(p) = 0$ and $g_*(\alpha) = \gamma$, then

$$(3.8) \quad \frac{\|g_*(\alpha)\|^2}{\|\alpha\|^2} = \frac{\|\gamma\|^2}{\|g_*^{-1}(\gamma)\|^2} = \frac{\text{vol}(B_k)}{\text{vol}(g_*^{-1}(B_k))} \leq \frac{\text{vol}(B_k)}{t}$$

since $g_*^{-1}(B_k)$ is an analytic ellipsoid which contains $L \cap I_C(p)$. (As the reader will surmise, by g_*^{-1} we really mean $(g_*|_L)^{-1}$.) The finiteness statement about t is obvious, and the assertion that there is a circumscribing ellipsoid of minimal volume follows from applying a normality argument to the set of linear maps from L to \mathbf{C}^k which map $L \cap I_C(p)$ into B_k .

4. The main results.

THEOREM 1. *Suppose that M is a complex manifold of dimension n and $p \in M$. Suppose that for some integer k , $1 \leq k \leq n$, there exists $\alpha \in D_p^k M$ such that $E_k(p; \alpha) = C_k(p; \alpha) \neq 0$. Let L be the k -dimensional complex subspace of $T_p M$ which is tangent to α . Then $E_1(p; v) = C_1(p; v)$ for all $v \in L$ and $I_K(p) \cap L = I_C(p) \cap L$ is an analytic ellipsoid.*

Proof. The assumption that $C_k(p; \alpha) \neq 0$ is easily seen to imply that $C_1(p; v) \neq 0$ for all $v \in L - \{0\}$. The continuity of C_1 implies that $I_C(p)$ is bounded, hence $I_K(p)$ is bounded as well since $I_K(p) \subset I_C(p)$. By Lemma 2 there exist k -dimensional analytic ellipsoids $R \subset L \cap I_K(p)$ and $S \supset L \cap I_C(p)$ such that R has maximal volume and S has minimal volume, volumes being computed with respect to some hermitian metric $\langle \cdot, \cdot \rangle$ on M (the choice of which does not affect the choice of R and S). Also by Lemma 2 we have

$$(4.1) \quad E_k(p; \alpha) \geq \|\alpha\|^2 \frac{\text{vol}(B_k)}{\text{vol}(R)}$$

and

$$(4.2) \quad C_k(p; \alpha) \leq \|\alpha\|^2 \frac{\text{vol}(B_k)}{\text{vol}(S)}.$$

These estimates, together with the assumption that $E_k(p; \alpha) = C_k(p; \alpha)$ and the obvious inequality $\text{vol}(R) \leq \text{vol}(S)$ lead us to the conclusion that $\text{vol}(R) = \text{vol}(S)$, from which it follows (since $R \subset S$ and both are analytic ellipsoids) that $R = I_K(p) \cap L = I_C(p) \cap L = S$.

THEOREM 2. *With hypotheses as in Theorem 1, suppose that in addition M is taut. Then there exists a k -dimensional complex submanifold A of M which is biholomorphic to B_k and tangent to α at p , and there exists a holomorphic retraction $\phi: M \rightarrow A$.*

Proof. In view of the assumption of tautness of M , there exists a holomorphic map $f: B_k \rightarrow M$ and $\gamma_1 \in D_0^k B_k$ such that $f(0) = p$, $f_*(\gamma_1) = \alpha$, and $\|\gamma_1\|^2 = E_k(p; \alpha)$. Normality also implies that there exists a holomorphic map $g: M \rightarrow B_k$ and $\gamma_2 \in D_0^k B_k$ such that $g(p) = 0$, $g_*(\alpha) = \gamma_2$, and $\|\gamma_2\|^2 = C_k(p; \alpha)$. By hypothesis $\|\gamma_1\|^2 = \|\gamma_2\|^2$; hence since the fibre of $\Lambda^k T_0 B_k = D_0^k B_k$ has dimension 1 we must have $\gamma_2 = e^{i\theta} \gamma_1$ for some choice of θ . By replacing g by $e^{-i\theta} \cdot g$ we may assume that $\gamma_1 = \gamma_2$. But this implies that the composed mapping $g \circ f: B_k \rightarrow B_k$ has Jacobian determinant equal to 1 at the origin and hence, since the origin is fixed, must be a unitary transformation U . By replacing g by $U^{-1} \circ g$ we may assume that $g \circ f$ is the identity transformation.

Now let $A = f(B_k)$. It is easy to see that A is a holomorphic retract of M and in fact the map $\phi = f \circ g$ gives the desired retraction. The mapping f must have maximal rank everywhere and be 1-1 since these statements are true of $g \circ f$. Hence f is an embedding, which completes the proof. (We note that a theorem of Rossi [18, Theorem 5.2] guarantees that any holomorphic retract of a complex manifold must be a complex submanifold.)

REMARK. It would certainly be of interest if the assumption of tautness could be removed from Theorem 2. It is not needed in the top-dimensional case [10]. Some results for the $k = 1$ case have been obtained by Fox [8] for general complex manifolds, but with a local tautness assumption.

THEOREM 3. *Suppose that M is an n -dimensional complex manifold ($n \geq 2$) and $p \in M$. Suppose that there is a point $p \in M$ and an integer k , $2 \leq k \leq n$ such that $E_k(p; \alpha) = C_k(p; \alpha) \neq 0$ for all $\alpha \in D_p^k M$. Then*

$E_1(p; v) = C_1(p; v)$ for all $v \in T_p M$ and $I_K(p) = I_C(p)$ is an analytic ellipsoid.

Proof. The assumptions of Theorem 3 together with Theorem 1 imply that $E_1(p; \cdot) = C_1(p; \cdot)$, and that whenever L is a k -dimensional complex subspace of $T_p M$ then $I_K(p) \cap L = I_C(p) \cap L$ is an analytic ellipsoid. The boundedness of $I_C(p) = I_K(p)$ follows from the continuity of $C_1(p; \cdot)$, and of course $I_C(p)$ is always a convex circled domain. We now wish to argue that a bounded convex circled domain in \mathbb{C}^n which intersects every k -dimensional complex subspace in a k -dimensional analytic ellipsoid ($k \geq 2$) must be an n -dimensional analytic ellipsoid. It clearly suffices to establish the result when $k = 2$, but in this case our statement is equivalent to the fact that a (finite-dimensional) Banach space in which the parallelogram law holds is a Hilbert space, for the parallelogram law is a statement about the Banach space norm on all two-dimensional subspaces.

THEOREM 4. *Suppose that M is an n -dimensional complex manifold ($n \geq 2$) and $p \in M$. Suppose that there is a point $p \in M$ such that for all $\alpha \in D_p^2 M$ there is a 2-dimensional holomorphic retract of M tangent to α at p . Then if M is Carathéodory-hyperbolic (respectively if M is hyperbolic and $I_K(p)$ is convex) then $I_C(p)$ is an analytic ellipsoid (respectively $I_K(p)$ is an analytic ellipsoid).*

Proof. It is easy to see that the hypotheses of the theorem imply that whenever L is a two-dimensional complex subspace of $T_p M$ there are (complex) linear retractions of $I_C(p)$ onto $I_C(p) \cap L$ and of $I_K(p)$ onto $I_K(p) \cap L$. We now invoke a result of Bohnenblust [3] concerning projections in complex Banach spaces: if X is a complex Banach space in which there is a projection of norm 1 onto any two-dimensional complex subspace, then X is a Hilbert space. (The real case of this result is due to Kakutani [11] and Phillips [16].) Of course the unit ball in any finite dimensional Hilbert space is an ellipsoid (an analytic ellipsoid in the complex case).

THEOREM 5. *Suppose that Ω is a bounded complete circular domain of holomorphy in \mathbb{C}^n , $n \geq 2$, and that for some k , $2 \leq k \leq n$, we have $E_k(0; \alpha) = C_k(0; \alpha)$ for all $\alpha \in D_0^k \Omega$. Then Ω is an analytic ellipsoid.*

Proof. From a result of Barth [1] we obtain the fact that $I_K(0) = \Omega$. We now simply invoke Theorem 3.

5. The Schwarz lemma on circular domains. A domain $\Omega \subset \mathbb{C}^n$ is called circled or circular if $z \in \Omega$ implies $e^{i\theta} \cdot z \in \Omega$ for all real θ . Ω is complete circular or star-shaped circular or disked if $z \in \Omega$ implies $cz \in \Omega$ for all $c \in \mathbb{C}$ such that $|c| \leq 1$. Ω is convex circled if Ω is simultaneously circular and convex.

In §6 we shall need a version of the Schwarz lemma for mappings from circular domains to convex circled domains (cf. [20, Theorem 8.1.2] for the case when the map is defined on a complete circular domain). The following result actually generalizes the Cauchy estimates; the idea for the proof comes from an argument of Lempert [13, p. 261].

LEMMA 3. *Suppose that $\Omega_1 \subset \mathbb{C}^{n_1}$ is circled and contains the origin and $\Omega_2 \subset \mathbb{C}^{n_2}$ is convex circled. Suppose that $f: \Omega_1 \rightarrow \Omega_2$ is a holomorphic map such that $f(0) = 0$. Let $f = \sum_{j=1}^{\infty} P_j$ be the series expansion of f in terms of (n_2 -tuples of) homogeneous polynomials. Then $P_j(\Omega_1) \subset \Omega_2$.*

Proof. We note that

$$(5.1) \quad P_j(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ij\theta} f(ze^{i\theta}) d\theta$$

and that the series expansion of f is valid for all $z \in \Omega_1$ [5, p. 17]. Now fix $z \in \Omega_1$. The set $\{f(ze^{i\theta}) \mid \theta \in \mathbb{R}\}$ is a compact subset K_1 of Ω_2 . Let K_2 be the circled subset of Ω_2 generated by K_1 (also compact), and let \hat{K}_2 be its convex hull, a convex circled set and a compact subset of Ω_2 . Now any value of the integrand in (5.1) lies in K_2 , and any Riemann sum for the right-hand side of (5.1) is a convex combination of points in K_2 . Hence $P_j(z) \in \hat{K}_2 \subset \Omega_2$.

6. Intrinsic measures on circular domains. Using the lemmas in §3, we show how to compute the Eisenman norm of k -vectors at the origin of a complete circular domain of holomorphy. For the Carathéodory norms we consider slightly more general domains (circular domains containing 0), but unless $k = 1$ or $k = n$ we obtain only a pair of inequalities, essentially because of the nonexistence in general of projections of norm 1 onto subspaces of a Banach space of dimension greater than one.

THEOREM 6. *Let Ω be a complete circular domain of holomorphy. Let $\alpha \in D_0^k \Omega$ and let L be the k -dimensional complex linear subspace of \mathbb{C}^n tangent to α at 0. Let s be the supremum of the volumes of the k -dimensional analytic ellipsoids (centered at 0 as always) which can be inscribed in*

$L \cap \Omega$. Then

$$(6.1) \quad E_k(0; \alpha) = \|\alpha\|^2 \operatorname{vol}(B_k) s^{-1}.$$

($\|\alpha\|$ and s may be computed using the Euclidean metric or any constant-coefficient hermitian metric.)

Proof. The inequality $E_k(0; \alpha) \geq \|\alpha\|^2 \operatorname{vol}(B_k) s^{-1}$ follows from Lemma 2 and the fact, already noted, that $I_K(0)$ coincides with $\Omega[1]$. The opposite inequality is obtained by considering complex linear maps from B_k into $\Omega \cap L$.

REMARKS. (1) A linear extremal map exists if Ω is bounded (cf. Lemma 2).

(2) Remark 1 shows that an extremal map for $\alpha \in D_0^k \Omega$ need not be proper unless $\Omega \cap L$ is a k -dimensional analytic ellipsoid. This is to be contrasted with the behaviour of extremal mappings for the Kobayashi metric on strictly convex domains [12].

In studying the Carathéodory norms of k -vectors at the origin of a circular domain, we first note the following:

LEMMA 4. Let Ω be a circular domain in \mathbb{C}^n and let $\hat{\Omega}$ denote its convex hull, a convex circled domain. Then

$$(6.2) \quad C_k^\Omega(0; \cdot) = C_k^{\hat{\Omega}}(0; \cdot) \quad k = 1, \dots, n.$$

Proof. From Lemma 3 we see that it suffices to consider linear maps from Ω to B_k in computing $C_k^\Omega(0; \cdot)$ and $C_k^{\hat{\Omega}}(0; \cdot)$. But if $T: \mathbb{C}^n \rightarrow \mathbb{C}^k$ is any linear map such that $T(\Omega) \subset B_k$, then from the fact that $T(\hat{\Omega}) = \overline{T(\Omega)}$ it follows that $T(\hat{\Omega}) \subset B_k$. Hence the class of linear maps under consideration is identical whether one is computing $C_k^\Omega(0; \alpha)$ or $C_k^{\hat{\Omega}}(0; \alpha)$.

THEOREM 7. (a) Let Ω be a circular domain in \mathbb{C}^n and let $\hat{\Omega}$ be its convex hull. Let $\alpha \in D_0^k \Omega$ and let L be the k -dimensional linear subspace of \mathbb{C}^n tangent to α at 0. Let t be the infimum of the volumes of the k -dimensional analytic ellipsoids in L which contain $L \cap \hat{\Omega}$ (t is finite iff $L \cap \hat{\Omega}$ is bounded, in which case t is assumed.) Then

$$(6.3) \quad C_k(0; \alpha) \leq \|\alpha\|^2 \operatorname{vol}(B_k) t^{-1}.$$

(b) If in addition Ω is bounded let m denote the minimal norm with respect to the Banach space structure on \mathbf{C}^n whose unit ball is $\hat{\Omega}$ of a projection of \mathbf{C}^n onto L . (Thus $m \geq 1$.) Then

$$(6.4) \quad m^{-2k} \|\alpha\|^2 \text{vol}(B_k) t^{-1} \leq C_k(0; \alpha) \leq \|\alpha\|^2 \text{vol}(B_k) t^{-1}.$$

($\|\alpha\|^2$ and t may be computed using the Euclidean metric or any constant-coefficient hermitian metric; $\text{vol}(B_k)$ denotes the Euclidean volume of the unit ball in \mathbf{C}^k .)

Proof. (a) The right-hand side of (6.3) is zero iff $L \cap \hat{\Omega}$ is unbounded. In this case there does not exist a linear map $T: \hat{\Omega} \rightarrow B_k$ such that $T_*(\alpha) \neq 0$. In view of Lemmas 3 and 4 it follows that if $g: \Omega \rightarrow B_k$ is any holomorphic map such that $g(0) = 0$ then $g_*(\alpha) = 0$. Hence $C_k(0; \alpha) = 0$ and (6.3) is valid in this case.

We may therefore assume that $L \cap \hat{\Omega}$ is bounded. For the fact that the infimum t is assumed in this case see Lemma 2. In view of Lemma 3 it suffices to consider linear maps $T: \Omega \rightarrow B_k$ in estimating $C_k^\Omega(0; \alpha)$. Since $T(L \cap \hat{\Omega}) \subset B_k$ for such a map we obtain, abbreviating $(T|_L)_*^{-1}$ to T_*^{-1} ,

$$(6.5) \quad \frac{\|T_*(\alpha)\|^2}{\|\alpha\|^2} = \frac{\text{vol}(B_k)}{\text{vol}(T_*^{-1}(B_k))} \leq \frac{\text{vol}(B_k)}{t}$$

since $T_*^{-1}(B_k)$ is an analytic ellipsoid containing $L \cap \hat{\Omega}$.

(b) To prove the remaining inequality in (b), let A be an analytic ellipsoid of minimal volume circumscribing $L \cap \hat{\Omega}$ and let $T: L \rightarrow \mathbf{C}^k$ be a linear transformation such that $T(A) = B_k$. Let $\pi: \mathbf{C}^n \rightarrow L$ be a projection of minimal norm $m \geq 1$ with respect to the Banach space structure on \mathbf{C}^n whose unit ball is $\hat{\Omega}$. It is elementary to see that a projection of minimal norm exists, and also, in view of the linearity of the transformations, that $m^{-1} \cdot T \circ \pi$ maps Ω into B_k . Hence

$$(6.6) \quad \frac{\|(m^{-1} \cdot T \circ \pi)_*(\alpha)\|^2}{\|\alpha\|^2} = \frac{\|(m^{-1} \cdot T)_*(\alpha)\|^2}{\|\alpha\|^2} \\ = m^{-2k} \frac{\|T_*(\alpha)\|^2}{\|\alpha\|^2} = m^{-2k} \text{vol}(B_k) t^{-1}$$

from which it follows that $C_k(0; \alpha) \geq m^{-2k} \|\alpha\|^2 \text{vol}(B_k) t^{-1}$. This completes the proof of Theorem 7.

REMARK. There always exists a projection of norm 1 onto a 1-dimensional subspace of a Banach space by the Hahn-Banach theorem. It is this fact which allowed Carathéodory [4] to compute the infinitesimal

Carathéodory metric at the origin of a convex-circled domain. Projections of norm 1 also play a role in the study of fixed points of holomorphic maps [21].

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