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GRUNSKY INEQUALITIES FOR UNIVALENT FUNCTIONS WITH PRESCRIBED HAYMAN INDEX

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The Grunsky inequalities in their standard formulation are a generalization of the area principle. Our purpose is to apply a variational method to obtain a stronger system of inequalities which involves both the logarithmic coefficients and the Hayman index of a univalent function f in the usual class S. One immediate consequence is the well-known inequality of Bazilevich on logarithmic coefficients. Another application gives a sharpened form of the Goluzin inequalities on the values of f at prescribed points of the disk.

1. Main results. The class S consists of all functions $f(z) = z + a_2 z^2 + \cdots$ analytic and univalent in the unit disk $\mathbf{D}$. Closely related is the class Σ of all functions

$$g(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$$

analytic and univalent in the exterior $\Delta = \mathbf{C} - \overline{\mathbf{D}}$ of the disk. Given $g \in \Sigma$ we construct the double power series

$$\log \frac{g(z) - g(\zeta)}{z - \zeta} = -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm} z^{-n} \zeta^{-m}, \qquad z, \zeta \in \Delta.$$

Then $d_{mn} = d_{nm}$ and the *Grunsky inequalities* ([6]; see [3], Chapter 4) take the form

$$\left| \sum_{n=1}^{N} \sum_{m=1}^{N} d_{nm} \lambda_n \lambda_m \right| \leq \sum_{n=1}^{N} \frac{1}{n} |\lambda_n|^2, \qquad \lambda_n \in \mathbf{C}.$$

Now let $f \in S$ and consider the analogous series

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = -\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} z^n \zeta^m \tag{1}$$

for z, $\zeta \in \mathbf{D}$. Note that $c_{mn} = c_{nm}$ and $c_{00} = 0$. If $\zeta = 0$ the series reduces to

$$\log \frac{f(z)}{z} = -\sum_{n=0}^{\infty} c_{n0} z^n.$$

The *inversion* of f is the function $g \in \Sigma$ defined by $g(1/z) = 1/f(z)$. Thus

$$\frac{g(1/z) - g(1/\zeta)}{1/z - 1/\zeta} = \frac{f(z) - f(\zeta)}{z - \zeta} \cdot \frac{z\zeta}{f(z)f(\zeta)}.$$

Taking logarithms, we find

$$\begin{aligned} -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm} z^n \zeta^m &= \log\frac{f(z)-f(\zeta)}{z-\zeta} - \log\frac{f(z)}{z} - \log\frac{f(\zeta)}{\zeta} \\ &= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm} z^n \zeta^m. \end{aligned}$$

This shows that $c_{nm} = d_{nm}$ for $n,\ m \geq 1$, and so the Grunsky inequalities take the form

$$\left|\sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\lambda_n\lambda_m\right| \leq \sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2, \qquad \lambda_n \in \mathbf{C}. \tag{2}$$

If f is chosen to be the Koebe function $k(z) = z(1-z)^{-2}$, an easy calculation gives $c_{n0} = -2/n$ and $c_{nm} = (1/n)\delta_{nm}$ for $n,\ m \geq 1$, where δ_{nm} is the Kronecker symbol. This shows that the Grunsky inequalilties are sharp for each N.

In more standard notation, the *logarithmic coefficients* of a function $f \in S$ are the numbers γ_n defined by

$$\log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\gamma_n z^n.$$

Thus $2\gamma_n = -c_{n0}$, and $\gamma_n = 1/n$ for the Koebe function. A result of Bazilevich ([**1**, **2**]; see [**3**], §5.6) asserts that the logarithmic coefficients of each function $f \in S$ satisfy the sharp inequality

$$\sum_{n=1}^{\infty} n\left|\gamma_n - \frac{1}{n}\right|^2 \leq \frac{1}{2}\log\frac{1}{\alpha}, \tag{3}$$

where $\alpha = \lim_{r\to 1}(1-r)^2|f(r)|$ is the *Hayman index* of f, assumed to be positive. The Hayman index of a function $f \in S$ is the number

$$\alpha = \lim_{r\to 1}(1-r)^2 M_\infty(r, f),$$

where $M_\infty(r, f)$ is the maximum of $|f(z)|$ on the circle $|z| = r$. It is easily verified (see [**3**], §5.5) that $0 \leq \alpha \leq 1$ and that $\alpha = 1$ if and only if f is a rotation of the Koebe function. If $\alpha > 0$ then f has a unique *direction of maximal growth* $e^{i\theta}$ defined by the property $(1-r)^2|f(re^{i\theta})| \to \alpha$ as $r \to 1$. In our statement of the Bazilevich inequality (3) we have supposed that f is rotated so that its direction of maximal growth is $e^{i\theta} = 1$.

Hayman proved ([**7**, **8**]; see [**3**], §5.7) that $|a_n|/n \to \alpha$ for each fixed function $f \in S$ with index α.

Under the additional assumption that f has Hayman index α, $0 < \alpha < 1$, the Grunsky inequalities can be strengthened. The following theorem is the main result of this paper.

THEOREM 1. *For functions $f \in S$ with Hayman index α $(0 < \alpha < 1)$ and direction of maximal growth $e^{i\theta} = 1$, the inequality*

$$\operatorname{Re}\left\{\sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\lambda_n\lambda_m\right\} \leq \sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2 - 4\left(\operatorname{Re}\left\{\sum_{n=1}^{N}\lambda_n\left(\gamma_n - \frac{1}{n}\right)\right\}\right)^2\left(\log\frac{1}{\alpha}\right)^{-1} \tag{4}$$

holds for each N and for all $\lambda_n \in \mathbf{C}$, where c_{nm} are the Grunsky coefficients and γ_n are the logarithmic coefficients of f. This inequality is sharp for each choice of N and λ_n.

The proof is given in §3. It is not clear whether the inequality (4) is sharp for arbitrary values of α. The method of proof suggests that an extremal value of α will be determined by the given parameters λ_n. A further remark on this question appears in §4, at the end of the paper.

It may be observed that the Bazilevich inequality (3) is an immediate corollary of the theorem. To see this, apply the Grunsky inequalities (2) to get

$$-\sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2 \leq \operatorname{Re}\left\{\sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\lambda_n\lambda_m\right\}$$
$$\leq \sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2 - 4\left[\operatorname{Re}\left\{\sum_{n=1}^{N}\lambda_n\left(\gamma_n - \frac{1}{n}\right)\right\}\right]^2\left(\log\frac{1}{\alpha}\right)^{-1},$$

which implies

$$\left[\operatorname{Re}\left\{\sum_{n=1}^{N}\lambda_n\left(\gamma_n - \frac{1}{n}\right)\right\}\right]^2 \leq \frac{1}{2}\log\frac{1}{\alpha}\sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2.$$

Now choose $\lambda_n = n(\overline{\gamma_n} - 1/n)$ to obtain (3).

Another interesting corollary is found by choosing $N = 1$ and $\lambda_1 = 1$. In terms of the coefficients of f, this gives the sharp inequality

$$(2 - \operatorname{Re}\{a_2\})^2 \leq \log\frac{1}{\alpha}\left(1 - \operatorname{Re}\{a_2^2 - a_3\}\right).$$

Theorem 1 may also be applied to derive a generalized form of the Goluzin inequalities ([5]; see [3], §4.4). We have the following result.

THEOREM 2. *For functions $f \in S$ with Hayman index α $(0 < \alpha < 1)$ and direction of maximal growth $e^{i\theta} = 1$, the sharp inequality*

$$\operatorname{Re}\left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j \log \frac{z_i z_j [f(z_i) - f(z_j)]}{f(z_i) f(z_j)(z_i - z_j)} \right\} \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \overline{\lambda_j} \log \frac{1}{1 - z_i \overline{z_j}} - \left[\operatorname{Re}\left\{ \sum_{i=1}^{N} \lambda_i \log \frac{f(z_i)}{k(z_i)} \right\} \right]^2 \left(\log \frac{1}{\alpha} \right)^{-1} \tag{5}$$

holds for N, for all $\lambda_n \in \mathbf{C}$, and for all systems of points $z_i \in \mathbf{C}$. Here k denotes the Koebe function.

Proof. Using the expansion (1), we may express the left-hand side of the inequality (5) in the form

$$\operatorname{Re}\left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} z_i^n z_j^m \right\} = \operatorname{Re}\left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \sum_{i=1}^{N} \lambda_i z_i^n \sum_{j=1}^{N} \lambda_j z_j^m \right\}.$$

In view of the inequality (4), this has the upper bound

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{i=1}^{N} \lambda_i z_i^n \right|^2 - 4 \left(\log \frac{1}{\alpha} \right)^{-1} \left[\operatorname{Re} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{N} \lambda_i z_i^n \right) \left(\gamma_n - \frac{1}{n} \right) \right]^2$$
$$= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^{N} \lambda_i z_i^n \sum_{j=1}^{N} \overline{\lambda_j} \overline{z_j^n} - 4 \left(\log \frac{1}{\alpha} \right)^{-1} \left[\operatorname{Re} \sum_{i=1}^{N} \lambda_i \sum_{n=1}^{\infty} \left(\gamma_n - \frac{1}{n} \right) z_i^n \right]^2$$

which is equal to the right-hand side of (5). The estimate is sharp for each choice of N, $\lambda_n \in \mathbf{C}$, and $z_i \in \mathbf{D}$, since it is derived by appeal to the sharp inequality (4).

The inequality (5) is similar to a version of the Goluzin inequalities with Hayman index previously obtained by Kamotskiĭ [**10**], but the two systems appear to be different. Kamotskiĭ does not discuss the question of sharpness.

By analogy with our deduction of the Bazilevich inequality from Theorem 1, we may combine Theorem 2 with the Goluzin inequalities to obtain

$$\left[\operatorname{Re}\left\{ \sum_{i=1}^{N} \lambda_i \log \frac{f(z_i)}{k(z_i)} \right\} \right]^2 \leq 2 \log \frac{1}{\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \overline{\lambda_j} \log \frac{1}{1 - z_i \overline{z_j}}.$$

In particular,

$$\left[\sum_{i=1}^{N}\left|\log\frac{f(z_i)}{k(z_i)}\right|\right]^2 \le 2\log\frac{1}{\alpha}\sum_{i=1}^{N}\sum_{j=1}^{N}\log\frac{1}{1-z_i\bar{z}_j}. \tag{6}$$

This shows that as α tends to 1 the values of f approach those of the Koebe function.

It seems likely that as α tends to 1 the "sharpening terms" on the right-hand sides of the inequalities (4) and (5) actually approach zero uniformly for all $f \in S$ with index α and maximal growth direction $e^{i\theta} = 1$, at least for each given set of complex parameters λ_n. More specifically, the uniform estimate (for each fixed n)

$$\left|\gamma_n - \frac{1}{n}\right|^2 = o\left(\log\frac{1}{\alpha}\right), \qquad \alpha \to 1, \tag{7}$$

and the uniform estimate (for each fixed $z \in \mathbf{D}$)

$$\left|\log\frac{f(z)}{k(z)}\right|^2 = o\left(\log\frac{1}{\alpha}\right), \qquad \alpha \to 1, \tag{8}$$

seem quite plausible. If "o" were replaced by "O", the estimate (7) would follow at once from the Bazilevich inequality (3), while (8) would follow from (6).

In this connection the example

$$f(z) = \frac{z + (\alpha - 1)z^2}{(1 - z)^2}, \qquad 0 < \alpha < 1,$$

is instructive. This function f belongs to S, has Hayman index α, and maps $\mathbf{D}$ onto the complement of a half-line. Its logarithmic coefficients are

$$\gamma_n = \frac{1}{n} - (1 - \alpha)^n \frac{1}{2n}, \qquad n = 1, 2, \ldots.$$

Note that f satisfies (7) and (8) as α tends to 1.

2. Faber polynomials. Before passing to the proof of Theorem 1, we recall some facts about the *Faber polynomials* of a function $f \in S$. These are the monic polynomials $F_n(w) = w^n + \cdots$ with $F_n(0) = 0$ determined by the relation

$$F_n(1/f(z)) = z^{-n} + \sum_{m=0}^{\infty} \beta_{nm} z^m, \qquad n = 1, 2, \ldots. \tag{9}$$

We claim that the Faber polynomials are generated by

$$\log[1 - wf(z)] = -\sum_{n=1}^{\infty} \frac{1}{n} F_n(w) z^n. \tag{10}$$

To prove this, we write

$$\begin{aligned} \log[1 - wf(z)] &= -\sum_{n=1}^{\infty} \frac{1}{n} w^n [f(z)]^n \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} P_n(w) z^n, \end{aligned}$$

where P_n is a monic polynomial of degree n without constant term. Now write

$$\frac{f(z) - f(\zeta)}{z - \zeta} = \frac{f(\zeta)}{\zeta} \cdot \frac{1 - f(z)/f(\zeta)}{1 - z/\zeta}$$

and take logarithms to obtain

$$\begin{aligned} &-\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} z^n \zeta^m + \log \frac{f(z)}{z} \\ &\qquad = -\sum_{n=1}^{\infty} \frac{1}{n} P_n(1/f(\zeta)) z^n + \sum_{n=1}^{\infty} \frac{1}{n} z^n \zeta^{-n}. \end{aligned}$$

Comparing coefficients of z^n, we have

$$P_n(1/f(\zeta)) = \zeta^{-n} - 2n\gamma_n + n \sum_{m=1}^{\infty} c_{nm} \zeta^m.$$

Thus P_n has the characteristic property (9) of the Faber polynomial F_n, and so $P_n = F_n$. This proves (10).

Differentiation of (10) with respect to w gives

$$\frac{f(z)}{1 - wf(z)} = \sum_{n=1}^{\infty} Q_n(w) z^n, \tag{11}$$

where for convenience we define

$$Q_n(w) = \frac{1}{n} F_n'(w) = w^{n-1} + \cdots.$$

3. Boundary variation. The proof of Theorem 1 is based on the construction of an extremal problem whose associated quadratic differential is a perfect square. Integration is then a simple matter. This approach generalizes the variational proof of the Grunsky inequalities previously found by one of the authors ([**12**, **4**]; see [**9**], §12).

For fixed N and fixed complex parameters λ_n, we define the functional

$$\phi(f) = \sum_{n=0}^{N} \sum_{m=0}^{N} c_{nm}\lambda_n\lambda_m + \lambda_0^2 \log \alpha, \tag{12}$$

where f is a function in S with Grunsky coefficients c_{nm} and Hayman index α. It will be assumed that λ_0 is real. Observe that ϕ involves the logarithmic coefficients $\gamma_n = -\frac{1}{2}c_{n0}$ of f.

Now let f be a function which maximizes $\mathrm{Re}\{\phi\}$ over S. Clearly, the extremal function must have positive index α. Without loss of generality, we may assume that f has $e^{i\theta} = 1$ as its direction of maximal growth. This may be achieved by a rotation, which simply rotates the Grunsky coefficients.

In order to describe the extremal function f, we construct a *boundary variation*

$$f^* = f + a\rho^2 \frac{f^2}{w_0^2(f - w_0)} + O(\rho^3), \qquad \rho \to 0, \tag{13}$$

with respect to an omitted point $w_0 \notin f(\mathbf{D})$. Then $f^* \in S$ and so $\mathrm{Re}\{\phi(f^*)\} \le \mathrm{Re}\{\phi(f)\}$. Let c_{nm}^* denote the Grunsky coefficients of f^*. We begin with the calculation of an asymptotic formula for c_{nm}^*.

Observe that the variational formula (13) gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (c_{nm} - c_{nm}^*) z^n \zeta^m \\
&\quad = \log \frac{f^*(z) - f^*(\zeta)}{z - \zeta} - \log \frac{f(z) + f(\zeta)}{z - \zeta} = \log \frac{f^*(z) - f^*(\zeta)}{f(z) - f(\zeta)} \\
&\quad = \log\left\{1 + \frac{a\rho^2}{w_0^2[f(z) - f(\zeta)]}\left(\frac{f(z)^2}{f(z) - w_0} - \frac{f(\zeta)^2}{f(\zeta) - w_0}\right) + O(\rho^3)\right\} \\
&\quad = \frac{a\rho^2}{w_0^2}\left\{1 - \left[1 - \frac{f(z)}{w_0}\right]^{-1}\left[1 - \frac{f(\zeta)}{w_0}\right]^{-1}\right\} + O(\rho^3).
\end{aligned}$$

On the other hand, the generating relation (11) gives

$$\frac{1}{1 - wf(z)} = 1 + \frac{wf(z)}{1 - wf(z)} = 1 + w \sum_{n=1}^{\infty} Q_n(w) z^n,$$

and so

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(c_{nm}-c_{nm}^{*}\right)z^{n}\zeta^{m}$$
$$=\frac{a\rho^2}{w_0^2}\left\{1-\left[1+\frac{1}{w_0}\sum_{n=1}^{\infty}Q_n\left(\frac{1}{w_0}\right)z^n\right]\left[1+\frac{1}{w_0}\sum_{m=1}^{\infty}Q_m\left(\frac{1}{w_0}\right)\zeta^m\right]\right\}$$
$$+O(\rho^3).$$

Comparison of coefficients gives

$$c_{nm}^{*}=c_{nm}+a\rho^2 w_0^{-4}Q_n(1/w_0)Q_m(1/w_0)+O(\rho^3), \qquad n,m=1,2,\ldots; \tag{14}$$
$$c_{n0}^{*}=c_{n0}+a\rho^2 w_0^{-3}Q_n(1/w_0)+Q(\rho^3), \qquad n=1,2,\ldots.$$

Next we consider the variation of the Hayman index. Observe first that the variation (13) preserves the direction of maximal growth. Thus the index of f^* is

$$\begin{aligned}\alpha^* &= \lim_{r\to1}(1-r)^2|f^*(r)|\\ &= \lim_{r\to1}(1-r)^2|f(r)|\left|1+a\rho^2\frac{f(r)}{w_0^2(f(r)-w_0)}+O(\rho^3)\right|\\ &= \alpha\left[1+\lim_{r\to1}\operatorname{Re}\left\{\frac{a\rho^2}{w_0^2}\frac{f(r)}{f(r)-w_0}+O(\rho^3)\right\}\right]\\ &= \alpha\left[1+\operatorname{Re}\{a\rho^2 w_0^{-2}+O(\rho^3)\}\right].\end{aligned}$$

It follows that

$$\log\alpha^*=\log\alpha+\operatorname{Re}\{a\rho^2 w_0^{-2}\}+O(\rho^3). \tag{15}$$

Introducing the variational formulas (14) and (15) into the functional (12), we obtain

$$\begin{aligned}\phi(f^*) &= \sum_{n=1}^{N}\sum_{m=1}^{N}c_{nm}^{*}\lambda_n\lambda_m+2\lambda_0\sum_{n=1}^{N}c_{n0}^{*}\lambda_n+\lambda_0^2\log\alpha^*\\ &= \phi(f)+a\rho^2 w_0^{-4}\sum_{n=1}^{N}\sum_{m=1}^{N}Q_n(1/w_0)Q_m(1/w_0)\lambda_n\lambda_m\\ &\quad+2\lambda_0 a\rho^2 w_0^{-3}\sum_{n=1}^{N}Q_n(1/w_0)\lambda_n+\lambda_0^2\operatorname{Re}\{a\rho^2 w_0^{-2}\}+O(\rho^3)\\ &= \phi(f)+a\rho^2\left[w_0^{-2}\sum_{n=1}^{N}Q_n(1/w_0)\lambda_n+\lambda_0 w_0^{-1}\right]^2\\ &\quad-a\rho^2\lambda_0^2 w_0^{-2}+\lambda_0^2\operatorname{Re}\{a\rho^2 w_0^{-2}\}+O(\rho^3).\end{aligned}$$

Bearing in mind that λ_0 is real, we therefore deduce from the inequality $\mathrm{Re}\{\phi(f^*)\} \le \mathrm{Re}\{\phi(f)\}$ that

$$\mathrm{Re}\{a\rho^2 s(w_0) + O(\rho^3)\} \le 0, \tag{16}$$

where

$$s(w) = \left[w^{-2}\sum_{n=1}^{N} Q_n(1/w)\lambda_n + \lambda_0 w^{-1}\right]^2. \tag{17}$$

We now appeal to the basic lemma of the method of boundary variation ([**11**]; see [**3**], §10.3) to conclude from (16) that the extremal function f maps the disk onto the complement of a system Γ of analytic arcs which satisfy

$$s(w)\,dw^2 > 0. \tag{18}$$

Because $s(w)$ is a perfect square, the differential equation (18) is readily integrated. Taking the square-root, we obtain from (17) and (18) with a suitable parametrization

$$\left[w^{-2}\sum_{n=1}^{N} Q_n(1/w)\lambda_n + \lambda_0 w^{-1}\right]\frac{dw}{dt} = 1. \tag{19}$$

Recalling the definition $Q_n(w) = 1/n\,F_n'(w)$ and noting that

$$\frac{d}{dw}F_n(1/w) = -\frac{1}{w^2}F_n'(1/w),$$

we may express (19) in the form

$$\frac{d}{dt}\left\{\lambda_0 \log w - \sum_{n=1}^{N}\frac{1}{n}\lambda_n F_n(1/w)\right\} = 1.$$

Thus, on Γ we have

$$\lambda_0 \log w - \sum_{n=1}^{N}\frac{1}{n}\lambda_n F_n(1/w) = t + ic, \tag{20}$$

where c is a real constant.

We now define the function

$$G(z) = \lambda_0 \log\frac{f(z)}{k(z)} - \sum_{n=1}^{N}\frac{1}{n}\lambda_n F_n\left(\frac{1}{f(z)}\right) - (c + \lambda_0\pi)i, \tag{21}$$

where $k(z) = z(1-z)^{-2}$ is the Koebe function and the branch of the logarithm is chosen for which

$$\log\frac{f(z)}{k(z)} - \pi i = \log f(z) - \log|k(z)|$$

on $|z| = 1$. In view of (20), $G(z)$ is real for $|z| = 1$. It therefore follows by Schwarz reflection that G has a meromorphic continuation to the Riemann sphere with the symmetry property $G(1/\bar{z}) = \overline{G(z)}$. On the other hand, the defining property (9) of the Faber polynomials shows that

$$F_n(1/f(z)) - z^{-n}$$

is analytic in the closed disk $\overline{\mathbf{D}}$. Thus

$$G(z) = -\sum_{n=1}^{N} \frac{1}{n}\lambda_n z^{-n} - \sum_{n=1}^{N} \frac{1}{n}\overline{\lambda_n} z^n + A, \tag{22}$$

where A is a real constant. In order to determine A, we let $z = r$ tend to 1 in (22) to get

$$\lim_{r\to 1} G(r) = -2\,\mathrm{Re}\left\{\sum_{n=1}^{N} \frac{1}{n}\lambda_n\right\} + A.$$

Comparing this with (21) and remembering that $F_n(0) = 0$, we see that

$$\lambda_0 \lim_{r\to 1} \log(1-r)^2 f(r) = A + (c + \lambda_0\pi)i - 2\,\mathrm{Re}\left\{\sum_{n=1}^{N} \frac{1}{n}\lambda_n\right\}.$$

In particular,

$$A = \lambda_0 \log\alpha + 2\,\mathrm{Re}\left\{\sum_{n=1}^{N} \frac{1}{n}\lambda_n\right\}. \tag{23}$$

The next step is to compare coefficients in the two forms (21) and (22) of $G(z)$. Recall first that the Faber polynomials have the property

$$F_n\left(\frac{1}{f(z)}\right) = z^{-n} - 2n\gamma_n + n\sum_{m=1}^{\infty} c_{nm} z^m.$$

Thus (21) takes the form

$$\begin{aligned} G(z) = 2\lambda_0 \sum_{n=1}^{\infty}\left(\gamma_n - \frac{1}{n}\right)z^n - \sum_{n=1}^{N}\frac{1}{n}\lambda_n z^{-n} \\ +2\sum_{n=1}^{N}\lambda_n\gamma_n - \sum_{n=1}^{N}\lambda_n\sum_{m=1}^{\infty} c_{nm}z^m - (c+\lambda_0\pi)i. \end{aligned}$$

Comparison with (22) gives

$$2\sum_{n=1}^{N}\lambda_n\gamma_n = (c+\lambda_0\pi)i + A \tag{24}$$

and

$$2\lambda_0\left(\gamma_m - \frac{1}{m}\right) - \sum_{n=1}^{N}\lambda_n c_{nm} = \begin{cases} -\frac{1}{m}\overline{\lambda_m}, & m = 1,2,\ldots,N; \\ 0, & m > N. \end{cases} \tag{25}$$

Now multiply (24) by λ_0 and (25) by λ_m, $1 \le m \le N$, and add the equations:

$$2\lambda_0 \sum_{n=1}^{N} \lambda_n \gamma_n + 2\lambda_0 \sum_{n=1}^{N} \lambda_m\left(\gamma_m - \frac{1}{m}\right) - \sum_{n=1}^{N}\sum_{m=1}^{N} \lambda_n \lambda_m c_{nm}$$
$$= \lambda_0(c + \lambda_0 \pi)i + \lambda_0 A - \sum_{m=1}^{N} \frac{1}{m}|\lambda_m|^2.$$

Using the relation (23) for A, we conclude that

$$(26)\quad \operatorname{Re}\left\{\sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\lambda_n\lambda_m\right\}$$
$$= \sum_{n=1}^{N} \frac{1}{n}|\lambda_n|^2 + 4\lambda_0 \operatorname{Re}\left\{\sum_{n=1}^{N} \lambda_n\left(\gamma_n - \frac{1}{n}\right)\right\} - \lambda_0^2 \log \alpha.$$

In terms of the functional ϕ defined by (12), we may write (26) in the form

$$\operatorname{Re}\{\phi(f)\} = \sum_{n=1}^{N} \frac{1}{n}|\lambda_n|^2 - 4\lambda_0 \operatorname{Re}\left\{\sum_{n=1}^{N} \frac{1}{n}\lambda_n\right\}.$$

But this equation holds for a function $f \in S$ which *maximizes* $\operatorname{Re}\{\phi\}$. Thus for arbitrary functions $f \in S$ with Hayman index $\alpha > 0$ and direction of maximal growth $e^{i\theta} = 1$ we have the sharp inequality

$$(27)\quad \operatorname{Re}\left\{\sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\lambda_n\lambda_m\right\}$$
$$\le \sum_{n=1}^{N} \frac{1}{n}|\lambda_n|^2 + 4\lambda_0 \operatorname{Re}\left\{\sum_{n=1}^{N} \lambda_n\left(\gamma_n - \frac{1}{n}\right)\right\} - \lambda_0^2 \log \alpha.$$

The right-hand side of (27) is a quadratic expression in λ_0. Choosing

$$(28)\qquad \lambda_0 = \frac{2}{\log \alpha} \operatorname{Re}\left\{\sum_{n=1}^{N} \lambda_n\left(\gamma_n - \frac{1}{n}\right)\right\}$$

to minimize the bound, we obtain the inequality (4). It may be remarked that the two conditions (23) and (24) for the extremal functions of (27) can also be combined to give the relation (28).

4. Extremal functions. Having proved the inequality (4), we now turn to the problem of describing the extremal functions. According to equation (20), each function $f \in S$ which maximizes $\operatorname{Re}\{\phi\}$ must map the disk onto the complement of a system of arcs described by a condition of the form

$$\operatorname{Im}\left\{\lambda_0 \log w - \sum_{n=1}^{N} \frac{1}{n}\lambda_n F_n(1/w)\right\} = \text{const.},$$

where F_n are the Faber polynomials of f.

Conversely, suppose a function $f \in S$ has Hayman index $\alpha > 0$ and direction of maximal growth $e^{i\theta} = 1$, and that f maps $\mathbf{D}$ onto $\mathbf{C} - \Gamma$, where the points $w \in \Gamma$ satisfy

$$\operatorname{Im}\{\lambda_0 \log w + P(1/w)\} = 0 \tag{29}$$

for some $\lambda_0 \in \mathbf{R}$ and some polynomial P. In terms of the Faber polynomials F_n of f, we can find a (unique) set of complex constants μ, $\lambda_1, \ldots, \lambda_N$ such that

$$P(\zeta) = -\mu - \sum_{n=1}^{N} \frac{1}{n}\lambda_n F_n(\zeta).$$

Thus the condition (29) says that for a suitable real constant C the function

$$G(z) = \lambda_0 \log \frac{f(z)}{k(z)} - \sum_{n=1}^{N} \frac{1}{n}\lambda_n F_n(1/f(z)) + iC$$

is real on the unit circle $|z| = 1$.

We now conclude as in the proof of Theorem 1 that G has the form (22) with the constant A given by (23). From this it follows as before that f satisfies the equation (26). To summarize, we have shown that if an admissible function $f \in S$ maps the disk onto the complement of a system of arcs with the property (29) for some polynomial P, then f maximizes $\operatorname{Re}\{\phi\}$ for some choice of the parameters λ_n.

One final remark will now be made concerning the sharpness of the basic inequality (27). As we observed at the end of §3, the extremal functions for the inequality (27) must satisfy (28). Applying the Cauchy-Schwarz inequality to (28), we find

$$\lambda_0^2 \left(\log \frac{1}{\alpha}\right)^2 \le 4 \sum_{n=1}^{N} \frac{1}{n}|\lambda_n|^2 \sum_{n=1}^{N} n\left|\gamma_n - \frac{1}{n}\right|^2.$$

Thus the Bazilevich inequality (3) gives

$$\lambda_0^2 \log \frac{1}{\alpha} \le 2 \sum_{n=1}^{N} \frac{1}{n}|\lambda_n|^2,$$

or

$$\alpha \geq \exp\left\{-2\lambda_0^{-2}\sum_{n=1}^{N}\frac{1}{n}|\lambda_n|^2\right\}. \tag{30}$$

This shows that the inequality (27) is not sharp if α is chosen smaller than the lower bound in (30).

References

[1] I. E. Bazilevich, *On the dispersion of coefficients of univalent functions*, Mat. Sb., **68** (110) (1965), 549–560 (in Russian) = Amer. Math. Soc. Transl., (2) **71** (1968), 168–180.

[2] ______, *On a univalence criterion for regular functions and the dispersion of their coefficients*, Mat. Sb., **74** (116) (1967), 133–146 (in Russian) = Math. USSR-Sb. **3** (1967), 123–137.

[3] P.L. Duren, *Univalent Functions* (Springer-Verlag, Heidelberg and New York, 1983).

[4] P. R. Garabedian and M. Schiffer, *The local maximum theorem for the coefficients of univalent functions*, Arch. Rational Mech. Anal., **26** (1967), 1–32.

[5] G. M. Goluzin, *A method of variation in conformal mapping*, II, Mat. Sb., **21** (63) (1947), 83–117. (in Russian)

[6] H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen*, Math. Z., **45** (1939), 29–61.

[7] W. K. Hayman, *The asymptotic behaviour of p-valent functions*, Proc. London Math. Soc., **5** (1955), 257–284.

[8] ______, *Multivalent Functions*, (Cambridge University Press, 1958).

[9] J. A. Hummel, *Lectures on Variational Methods in the Theory of Univalent Functions* (Lecture Notes, Univ. of Maryland, 1972).

[10] V. I. Kamotskiĭ, *Application of the area theorem for investigation of the class $S(\alpha)$*, Mat. Zametki, **28** (1980), 695–706 (in Russian) = Math. Notes **28** (1980), 803–808.

[11] M. Schiffer, *A method of variation within the family of simple functions*, Proc. London Math. Soc., **44** (1938), 432–449.

[12] ______, *Faber polynomials in the theory of univalent functions*, Bull. Amer. Math. Soc., **54** (1948), 503–517.

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