PACIFIC JOURNAL OF MATHEMATICS
Vol. 135, No. 1, 1988

SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH NONLINEAR VISCOSITY

AVNER FRIEDMAN AND JINDRICH NECAS

An equation of the form

$$\ddot{u}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial W(p)}{\partial p_i}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial V(q)}{\partial q_i}=f$$

where $p=\nabla u$, $q=\nabla\dot{u}$, $\dot{u}=\partial u/\partial t$, $\ddot{u}=\partial^2 u/\partial t^2$ represents, for suitable functions $W(p)$, $V(q)$, a nonlinear hyperbolic equation with nonlinear viscosity and it appears in models of nonlinear elasticity. In this paper existence and regularity of solutions for the Cauchy problem will be established. In particular, if $n=2$, or if $n\geq 3$ and the eigenvalues of $(\partial^2 V/\partial q_j\partial q_j)$ belong to a "small" interval, then the solution is classical. These results will actually be established for a system of equations of the above type.

Introduction. Consider a system of N nonlinear equations

$$\ddot{u}_k-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial W(p)}{\partial p_{ki}}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial V(q)}{\partial q_{ki}}=f_k\qquad(1\leq k\leq N)\tag{0.1}$$

in a cylinder $\Omega\times(0,\infty)$, with initial data

$$u_k(x,0)=u_{k0}(x),\quad \dot{u}_k(x,0)=u_{k1}(x)\tag{0.2}$$

and boundary conditions

$$u=0\quad\text{if } x\in\partial\Omega,\ t>0;\tag{0.3}$$

here Ω is a bounded domain in $\mathbf{R}^n$,

$$p=(p_{li}),\quad q=(q_{li})\quad\text{and}$$
$$p_{li}=\frac{\partial u_l}{\partial x_i},\quad q_{li}=\frac{\partial \dot{u}_l}{\partial x_i},\quad \dot{w}=\frac{\partial w}{\partial t}.$$

The special case

$$N=1,\quad \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial V(q)}{\partial q_{ki}}=\Delta\dot{u}\qquad(k=1)\tag{0.4}$$

has been studied by several authors. For $n=1$, existence and uniqueness of a classical solution was established in [1], [2], [6], [7]. For general n Engler [3] has recently established the existence of a strong solution when $W(p)$ is a general nonlinear isotropic function, that is, $\nabla W(p) = g(|p|^2)p$, and $g(s)$ satisfies:

$$-\frac{1}{2} < k_0 \leq \frac{sg'(s)}{g(s)} \leq k_1 < \infty; \tag{0.5}$$

in case $g(s)$ is globally Lipschitz, the solution is classical. His results depend in a crucial way on the assumptions in (0.4) (especially the second one). For earlier work on weak solutions see also the references given in [3].

For $n = 2$ Petcher [10] established the smoothness of the weak solutions, in case (0.4) and $\partial W(p)/\partial p_i = \sigma_i(p_i)$, $\sigma_i'(s) \geq 0$, $\sigma_i'(s) \geq c_1|s|^2$ if $|s| \geq 1$, $\sigma_i''(s) \leq c_2|s|$ if $|s| \geq 1$ and c_1, c_2 are positive constants.

Systems of the form (0.1)–(0.3) may be considered to represent models of wave propagation in elastic material with nonlinear Hook's Law (corresponding to the internal energy function W) and nonlinear viscosity (corresponding to V). In case (0.4), when $n = 1$ the equation models simple shear motion of a beam (see [6]), and when $n = 2$ it models antiplane shear motion of a column with cross section Ω (see [4], [8]); u is the displacement from the rest position. Nonlinear viscosity terms appear in various models of elasticity; see [11].

In this paper we consider (0.1)–(0.3) with both $W(p)$ and $V(q)$ nonlinear functions. In §§1–5 we assume that

(0.6) $\left(\frac{\partial^2 W(p)}{\partial p_{ki}\partial p_{lj}}\right)$ and $\left(\frac{\partial^2 (V(q))}{\partial q_{ki}\partial q_{lj}}\right)$ are uniformly positive definite and bounded matrices.

It is well known that under these conditions there exists a unique global weak solution; our interest is to derive regularity of the solution. In fact we prove (in §5) that the solution is classical if either $n \leq 2$ or if $n \geq 3$ and the eigenvalues of the second matrix in (0.6) lie in an interval (λ_1, λ_2) with $\lambda_2 - \lambda_1$ small enough.

Our proof is based on establishing estimates on

$$\|\nabla \ddot{u}\|_{L^s(Q_T)} \qquad \text{(in §2)},$$
$$\|\nabla^2 \dot{u}\|_{L_2(Q_T)} \qquad \text{(in §3)}, \qquad \|\nabla^2 \dot{u}\|_{L^s(Q_T)} \qquad \text{(in §4)}$$

where $Q_T = \Omega \times (0, T)$, and finally on

$$\|\nabla^2 \dot{u}\|_{L^\infty((0,T);L^s(\Omega))} \qquad \text{(in §5)},$$

for some $s > 2$, or for any $s > 2$ if $\lambda_2 - \lambda_1$ is small enough.

Finally in §6 we consider the case where

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial V(q)}{\partial q_{ki}} = \Delta \dot{u}_k,$$

but $W(p)$ is a general nonlinear function of p (not necessarily isotropic) satisfying nonuniform ellipticity condition, and extend the result of Engler [3] by proving the existence of a strong solution for this case.

1. Existence and uniqueness. Let N, n be positive integers and let $p = (p_1, \cdots, p_N)$, $q = (q_1, \cdots, q_N)$ where $p_k = (p_{k1}, \ldots, p_{kn})$, $q_k = (p_{k1}, \cdots, q_{kn})$ are variable points in $\mathbf{R}^n$. We are given two functions

$$W = W(p), \quad V = V(q) \quad \text{from } \mathbf{R}^{nN} \text{ into } \mathbf{R}^1,$$

satisfying the following conditions:

$$W \text{ and } V \text{ belong to } C^2(\mathbf{R}^{nN}), \tag{1.1}$$

$$\tilde{\lambda}_1 |\xi|^2 \leq \sum_{k,l=1}^{N} \sum_{i,j=1}^{n} \frac{\partial^2 W(p)}{\partial p_{ki} \partial p_{lj}} \xi_i^k \xi_j^l \leq \tilde{\lambda}_2 |\xi|^2 \qquad (0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 < \infty), \tag{1.2}$$

$$\lambda_1 |\xi|^2 \leq \sum_{k,l=1}^{N} \sum_{i,j=1}^{n} \frac{\partial^2 V(q)}{\partial q_{ki} \partial q_{lj}} \xi_i^k \xi_j^l \leq \lambda_2 |\xi|^2 \quad (0 < \lambda_1 < \lambda_2 < \infty) \tag{1.3}$$

for all p, q and $\mathbf{R}^{nN}$ and ξ_m^λ real, where

$$|\xi|^2 = \sum_{k=1}^{N} \sum_{i=1}^{n} (\xi_i^k)^2.$$

Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 boundary $\partial\Omega$; for any $T > 0$ we set

$$\Omega_T = \{(x, T); x \in \Omega\}, \quad Q_T = \{(x, t); x \in \Omega, \ 0 < t < T\}.$$

We write $\partial/\partial t$ also as "$\cdot$", i.e., $\dot{u} = \partial u/\partial t$, $\ddot{u} = \partial^2 u/\partial t^2$, etc.

Consider the system of N nonlinear partial differential equations

$$\ddot{u}_k - \sum_{i=1}^{n} \frac{\partial}{\partial p_{ki}} \frac{\partial W(p)}{\partial p_{ki}} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial V(q)}{\partial q_{ki}} = f_k \quad \text{in } Q_T \qquad (1 \leq k \leq N) \tag{1.4}$$

where $p_k = \nabla u_k$, $q_k = \nabla \dot{u}_k$ (thus $p_{ki} = \partial u_k/\partial x_i$, $q_{ki} = \partial \dot{u}_k/\partial x_i$), with initial conditions

$$u_k(x, 0) = u_{k0}(x), \qquad \dot{u}_k(x, 0) = u_{k1}(x) \quad \text{for } x \in \Omega \tag{1.5}$$

and boundary conditions

$$u_k(x,t) = 0 \quad \text{for } x \in \partial\Omega,\ 0 < t < T. \tag{1.6}$$

We assume that

$$\begin{gathered} f_k \in L^2(Q_T) \quad \forall T > 0, \\ u_{k0}, u_{k1} \text{ belong to } W_0^{1,2}(\Omega). \end{gathered} \tag{1.7}$$

DEFINITION 1.1. A function $u = (u_1, \ldots, u_N)$ is called a *weak solution* of (1.4)–(1.6) if

$$\begin{gathered} u \in C([0,T]; W_0^{1,2}(\Omega)), \\ \dot{u} \in L^2((0,T); W_0^{1,2}(\Omega)), \\ \ddot{u} \in L^2((0,T); W^{-1,2}(\Omega)) \end{gathered} \tag{1.8}$$

where $W^{-1,2}(\Omega)$ is the conjugate of $W_0^{1,2}(\Omega)$, if (1.5) is satisfied (observe that $\dot{u} \in C([0,T];\ W^{-1,2}(\Omega))$ and thus $\dot{u}_k(x,0)$ is well defined in the trace class $W^{-1,2}(\Omega)$), and if (1.4) is satisfied in the following sense:

$$\iint_{Q_T} \left[\ddot{u}_k \phi + \sum \frac{\partial W(p)}{\partial p_{ki}} \frac{\partial \phi}{\partial x_i} + \sum \frac{\partial V(q)}{\partial q_{ki}} \frac{\partial \phi}{\partial x_i} \right] = \iint_{Q_T} f_k \phi$$

for any $\phi \in L^2((0,T);\ W_0^{1,2}(\Omega))$.

THEOREM 1.1. *There exists a unique weak solution* $u = (u_1, \ldots, u_N)$ *of* (1.4)–(1.6); *further, if* u_{k0}, u_{k1} *belong to* $W^{2,2}(\Omega)$ *then*

$$u \in C^1((0,T); W_0^{1,2}(\Omega)). \tag{1.9}$$

Indeed, existence and uniqueness have been established, for instance, in general Hilbert space framework in [5; Chapter 7, Theorem 1.2], and (1.9) follows from [5; Chapter 7, Theorem 3.2].

By taking a sequence $T_m \uparrow \infty$ and the corresponding solutions $u = u_{T_m}$, and extracting a convergent subsequence, we obtain a solution u of (1.4)–(1.6) for all $T > 0$; by Theorem 1.1, the solution is unique.

We conclude this section with a conservation law. Multiplying (1.4) by $\dot{u}_k$, integrating over Q_t and summing over k, we get

$$\begin{aligned} \frac{1}{2} \sum \int_{\Omega_t} \dot{u}_k^2 - \frac{1}{2} \sum \int_{\Omega_0} \dot{u}_{k0}^2 + \iint_{Q_t} \sum \frac{\partial W(p)}{\partial p_{ki}} \frac{\partial p_{ki}}{\partial t} \\ + \iint_{Q_t} \sum \frac{\partial V(q)}{\partial q_{ki}} q_{ki} = \sum \iint_{Q_t} f_k \dot{u}_k, \end{aligned}$$

and the third integral on the left-hand side is equal to

$$\iint_{Q_t} \frac{d}{dt} W(p) = \int_{\Omega_t} W(p) - \int_{\Omega_0} W(p).$$

Since, by (1.2), (1.3),

$$\sum \frac{\partial V(q)}{\partial q_{ki}} q_{ki} \geq \frac{1}{2}\lambda_1 |q|^2 - C, \qquad W(p) \geq \frac{1}{2}\tilde{\lambda}_1 |p|^2 - C, \tag{1.10}$$

we obtain the estimate

$$\int_{\Omega_t} |\dot{u}|^2 + \int_{\Omega_t} |\nabla u|^2 + \iint_{Q_T} |\nabla \dot{u}|^2 \leq C \qquad (C = C(T)) \tag{1.11}$$

for $0 \leq t \leq T$.

In §§2–5 we assume that

$$W(p) \text{ and } V(q) \text{ belong to } C^3, \tag{1.12}$$

$$f_k \in W^{1,r}(Q_T) \quad \forall T > 0, \quad 1 < r < \infty, \tag{1.13}$$

$$u_{k0}, u_{k1} \text{ belong to } W_0^{2,2}\Omega) \cap W^{3,r}(\Omega) \quad \forall 1 < r < \infty, \tag{1.14}$$

and derive additional regularity results for the weak solution (which already satisfies (1.8), (1.9) for all $T > 0$); in particular, the solution will be shown to be classical in case $n \leq 2$.

2. $\nabla \ddot{u}$ is in L^s.

LEMMA 2.1. *Suppose v is a weak solution of*

$$\dot{v} - \gamma \Delta v = f_0 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \quad \textit{in } Q_T,$$

$v = 0$ on $\partial_p Q_T$, the parabolic boundary of Q_T, $f_i \in L^q(Q_T)$ for some $1 < q < \infty$ $(0 \leq i \leq n)$, $0 < \gamma_0 \leq \gamma \leq \gamma_1 < \infty$ (γ, γ_i constants). Then, for every $0 < t < T$,

$$\iint_{Q_t} \left\{ \sum_{i=1}^{n} \left| \frac{\partial v}{\partial x_i} \right|^q \right\}^{1/q} \leq C_q \iint_{Q_t} \left\{ \sum_{i=0}^{n} |f_i|^q \right\}^{1/q} \tag{2.1}$$

where C_q is a constant dependent only on Ω, γ_0, γ_1, T and q.

Proof. Consider first the case $\Omega = \{x_n > 0\}$. Let u_i be the solutions of

$$\dot{u}_i - \gamma \Delta u_i = f_i \quad \text{in } Q_T,$$

$$u_i = 0 \quad \text{on } \{x_n = 0\} \times (0, T) \text{ if } 0 \le i \le n-1,$$
$$\frac{\partial u_n}{\partial x_n} = 0 \quad \text{on } \{x_n = 0\} \times (0, T),$$
$$u_i(x, 0) = 0 \quad \text{if } x \in \Omega, \qquad 0 \le i \le n.$$

Then

$$v = u_0 + \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}.$$

Each u_i $(0 \le i \le n-1)$ can be represented by means of Green's function, and u_n can be represented by means of Neumann's function. Applying L^q parabolic estimates to the u_i [**9**; IV, §3] the assertion (2.1) follows. In order to extend (2.1) to general domains Ω, we use partition of unity and proceed as in the derivation of L^q estimates for parabolic equations in non-divergence form (see, e.g., [**9**]).

From now on we assume that (1.12)–(1.14) hold, and set

$$A_{kl}^{ij}(q) = \frac{\partial^2 V(q)}{\partial q_{ki} \partial q_{lj}}, \quad B_{kl}^{ij}(p) = \frac{\partial^2 W(p)}{\partial p_{ki} \partial p_{lj}}. \tag{2.2}$$

THEOREM 2.2. *For any $T > 0$ there exist constants $p_0 > 2$ and $C > 0$ such that*

$$\sum_{k=1}^{N} \left[\iint_{Q_T} |\nabla \ddot{u}_k|^s + \iint_{Q_T} |\nabla \dot{u}_k|^s \right] \le C \quad \textit{if } 2 \le s \le p_0. \tag{2.3}$$

Proof. We first proceed formally, assuming that the solution is smooth. Let $z_k = \ddot{u}_k$. Differentiating (1.4) in t we get

$$\dot{z}_k - \sum_{i,j} \frac{\partial}{\partial x_i} \left(A_{kl}^{ij}(q) \frac{\partial z_l}{\partial x_j} \right) = \dot{f}_k - \sum_{i,j} \frac{\partial}{\partial x_i} \left(B_{kl}^{ij}(p) \frac{\partial \dot{u}_l}{\partial x_j} \right), \tag{2.4}$$

and, by (1.12)–(1.14),

(2.5) $z_k(x, 0)$ belongs to $W_0^{1,2}(\Omega) \cap W^{1,r}(\Omega)$ for any $1 < r < \infty$.

Let

$$\gamma = \frac{\lambda_1 + \lambda_2}{2} \tag{2.6}$$

and rewrite (2.4) in the form

$$\dot{z}_k - \gamma \Delta z_k = -\sum \frac{\partial}{\partial x_i} (\gamma \delta_{ij} \delta_{kl} - A_{kl}^{ij}(q)) \frac{\partial z_l}{\partial x_j} + G_k \tag{2.7}$$

where

$$G_k = \dot{f}_k - \sum \frac{\partial}{\partial x_i}\left(B^{ij}_{kl}(p)\frac{\partial \dot{u}_l}{\partial x_j}\right). \tag{2.8}$$

In view of the choice (2.6), we have

$$-\frac{(\lambda_2-\lambda_1)}{2}|\xi|^2 \le \sum(\gamma\delta_{ij}\delta_{kl} - A^{ij}_{kl}(q))\xi^k_i\xi^l_j \le \frac{\lambda_2-\lambda_1}{2}|\xi|^2 \tag{2.9}$$

where $|\xi|^2 = \sum_{k,i}(\xi^k_i)^2$. Since $A^{ij}_{kl} = A^{ji}_{lk}$ by (2.2), the matrix $A_{\alpha,\beta} \equiv A^{ij}_{kl}$ where $\alpha = (i,k), \beta = (j,l)$ is symmetric, i.e., $A_{\alpha\beta} = A_{\beta\alpha}$. Hence (2.9) implies that for any real vectors $\xi = (\xi^k_i)$, $\zeta = (\zeta^k_i)$,

$$\left|\sum(\gamma\delta_{ij}\delta_{kl} - A^{ij}_{kl}(q)\xi^k_i\xi^l_j)\right| \le \tfrac{1}{2}(\lambda_2-\lambda_1)|\xi|\,|\zeta|. \tag{2.10}$$

Consider the linear parabolic system

$$\begin{aligned}\dot{z}_k - \gamma\Delta z_k = &-\sum\frac{\partial}{\partial x_i}[(\gamma\delta_{ij}\delta_{kl} - A^{ij}_{kl}(q))h_{lj}] \\ &+\sum\frac{\partial}{\partial x_j}g_{kj} + g_{k0} \quad \text{in } Q_T,\end{aligned} \tag{2.11}$$

$$z_k = 0 \quad \text{on } \partial_p Q_T \tag{2.12}$$

where $q = \nabla\dot{u}$, and $G \equiv (h_{lj}, g_{ki})$ ($1 \le l,k \le N$ and $1 \le j \le n$, $0 \le i \le n$) is any vector with components in $L^s(Q_T)$. We introduce the norm

$$\|G\|_s = \left\{\iint_{Q_T}\left[\sum|h_{lj}|^s + M\sum|g_{ki}|^s\right]\right\}^{1/s} \tag{2.13}$$

where M is a positive constant to be determined.

For any G, there is a unique solution $z = (z_1, \dots, z_N)$ of (2.11), (2.12). We denote the vector $(\nabla z_1, \dots, \nabla z_N)$ by ∇z and set $\nabla z = SG$; we also define the norm

$$\|\nabla z\|_s = \left\{\iint_{Q_T}\sum|\nabla z_k|^s\right\}^{1/s}. \tag{2.14}$$

Then S is a linear mapping from L^s into L^s (with the norms defined by (2.13), (2.14)), and by Lemma 2.1

$$\|S\|_s \le A_s < \infty \tag{2.15}$$

where A_s is a constant depending on s, as well as on λ_1, λ_2, T and Ω.

We claim that

$$\|S\|_2 \le \frac{\lambda_2-\lambda_1}{\lambda_2+\lambda_1} < 1. \tag{2.16}$$

Indeed, if we multiply (2.11) by z_k and integrate over Ω_t, we get, after summing on k,

$$\int_{\Omega_T} \sum |z_k|^2 + \gamma \iint_{Q_T} \sum |\nabla z_k|^2$$
$$= \iint_{Q_T} \sum (\gamma \delta_{ij}\delta_{kl} - A_{kl}^{ij}(q)) h_{lj} \frac{\partial z_k}{\partial x_i}$$
$$- \iint_{Q_T} \left(\sum_{k=1}^{N} \sum_{j=1}^{n} g_{kj} \frac{\partial z_k}{\partial x_j} \right) + \iint_{Q_T} \sum g_{k0} z_k.$$

In view of (2.10), the first integral on the right-hand side is bounded by

$$\frac{1}{2}(\lambda_2 - \lambda_2) \iint_{Q_T} \left(\sum |h_{lj}|^2 \right)^{1/2} \left(\sum \left| \frac{\partial z_k}{\partial x_i} \right|^2 \right)^{1/2}$$
$$= \frac{1}{4}(\lambda_2 - \lambda_1) \left\{ \iint_{Q_T} \sum |h_{lj}|^2 + \iint_{Q_T} |\nabla z_k|^2 \right\}.$$

Recalling (2.6) we obtain

$$\begin{aligned} &\iint_{Q_T} \left(\sum |\nabla z_k|^2 \right) \\ &\quad \le \frac{\lambda_2 - \lambda_1}{\lambda_2 + 3\lambda_1} \iint_{Q_T} \left(\sum |h_{lj}|^2 \right) \\ &\quad + C \left[\iint_{Q_T} \left(\sum_{k=1}^{N} \sum_{j=1}^{n} |g_{kj}|^2 \right)^2 \right]^{1/2} \left[\iint_{Q_T} \sum |\nabla z_k|^2 \right]^{1/2} \\ &\quad + C \left[\iint_{Q_T} \sum |g_{k0}|^2 \right]^{1/2} \left[\iint_{Q_T} \sum z_k^2 \right]^{1/2}. \end{aligned} \tag{2.17}$$

Since

$$\iint_{Q_T} |z_k|^2 \le C \iint_{Q_T} |\nabla z_k|^2,$$

the assertion (2.16) now follows provided M is chosen sufficiently large (depending on the constant C in (2.17)).

By the Riesz-Thorin theorem, $\log \|S\|_{1/\beta}$ is a convex function of β, $0 < \beta < 1$. Hence,

$$\|S\|_s \le \|S\|_2^\alpha \|S\|_{q_0}^{1-\alpha} \quad \text{if } \frac{1}{s} = \frac{1-\alpha}{q_0} + \frac{\alpha}{2}, \quad q_0 > 2, \quad 0 \le \alpha \le 1.$$

Using (2.15), (2.16) we conclude that

$$(2.18)\qquad \|S\|_s \le \left(\frac{\lambda_2-\lambda_1}{\lambda_2+\lambda_1}\right)^{\alpha}(A_{q_0})^{1-\alpha} \equiv \vartheta_s \quad \text{if } \frac{1}{s} = \frac{1-\alpha}{q_0} + \frac{\alpha}{2};$$

notice that $\vartheta_s < 1$ if $s-2$ is small enough.

In view of (2.7), (2.8), the functions $\tilde{z}_k(x,t) \equiv z_k(x,t) - z_k(x,0)$ satisfy a system of the form (2.11), (2.12) with

$$h_{lj} = \frac{\partial \tilde{z}_l}{\partial x_j}, \quad g_{kj} = -\sum B^{ij}_{kl}(p)\frac{\partial \dot{u}_l}{\partial x_j} \qquad (j>0),$$

$$g_{k0} = \dot{f}_k + \tilde{g}_k$$

and $\tilde{g}_k$ depending on the data $z_i(x,0)$. By (2.18),

$$\iint_{Q_t} |\nabla \tilde{z}|^s \le (\|S\|_s)^s \iint_{Q_t}\left\{\sum |h_{lj}|^s + M\sum |g_{ki}|^s\right\}$$
$$\le (\vartheta_s)^s \iint_{Q_t} |\nabla \tilde{z}|^s + (\vartheta_s)^s M \iint_{Q_t}\sum |g_{ki}|^s$$

where $\vartheta_s < 1$. Hence

$$\iint_{Q_t} |\nabla \tilde{z}|^s \le \frac{M}{1-(\vartheta_s)^s}\iint_{Q_t}\sum |g_{ki}|^s.$$

Recalling that $z_k = \ddot{u}_k$, we conclude that

$$(2.19)\qquad \iint_{Q_t}\sum |\nabla \ddot{u}_k|^s \le C + C\iint_{Q_t}\sum |\nabla \dot{u}_k|^s$$

provided $t = T$ and s is such that $\vartheta_s < 1$; this estimate is valid also for any $t \in (0,T)$, since A_{q_0} and ϑ_s are independent of t.

Set

$$\phi(t) = \iint_{Q_t}\sum |\nabla \ddot{u}_k|^s.$$

Since

$$\nabla \dot{u}_k(x,t) = \nabla \dot{u}_k(x,0) + \int_0^t \nabla \ddot{u}_k(x,\tau)\,d\tau,$$

we have

$$\iint_{Q_t} |\nabla \dot{u}_k|^s \le C + C\int_0^t \phi(\tau)\,d\tau.$$

Substituting this into (2.19), we get

$$\phi(t) \le C + C\int_0^t \phi(\tau)\,d\tau.$$

Hence, by Gronwall's inequality, $\phi(t) \le C$, and (2.3) follows.

So far we have assumed that the solution was smooth enough. In order to prove (2.3) rigorously, we approximate the initial values by smooth initial values, say u_{k0}^λ, u_{k1}^λ $(\lambda \to 0)$ in $C_0^2(\Omega)$. By a fixed point argument one can prove that (1.4)–(1.6) with initial data u_{k0}^λ, u_{k1}^λ has a classical solution in Q_σ if $\sigma = \sigma_\lambda$ is small enough; $\sigma_\lambda \to 0$ if $\lambda \to 0$. The unique weak solution u_k^λ asserted in Theorem 1.1 which corresponds to the data u_{k0}^λ, u_{k1}^λ must coincide with the classical solution for $t < \sigma_\lambda$, and, as $\lambda \to 0$,

$$u_k^\lambda \to u_k \quad \text{in } C^1([0,T]; W_0^{1,2}(\Omega)). \tag{2.20}$$

We now work with finite differences in time; that is, setting

$$z_k(x,t) = \Delta_h \dot{u}_k(x,t) \equiv \frac{\dot{u}_k(x,t+h) - \dot{u}_k(x,t)}{h}$$

and finite-differencing (1.4) in time, we obtain a system similar to (2.4) with A_{kl}^{ij} replaced by

$$\tilde{A}_{kl}^{ij} = \int_0^1 \frac{\partial^2 V}{\partial q_{ki} \partial q_{lj}} (\nabla \dot{u}(x,t) + \tau(\nabla \dot{u}(x,t+h) - \nabla \dot{u}(x,t)))\, d\tau$$

and similarly B_{kl}^{ij} replaced by $\tilde{B}_{kl}^{ij}$ and $\dot{f}_k$ replaced by $\Delta_h f_k$. Since u_k^λ is smooth in $\overline{Q}_{\sigma\lambda}$, (2.5) is satisfied.

Proceeding as before, we deduce that

$$\sum \iint_{Q_T} |\Delta_h(\nabla \dot{u}_k^\lambda)|^s \le C + C \sum \int_\Omega |\Delta_h \nabla \dot{u}_k^\lambda(x,0)|^s. \tag{2.21}$$

As $h \to 0$ the right-hand side converges to

$$C + C \sum \int_\Omega |\nabla \ddot{u}_k^\lambda(x,0)|^s. \tag{2.22}$$

Since, in view of (1.14), we can choose the approximations u_{k0}^λ, u_{kl}^λ such that the right-hand side of (2.22) is bounded independently of λ, it follows from (2.21) that

$$\sum \iint_{Q_T} |\nabla \ddot{u}_k^\lambda|^s \le C,$$

where C is a constant independent of λ. Letting $\lambda \to 0$ and recalling (2.20), the assertion (2.3) follows.

The proof of Theorem 2.2 yields:

COROLLARY 2.3. *For any* $2 < p_0 < q_0$, *if* $\lambda_2 - \lambda_1$ *is sufficiently small so that* $\vartheta_{p_0} < 1$ (ϑ_{p_0} *as in* (2.18)), *then* (2.3) *holds for all* $2 \le s \le p_0$.

3. $\nabla^2 u$ is in L^2.

LEMMA 3.1. *There exists a constant C such that*

$$\sum_{k=1}^{N}\left\{\iint_{Q_T}|\nabla^2\dot u_k|^2+\iint_{Q_T}|\nabla^2 u_k|^2\right\}\le C. \tag{3.1}$$

Proof. We first establish that $\nabla^2 u$ and $\nabla^2 \dot u$ belong to $L^2(\Omega_0\times(0,T))$ where Ω_0 is any compact subdomain of Ω. We shall begin be deriving a priori estimates, assuming for the moment that $\nabla^2 u$ and $\nabla^2\dot u$ are indeed in L^2.

Let B_R and $B_{R'}$, be two concentric balls with radii R and R' respectively, $R<R'$, and $\overline{B_{R'}}\subset\Omega$. Let η be a cut-off function: $\eta\in C_0^\infty(B_{R'})$, $\eta=1$ in B_R. Fix any integer m, $1\le m\le n$, and set $z_k=\partial\dot u_k/\partial x_m$. Differentiating (1.4) with respect to x_m we get

$$\dot z_k-\sum\frac{\partial}{\partial x_i}\left(A_{kl}^{ij}(q)\frac{\partial z_l}{\partial x_j}\right)=\frac{\partial f_k}{\partial x_m}+\sum\frac{\partial}{\partial x_i}\left(B_{kl}^{ij}(p)\frac{\partial}{\partial x_j}\frac{\partial u_l}{\partial x_m}\right). \tag{3.2}$$

The functions $Z_k=\eta z_k$ satisfy:

$$\begin{aligned}\dot Z_k-\sum_{i,j,l}\frac{\partial}{\partial x_j}\left(A_{kl}^{ij}(q)\frac{\partial Z_l}{\partial x_j}\right)\\ =\sum_{i,j,l}\frac{\partial}{\partial x_i}\left(B_{kl}^{ij}(p)\frac{\partial}{\partial x_j}\left(\eta\frac{\partial u_l}{\partial x_m}\right)\right)+F_k\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}F_k=\eta\frac{\partial F_k}{\partial x_m}&-\sum\frac{\eta_{x_i}}{\eta}A_{kl}^{ij}\frac{\partial Z_l}{\partial x_j}\\ &-\sum\frac{\partial}{\partial x_i}(A_{kl}^{ij}\eta_{x_j}z_l)\sum\frac{\eta_{x_i}\eta_{x_j}}{\eta}A_{kl}^{ij}z_l\\ &-\sum\frac{\eta_{x_i}}{\eta}B_{kl}^{ij}\frac{\partial}{\partial x_j}\left(\eta\frac{\partial u_l}{\partial x_m}\right)-\sum\frac{\partial}{\partial x_i}\left(B_{kl}^{ij}\eta_{x_j}\frac{\partial u_l}{\partial x_m}\right)\\ &+\sum\frac{\eta_{x_i}\eta_{x_j}}{\eta}B_{kl}^{ij}\frac{\partial u_l}{\partial x_m}.\end{aligned} \tag{3.4}$$

Multiplying (3.3) by Z_k and integrating over Q_t, we get, after summing over k,

$$\begin{aligned}\frac12\sum\int_{\Omega_t}Z_k^2-\frac12\sum\int_{\Omega_0}Z_k^2+\sum\iint_{Q_t}A_{kl}^{ij}\frac{\partial Z_l}{\partial x_j}\frac{\partial Z_k}{\partial x_i}\\ =\sum\iint_{Q_t}B_{kl}^{ij}\frac{\partial}{\partial x_j}\left(\eta\frac{\partial u_l}{\partial x_m}\right)\frac{\partial Z_k}{\partial x_j}+\sum\iint_{Q_t}F_kZ_k.\end{aligned} \tag{3.5}$$

Recalling that $Z_k = \eta z_k$ and using (1.11) and (3.4), we easily estimate

$$\left|\sum \iint_{Q_t} F_k Z_k\right| \le C + C\sum \iint_{Q_t} |\nabla Z|\,|z| + C\sum \iint_{Q_t} \left|\nabla\left(\eta \frac{\partial u_k}{\partial x_m}\right)\right|\,|z|$$

where $|z|^2 = \sum |z_k|^2$, $|\nabla Z|^2 = \sum |\nabla Z_k|^2$. Hence

$$(3.6)\quad \left|\sum \iint_{Q_t} F_k Z_k\right| \le C + C\left[\iint_{Q_t} |\nabla z|^2\right]^{1/2} + C\left[\sum_{k=1}^{N} \iint_{Q_t} \left|\nabla\left(\eta\frac{\partial u_k}{\partial x_m}\right)\right|^2\right]^{1/2}.$$

Substituting this into (3.5) and using (1.2), (1.3), we easily obtain, after using the Schwarz inequality,

$$(3.7)\qquad \sum \iint_{Q_t} |\nabla Z_k|^2 \le C + \sum \iint_{Q_t} \left|\nabla\left(\eta\frac{\partial u_k}{\partial x_m}\right)\right|^2.$$

Since

$$Z_k = \eta \frac{\partial \dot{u}_k}{\partial x_m} = \frac{\partial}{\partial t}\left(\frac{\partial u_k}{\partial x_m}\right),$$

we have

$$\begin{aligned}\left|\nabla\left(\eta\frac{\partial u_k}{\partial x_m}\right)\right|^2 &= \left|\nabla\left(\eta\frac{\partial u_{k0}}{\partial x_m}\right) + \int_0^t Z_k(x,s)\,ds\right|^2\\ &\le 2\left|\nabla\left(\eta\frac{\partial u_{k0}}{\partial x_m}\right)\right|^2 + 2T\int_0^t |Z_k(x,s)|^2\,ds.\end{aligned}$$

Substituting this into (3.7) and setting

$$\phi(t) = \sum \iint_{Q_t} |\nabla Z_k|^2,$$

we obtain

$$\phi(t) \le C + C\int_0^t \phi(s)\,ds.$$

Hence, by Gronwall's inequality,

$$(3.8)\qquad \sum_k \iint_{Q_T} \left|\nabla\left(\eta\frac{\partial \dot{u}_k}{\partial x_m}\right)\right|^2 \le C,$$

and then also

$$(3.9)\qquad \sum_k \iint_{Q_T} \left|\nabla\left(\eta\frac{\partial u_k}{\partial x_m}\right)\right|^2 \le C.$$

In order to prove (3.8), (3.9) rigorously, we work with finite differences, replacing $\partial u_k/\partial x_m$ by

$$(\Delta_m^k u_k)(x,t) = \frac{u_k(x+he_m,t) - u_k(x,t)}{h}$$

where e_m is the unit vector in the x_m-direction. Then, with $z_k = \Delta_m^h \dot{u}_k$, $Z_k = \eta z_k$, the relations (3.2)–(3.4) hold with minor differences, namely, $A_{kl}^{ij}(p)$ is replaced by

$$\tilde{A}_{kl}^{ij} = \int_0^1 \frac{\partial^2 V}{\partial q_{ki}\partial q_{lj}}(\nabla\dot{u}(x,t) + \tau(x+he_m,t) - \nabla\dot{u}(x,t))\,d\tau$$

and similarly for B_{kl}^{ij}, and $\partial f_k/\partial x_m$ is replaced by $\Delta_m^h f_k$; notice that ∇z_k and $\nabla \dot{u}_k$ belong to $L^2(Q_T)$ (by Theorem 2.2). We can now proceed as before (but this time rigorously), to establish analogously to (3.8), (3.9), that

$$\sum_k \left[\iint_{Q_T} |\nabla(\eta\Delta_m^h \dot{u}_k)|^2 + \iint_{Q_T} |\nabla(\eta\Delta_m^h u_k)|^2\right] \le C; \tag{3.10}$$

C is independent of h. By standard lemma in calculus it follows that, in $B_R \times (0,T)$ (where $\eta = 1$), the derivatives

$$\nabla\frac{\partial \dot{u}_k}{\partial x_m}, \quad \nabla\frac{\partial u_k}{\partial x_m}$$

exist and belong to $L^2(B_R \times (0,T))$, and

$$\sum_k \left[\int_0^T \int_{B_R} \left|\nabla\frac{\partial \dot{u}_k}{\partial x_m}\right|^2 + \int_0^T \int_{B_R} \left|\nabla\frac{\partial u_k}{\partial x_m}\right|^2\right] \le C; \tag{3.11}$$

this holds for every $1 \le m \le n$.

We next proceed to extend the interior estimates (3.11) to the boundary, replacing B_R by $B_R(x_0) \cap \Omega$ where $B_R(x_0) = \{x \in \mathbf{R}^n, |x - x_0| < R\}$ and x_0 is any point on $\partial\Omega$. Suppose $\partial\Omega \cap B_{R_0}(x_0)$ (for some $0 < R < R_0$) is given by $x_n = h(x_1, \ldots, x_{n-1})$ with $h \in C^2$, such that $x_n > h$ in $\Omega \cap B_{R_0}(x_0)$. Take for simplicity $x_0 = (0, \ldots, 0)$ and introduce new variables

$$x_i' = x_i \quad \text{if } 1 \le i \le n-1,$$
$$x_n' = x_n - h(x_1, \ldots, x_{n-1});$$

it will be convenient to write $x_i' = h_i(x)$. Setting

$$U_k(x', t) = u_k(x, t),$$

(1.4) becomes

$$\ddot{U}_k - \sum_{\lambda=1}^{n} \frac{\partial}{\partial x'_\lambda}\left(\frac{\partial h_\lambda}{\partial x_i}\frac{\partial W(P)}{\partial p_{ki}}\right) - \sum_{\lambda=1}^{n} \frac{\partial}{\partial x'_\lambda}\left(\frac{\partial h_\lambda}{\partial x_i}\frac{\partial V(Q)}{\partial q_{ki}}\right) = f_k \tag{3.12}$$

where $P = (P_1, \dots, P_N)$, $Q = (Q_1, \dots, Q_N)$, $P_k = (P_{k1}, \dots, P_{kn})$, $Q_k = (Q_{k1}, \dots, Q_{kn})$, and

$$P_{ki} = \sum_{\mu=1}^{n} \frac{\partial U_k}{\partial x'_\mu}\frac{\partial h_\mu}{\partial x_i}, \quad Q_{ki} = \sum_{\mu=1}^{n} \frac{\partial \dot{U}_k}{\partial x'_\mu}\frac{\partial h_\mu}{\partial x_i}.$$

As before we first proceed formally, differentiating (3.11) in any tangential direction x_m ($1 \le m \le n-1$). Setting $z_k = \partial\dot{U}/\partial x_m$, $Z_k = \eta z_k$ where $\eta(x')$ is a cut-off function $\eta \in C_0^\infty(B_{\rho'})$, $\eta = 1$ in B_ρ for some $0 < \rho < \rho'$, and defining

$$\tilde{A}_{kl}^{\lambda\mu} = \sum_{i,j=1}^{n} A_{kl}^{ij}\frac{\partial h_\lambda}{\partial x_i}\frac{\partial h_\mu}{\partial x_j}, \qquad \tilde{B}_{kl}^{\lambda\mu} = \sum_{i,j=1}^{n} B_{kl}^{ij}\frac{\partial h_\lambda}{\partial x_i}\frac{\partial h_\mu}{\partial x_j},$$

we find that

$$\begin{aligned} \dot{Z}_k - \sum_{i,j} \frac{\partial}{\partial x'_i}\left(\tilde{A}_{kl}^{ij}(Q)\frac{\partial Z_l}{\partial x'_j}\right) \\ = \sum_{i,j} \frac{\partial}{\partial x'_i}\left(\tilde{B}_{kl}^{ij}(P)\frac{\partial}{\partial x'_j}\left(\eta\frac{\partial U_l}{\partial x'_m}\right)\right) + \tilde{F}_k \end{aligned} \tag{3.13}$$

where $\tilde{F}_k$ is defined similarly to F_k in (3.4); the difference is that A_{kl}^{ij},B_{kl}^{ij} are replaced by $\tilde{A}_{kl}^{ij}$, $\tilde{B}_{kl}^{ij}$, that u_l is replaced by U_l, and that the variables x_j are replaced by the variables x'_j.

Multiplying (3.13) by Z_k and integrating over (x, t), and proceeding as before, we arrive at the analogs of (3.8), (3.9). If we work with finite differences Δ_m^h (instead of with $\partial/\partial x_m$), then we can establish rigorously the analogs of (3.10) and (3.11); thus

$$\nabla\frac{\partial}{\partial x'_m}\dot{U}_k \quad \text{and} \quad \nabla\frac{\partial}{\partial x'_m}U_k$$

belong to $L^2(B_\rho \times (0, T))$ and

$$\sum_k \int_0^t \int_{B_\rho} \left[\left|\nabla\frac{\partial}{\partial x'_m}\dot{U}_k\right|^2 + \left|\nabla\frac{\partial}{\partial x'_m}U_k\right|^2\right] \le C \tag{3.14}$$

$$(1 \le m \le n-1).$$

It remains to consider the normal derivatives $\partial(\nabla \dot{U}_k)/\partial x'_n$, $\partial(\nabla U_k)/\partial x'_n$. Since we have already proved that $\nabla^2 \dot{u}_k$, $\nabla^2 u_k$, $\nabla^2 u_k$ are locally in $L^2(Q_T)$, the same is true of $\nabla^2 \dot{U}_k$, $\nabla^2 U_k$. Hence we can apply the $\partial/\partial x'_\lambda$ differentiation to $\partial W/\partial p_{ki}$, $\partial V/\partial q_{ki}$ in (3.12), thus obtaining the relation

$$\ddot{U}_k - \sum \tilde{B}^{ij}_{kl} \frac{\partial^2 \dot{U}_l}{\partial x'_i \partial x'_j} - \sum \tilde{A}^{ij}_{kl} \frac{\partial^2 U_l}{\partial x'_i \partial x'_j} = f_k \quad \text{in } B_\rho \times (0, T).$$

In view of Theorem 2.2 and (3.14), we can estimate the $L^2(B_\rho \times (0,T))$-norm of $\ddot{U}_k$ and of

$$\frac{\partial^2 \dot{U}_l}{\partial x'_i \partial x'_j}, \quad \frac{\partial^2 U_l}{\partial x'_i \partial x'_j} \quad \text{for all } (i, j) \neq (n, n).$$

It follows that

$$\sum_l \tilde{B}^{nn}_{kl} \frac{\partial^2 \dot{U}_l}{\partial (x'_n)^2)} + \sum_l \tilde{A}^{nn}_{kl} \frac{\partial^2 U_l}{\partial (x'_n)^2} = \tilde{g}_k, \tag{3.15}$$

where

$$\sum \int_0^T \int_{B_\rho} \tilde{g}_k^2 \leq C. \tag{3.16}$$

By ellipticity, the matrix $(\tilde{B}^{nn}_{kl})^N_{k,l=1}$ is uniformly positive. Hence we can solve from (3.15),

$$\frac{\partial^2 \dot{U}_k}{\partial (x'_n)^2} = \sum_l a_{kl} \frac{\partial^2 U_l}{\partial (x'_n)^2} + \tilde{g}_k \tag{3.17}$$

where $\tilde{g}_k$ is another function, still satisfying (3.16), and a_{kl} are uniformly bounded functions. Setting

$$\phi(t) = \sum_k \int_0^t \int_{B_\rho} \left| \frac{\partial^2 \dot{U}_k}{\partial (x'_n)^2} \right|^2$$

we easily deduce from (3.17) that

$$\phi(t) \leq C + C^t \int_0 \phi(s)\, ds$$

and therefore

$$\sum_k \int_0^T \int_{B_\rho} \left[\left| \frac{\partial^2 \dot{U}_k}{\partial (x'_n)^2} \right|^2 + \left| \frac{\partial^2 U_k}{\partial (x'_n)^2} \right|^2 \right] \leq C.$$

Combining this with (3.14) we find, after going back to the variables x_i, that (3.11) holds with B_R replaced by $B_R(x_0) \cap \Omega$ (for R small enough), for all $1 \le m \le n$. This completes the proof of Theorem 3.1.

4. $\nabla^2 \dot{u}$ is in L^s. In this section we prove:

THEOREM 4.1. *For any $T > 0$ there exist constants $p_0 > 2$ and $C > 0$ such that*

$$\sum_{k=1}^{N} \left[\iint_{Q_T} |\nabla^2 \dot{u}_k|^s + \iint_{Q_T} |\nabla^2 u_k|^s \right] \le C \quad \text{if } 2 \le s \le p_0. \tag{4.1}$$

Proof. We first establish the assertion of the theorem in $\Omega_0 \times (0, T)$ where Ω_0 is any compact subdomain of Ω. We begin by taking any two concentric balls B_R, $B_{R'}$ with $R < R'$, $\overline{B_{R'}} \subset \Omega$ as in the proof of Theorem 3.1, and introducing the functions $z_k = \partial \dot{u}_k / \partial x_m$, $Z_k = \eta z_k$ as before. We assume for the moment that ∇z_k is in $L^s(Q_T)$. By (3.3), (3.4) we have,

$$\dot{Z}_k - \gamma \Delta Z_k = -\sum_{i,j,l} \frac{\partial}{\partial x_i} \left((\gamma \delta_{ij} \delta_{kl} - A_{kl}^{ij}(q)) \frac{\partial Z_l}{\partial x_j} \right) + G_k \tag{4.2}$$

where γ is as in (2.6) and

$$\begin{aligned} G_k = {} & \sum \frac{\partial}{\partial x_j} \left(B_{kl}^{ij}(p) \frac{\partial}{\partial x_j} \left(\eta \frac{\partial u_l}{\partial x_m} \right) \right) + \eta \frac{\partial f_k}{\partial x_m} \\ & - \sum \eta_{x_i} A_{kl}^{ij} \frac{\partial z_l}{\partial x_j} - \sum \frac{\partial}{\partial x_i} (A_{kl}^{ij} \eta_{x_j} z_l) \\ & - \sum \eta_{x_i} B_{kl}^{ij} \frac{\partial}{\partial x_j} \frac{\partial u_l}{\partial x_m} - \sum \frac{\partial}{\partial x_i} \left(B_{kl}^{ij} \eta_{x_j} \frac{\partial u_l}{\partial x_m} \right). \end{aligned} \tag{4.3}$$

Let

$$\tilde{p} = \frac{n}{n-2} \quad \text{if } n \ge 3, \qquad \tilde{p} = \infty \quad \text{if } n \le 2.$$

Set

$$\begin{aligned} h_k &= \eta \frac{\partial f_k}{\partial x_m} - \sum_{i,j,l} \eta_{x_i} A_{kl}^{ij}(q) \frac{\partial z_l}{\partial x_j} - \sum_{i,j,l} \eta_{x_i} B_{kl}^{ij}(p) \frac{\partial}{\partial x_j} \frac{\partial u_l}{\partial x_m}, \\ g_{jk,1} &= -\sum_{i,l} A_{kl}^{ij}(q) \eta_{x_j} z_l - \sum_{i,l} B_{kl}^{ij}(p) \eta_{x_j} \frac{\partial u_l}{\partial x_m}. \end{aligned} \tag{4.4}$$

By Theorem 3.1 and Sobolev's imbedding,

$$\iint_{Q_T} h_k^2 \le C, \quad \iint_{Q_T} |g_{jk,1}|^r \le C \quad \forall r < \tilde{p}. \tag{4.5}$$

Let U_k and V_k be the solutions of

$$\dot{U}_k - \gamma\Delta U_k = h_k \quad \text{in } Q_T, \qquad U_k = Z_k(x,0) \quad \text{on } \partial_p Q_T, \tag{4.6}$$

and

$$\dot{V}_k - \gamma\Delta V_k = \sum \frac{\partial}{\partial x_j} g_{jk,1} \quad \text{in } Q_T, \qquad V_k = 0 \quad \text{on } \partial_p Q_T. \tag{4.7}$$

By L^2 estimates for the heat operator,

$$\sum_{j=0}^{2} \iint_{Q_T} |\nabla^j U_k|^2 \le C \iint_{Q_T} |h_k|^2 + C \sum \|Z_k(\cdot,0)\|_{W^{2,2}(\Omega)} \le C \quad \text{(by (4.5), (1.14))}.$$

Consequently, by Sobolev's imbedding,

$$\iint_{Q_T} |\nabla U_k|^r \le C \quad \forall r < \tilde{p}. \tag{4.8}$$

Next, by Lemma 2.1 and (4.5),

$$\iint_{Q_T} |\nabla V_k|^r \le \sum_j \iint_{Q_T} |g_{jk,1}|^r \le C \quad \forall r < \tilde{p}. \tag{4.9}$$

Consider the functions $\tilde{Z}_k = Z_k - U_k - V_k$. From (4.2), (4.3) and (4.4)–(4.6) we see that the $\tilde{Z}_k$ satisfy:

$$\frac{\partial}{\partial t}\tilde{Z}_k - \gamma\Delta\tilde{Z}_k = -\sum \frac{\partial}{\partial x_i}\left((\gamma\delta_{ij}\delta_{kl} - A_{kl}^{ij}(q))\frac{\partial \tilde{Z}_l}{\partial x_j}\right) + \sum \frac{\partial}{\partial x_j}\hat{g}_{kj} \quad \text{in } Q_T, \tag{4.10}$$

$$\tilde{Z}_k = 0 \quad \text{on } \partial_p Q_T \tag{4.11}$$

where

$$\hat{g}_{kj} = -\sum (\gamma\delta_{ij}\delta_{kl} - A_{kl}^{ij}(q))\frac{\partial(U_l + V_l)}{\partial x_j} + \sum B_{kl}^{ij}(p)\frac{\partial}{\partial x_j}\left(\eta\frac{\partial u_l}{\partial x_m}\right). \tag{4.12}$$

In order to estimate $\nabla \tilde{Z}_k$ we consider, more generally, the system

$$\dot{Y}_k - \gamma \Delta Y_k = \sum_{i,j,l} \frac{\partial}{\partial x_i}((\gamma \delta_{ij}\delta_{kl} - A^{ij}_{kl}(q))h_{lj}) + \sum \frac{\partial}{\partial x_j} g_{kj} \quad \text{in } Q_T, \tag{4.13}$$

$$Y_k = 0 \quad \text{on } \partial_p Q_T, \tag{4.14}$$

and proceed as in §2 to define an operator S mapping vectors $G = (h_{lj}, g_{kj})$ into vectors $\nabla Y = (\nabla Y_1, \dots, \nabla Y_N)$ where $Y_1, \dots, Y_N$ is the solution of (4.13), (4.14); we use the L^s norms

$$\|G\| = \left\{ \iint_{Q_T} \left(\sum |h_{lj}|^s + M \sum |g_{kj}|^s \right) \right\}^{1/s},$$

$$\|\nabla Y\|_s = \left\{ \iint_{Q_T} \sum |\nabla Y_k|^s \right\}^{1/s}.$$

By Lemma 2.1

$$\|S\|_s \le A_s,$$

where A_s is a constant. Using (2.10) we can derive the same estimate (2.16) as before, provided M is sufficiently large, and then (2.18) also holds. Recalling (4.10)–(4.12) we deduce that if $p_0 > 2$, $p_0 - 2$ small enough, then

$$\sum \iint_{Q_T} |\nabla \tilde{Z}_k|^s \le C \sum \iint_{Q_T} |\hat{g}_{kj}|^s \quad \forall\, 2 \le s \le p_0. \tag{4.15}$$

This inequality can also be established with Q_T replaced by Q_t, for any $0 < t \le T$. Substituting $\hat{g}_{ki}$ from (4.12) and using (4.8) (4.9), we get

$$\sum \iint_{Q_T} |\nabla Z_k|^s \le C + \sum \iint_{Q_t} \left| \nabla \left(\eta \frac{\partial u_k}{\partial x_m} \right) \right|^s, \quad 2 \le 2 \le p_0. \tag{4.16}$$

Hence, by Gronwall's inequality (cf. the argument following (2.19)),

$$\sum \iint_{Q_T} |\nabla Z_k|^s \le C.$$

It follows that

$$\sum_k \iint_{Q_T} \left| \nabla \left(\eta \frac{\partial \dot{u}_k}{\partial x_m} \right) \right|^s + \sum_k \iint_{Q_T} \left| \nabla \left(\eta \frac{\partial u_k}{\partial x_m} \right) \right|^s \le C. \tag{4.17}$$

In deriving (4.17) we have assumed that $\nabla^2\dot{u}_k, \nabla^2 u_k$ belong to L^s. In order to derive (4.17) without making this a priori restriction, we work with finite differences, as in §3, and establish (4.17) with $\partial/\partial x_m$ replaced by Δ_m^h. It then follows that $\nabla^2\dot{u}_k$, $\nabla^2 u_k$ belong to $L^s(\Omega_0\times(0,T))$ for any compact subdomain Ω_0 of Ω, and (4.17) holds.

In order to derive the L^s estimates near the lateral boundary of Q_T, we apply the same arguments used in the derivation of (4.17) to the system (3.3), and thus derive L^s estimates, which extend the L^2 estimates of (3.14). Finally, using (3.17), we can estimate the L^s norm of $\partial^2\dot{U}_k/\partial(x_n')^2$, $\partial^2 U_k/\partial(x_n')^2$ near the boundary. This completes the proof of Theorem 4.1.

Analogously to Corollary 2.3 we have:

COROLLARY 4.2. *For any* $2 < p_0 < q_0$, *if* $\lambda_2 - \lambda_1$ *is sufficiently small so that* $\vartheta_{p_0} < 1$ (ϑ_{p_0} *as in* (2.18)) *then* (4.1) *holds for any* $2 \le s \le p_0$.

5. Additional regularity.

LEMMA 5.1. *If* $p_0 > 2$ *and* $p_0 - 2$ *is sufficiently small, then*

$$\sup_{0<t<T}\sum_k\int_{\Omega_t}|\ddot{u}_k|^s\,ds \le C \quad \forall\, 2 < s \le p_0. \tag{5.1}$$

Proof. Multiplying (2.4) by $|\dot{z}_k|^\alpha z_k$ ($\alpha = s-2 > 0$) and integrating over Q_t, we obtain

$$\begin{aligned}
&\frac{1}{s}\int_{\Omega_t}|\ddot{u}_k|^s - \frac{1}{s}\int_{\Omega_0}|\ddot{u}_k|^s \\
&\quad = -(1+\alpha)\iint_{Q_t}\sum A_{kl}^{ij}(q)\frac{\partial\ddot{u}_l}{\partial x_j}\frac{\partial\ddot{u}_k}{\partial x_i}|\ddot{u}_k|^\alpha \\
&\qquad -(1+\alpha)\iint_{Q_t}\sum B_{kl}^{ij}(p)\frac{\partial\dot{u}_l}{\partial x_j}\frac{\partial\ddot{u}_k}{\partial x_i}|\ddot{u}_k|^\alpha + \iint_{Q_t}\dot{f}_k|\ddot{u}_k|^\alpha\ddot{u}_k.
\end{aligned}$$

By Theorems 2.2, 4.1,

$$|\nabla\dot{u}|\,|\nabla\ddot{u}| \quad \text{and} \quad |\nabla u|\,|\nabla\ddot{u}|$$

belong to $L^r(Q_t)$ for some $r > 1$. Choosing $s = 2r$ and using Hölder's inequality, we get

$$\sum_k\int_{\Omega_t}|\ddot{u}_k|^s \le C + C\sum_k\iint_{Q_t}|\ddot{u}_k|^s$$

and (5.1) then follows by Gronwall's inequality.

THEOREM 5.2. *There exists a* $p_0 > 2$ *such that*

$$\sup_{0<t<T} \sum_k \int_\Omega |\nabla^2 \dot{u}_k|^s \le C \quad \forall\, 2 \le s \le p_0. \tag{5.2}$$

Proof. Notice that

$$\sum \int_{\Omega_t} |\nabla^2 u_k|^s \le C + C \sum \iint_{Q_t} |\nabla^2 \dot{u}_k|^s \le C' \qquad (C' \text{ constant}). \tag{5.3}$$

We can write (1.4) as an elliptic system

$$-\sum \frac{\partial}{\partial x_i} \frac{\partial V(q)}{\partial q_{ki}} = \tilde{g}_k \tag{5.4}$$

where, by (5.3) and Lemma 5.1,

$$\sum \int_{\Omega_t} |\tilde{g}_k|^s \le C \tag{5.5}$$

if $2 \le s \le p_0$ for some $p_0 > 2$. We now argue as in §4; we set $z_k = \partial \dot{u}_k / \partial x_m$, $Z_k = \eta z_k$ (η a cut-off function for $B_{R'}$), differentiate (5.4) with respect to x_m, multiply by η and integrate over Ω_t. We obtain, analogously to (4.2),

$$-\gamma \Delta Z_k = -\sum \left(\frac{\partial}{\partial x_i} \left(\gamma \delta_{ij} \delta_{kl} - A_{kl}^{ij}(q) \frac{\partial Z_l}{\partial x_j} \right) + \tilde{G}_k \right.$$

where the $L^q(\Omega_t)$-norm of $\tilde{G}_k$ can be estimated using (5.5). We then deduce, as in the parabolic case, that (2.18) holds and, consequently,

$$\int_{\Omega_t} |\nabla Z_k|^s \le C \quad \forall\, 2 \le s \le p_0.$$

Thus

$$\sup_{0<t<T} \sum \int_{B_R} |\nabla^2 \dot{u}_k(x,t)|^s \le C \quad \text{if } \overline{B}_R \subset \Omega.$$

Similarly we obtain the corresponding estimate with B_R replaced by $B_R(x_0) \cap \Omega$, where $x_0 \in \partial\Omega$, and (5.2) follows.

THEOREM 5.3. *If* $n \le 2$ *then* ∇u_k *and* $\nabla \dot{u}_k$ *are Hölder continuous in* $\overline{\Omega}_T$; *consequently* $(u_1, \dots, u_N)$ *is a classical solution in* $\overline{\Omega}_T$ *for all* $T > 0$.

Proof. From Theorem 5.2 and Sobolev's imbedding it follows that

$$|\nabla \dot{u}(x_1, t) - \nabla \dot{u}(x_2, t)| \le C |x_1 - x_2|^2 \qquad (\alpha = 1 - 2/s). \tag{5.6}$$

We next prove that, for any $0 < t_1 < t_2 < T$,

(5.7) $|\nabla\dot{u}(x,t_2)-\nabla\dot{u}(x,t_1)| \le C|t_2-t_1|^{\beta} \qquad (\beta = (1-2/s)(1-1/s)).$

Let $\rho = (t_2 - t_1)^{\gamma}$ ($\gamma = 1 - 1/s$) and consider first the case where $B_\rho(x)$ is contained in Ω. Let

$$v(x,t) = \frac{\partial^2 u_k}{\partial t \partial x_m}, \quad v_\rho(x,t) = \fint_{B_\rho(x)} v(y,t)\,dy$$

where $\fint$ means the average. Then

$$\begin{aligned}|v(x,t_2) - v(x,t_1)| &\le \sum_{i=1}^{2} |v(x,t_i) - v_\rho(x,t_i)| + |v_\rho(x,t_2) - v_\rho(x,t_1)| \\ &\le C\rho^{\alpha} + |B_\rho|^{-1} \int_{t_1}^{t_2} \int_{B_\rho(x)} |v_t(y,t)| \qquad \text{(by (5.6))} \\ &\le C\rho^{\alpha} + |t_2 - t_1|^{1-1/s} |B_\rho|^{-1/s} \left(\iint_{Q_T} |v_t|^s \right)^{1/s} \\ &\quad + C|t_2 - t_1|^{\gamma(1-2/s)} + C|t_2 - t_1|^{1-1/s-2\gamma/s}\end{aligned}$$

by Theorem 2.2, and (5.7) follows. In case $B_\rho(x)$ is not contained in Ω, we work with

$$\fint_{B_\rho(x)\cap\Omega} v(y,t)\,dy \quad \text{instead of} \quad \fint_{B_\rho(x)} v(y,t)\,dy.$$

The Hölder continuity of $\nabla\dot{u}$ follows from (5.6), (5.7). The Hölder continuity of ∇u follows from that of $\nabla\dot{u}$, and thus the proof of Theorem 5.3 is complete.

Using Corollaries 2.3, 4.2 and an extension of Theorem 5.2 to any p_0 ($p_0 > n$) if $\lambda_2 - \lambda_1$ is small enough we also have:

THEOREM 5.4. *Let $n \ge 3$. If $\lambda_2 - \lambda_1$ is small enough then ∇u_k and $\nabla\dot{u}_k$ are Hölder continuous in $\overline{Q}_T$ and, consequently, $(u_1, \dots, u_N)$ is a classical solution in $\overline{\Omega}_T$.*

6. General W and linear viscosity term. In this section we consider systems of the form

$$\ddot{u}_k - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial W(p)}{\partial p_{ki}} - \Delta\dot{u} = f_k \qquad (1 \le k \le N) \tag{6.1}$$

instead of (1.4), but replace the condition (1.2) by a much weaker condition which allows quite general nonlinear growth for W, namely:

$$(6.2)\quad \begin{aligned} \tilde{\lambda}_1(\delta+|p|)^{r-2}|\xi|^2 \le \sum_{k,l=1}^{N}\sum_{i,j=1}^{n} \frac{\partial^2 W(p)}{\partial p_{ki}\partial p_{lj}}\xi_i^k\xi_j^l \\ \le \tilde{\lambda}_2(\delta+|p|)^{r-2}|\xi|^2, \\ \text{where } 0<\tilde{\lambda}_1<\tilde{\lambda}_2<\infty, \delta\ge 0,\ r>2. \end{aligned}$$

We also suppose that

$$(6.3)\qquad W(0)\ge 0,\quad \frac{\partial W(0)}{\partial p_{ki}}=0;$$

then $W(p)>0$ if $|p|>0$.

DEFINITION 6.1. A strong solution $u=(u_1,\dots,u_N)$ of (6.1), (1.5), (1.6) in Q_T is a function u satisfying: (i) the first two relations in (1.8) hold; (ii) $W(\nabla u)\in L^\infty((0,T);\, L^1(\Omega))$; (iii) (6.1) holds in the distribution since in Q_T and (1.6) holds in the trace sense; finally, (iv) the functions

$$\ddot{u}_k,\quad \nabla^2\dot{u}_k,\quad \frac{\partial}{\partial x_i}\frac{\partial W(p)}{\partial p_{ki}}$$

belong to $L^{r'}_{\mathrm{loc}}(Q_T)$ and (6.1) holds pointwise a.e.; here $1/r'+1/r=1$.

THEOREM 6.1. *Assume that* (1.12)–(1.14) *and* (6.2), (6.3) *hold. Then there exists a function* $u=(u_1,\dots,u_N)$ *which is a strong solution of* (6.1), (1.5), (1.6) *in* Q_T, *for every* $T>0$.

Proof. We begin by introducing a family of truncations of W. For any small $\varepsilon>0$, let $M=[1/\varepsilon]$ (the largest integer $\le 1/\varepsilon$), and define

$$(6.4)\qquad W_\varepsilon(p)=W(p)\eta_\varepsilon(|p|)+C_0M^{r-2}|p|^2\chi_\varepsilon(|p|)+\varepsilon|p|^2$$

where $\eta_\varepsilon(Ms)=\phi(s)$, $\chi_\varepsilon(Ms)=\psi(s)$, $\phi\in C^3(\mathbf{R}^1)$, $\phi(s)=1$ if $s\le\beta$, $\phi(s)=0$ if $s\ge\gamma$ and $\psi\in C^3(\mathbf{R})$, $\psi(s)=0$ if $s\le\alpha$, $\psi(s)=1$ if $s\ge\beta$, $\psi'(s)\ge 0$ if $\alpha<s<\beta$, and

$$(6.5)\qquad 2\psi(s)+s\psi'(s)+s^2\psi''(s)\ge 0\quad \text{if } \alpha<s<\beta.$$

We can choose, for instance

$$\alpha=e^{\pi/(4\sqrt{2})},\quad \beta=e^{\pi/\sqrt{2}},\quad \gamma=\beta+1$$

and take $\psi_0(s)=(\mu-\cos(\sqrt{2}\log s))^+/(1+\mu)$ if $\alpha\le s\le\beta$, where μ is a sufficiently small positive constant, and then mollify ψ_0 and shift it slightly to the left to obtain a function ψ satisfying all of the above conditions.

Using (6.5) we compute that

$$\sum \frac{\partial^2(\chi_\varepsilon(|p|)|p|^2)}{\partial p_{ki}\partial p_{lj}}\xi_i^k\xi_j^l \geq 0 \quad \text{if } |p| \leq \beta M$$

and, therefore, if C_0 is chosen large enough (independently of ε),

$$\text{(6.6)} \qquad \lambda_1\Gamma_\varepsilon(p)|\xi|^2 \leq \sum \frac{\partial^2 W_\varepsilon(p)}{\partial p_{ki}\partial p_{lj}}\xi_i^k\xi_j^l \leq \lambda_2\Gamma_\varepsilon(p)|\xi|^2$$

where λ_1, λ_2 are positive constants independent of ε and

$$\text{(6.7)} \qquad \Gamma_\varepsilon(p) = \begin{cases} (\delta + |p|)^{r-2} + \varepsilon & \text{if } |p| < \gamma M, \\ (\delta + \gamma M)^{r-2} + \varepsilon & \text{if } |p| \geq \gamma M. \end{cases}$$

We can also easily verify that

$$\text{(6.8)} \qquad \Gamma_\varepsilon(p)^{r'/(2-r')} \leq C(1 + W_\varepsilon(p)),$$

$$\text{(6.9)} \qquad \left|\frac{\partial W_\varepsilon(p)}{\partial p_{ki}}\right| \leq C\,[W_\varepsilon(p)\Gamma_\varepsilon(p)]^{1/2},$$

and, as $\varepsilon \to 0$,

$$\text{(6.10)} \qquad W_\varepsilon(p) \to W(p) \quad \text{in } C^3_{\text{loc}}(\mathbf{R}^{nN}).$$

Consider the system

$$\text{(6.11)} \qquad \ddot{u}_k - \sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial W_\varepsilon(p)}{\partial p_{ki}} - \Delta\dot{u}_k = f_k \qquad (1 \leq k \leq N).$$

Since W_ε is a C^3 function, we can apply Theorem 5.4 to conclude that there exists a function $u^\varepsilon = (u_1^\varepsilon, \dots, u_N^\varepsilon)$ which is a classical solution of (6.11), (1.5), (1.6) in Q_∞. We shall proceed to derive estimates on u^ε which are independent of ε, and then complete the proof of the theorem by taking $\varepsilon \to 0$.

First, analogously to the derivation of (1.11) we have

$$\text{(6.12)} \qquad \frac{1}{2}\int_{\Omega_t} |\dot{u}^\varepsilon|^2\,dx + \int_{\Omega_t} W_\varepsilon(\nabla u^\varepsilon)\,dx + \iint_{Q_T} |\nabla\dot{u}^\varepsilon|^2\,dx\,dt \leq C.$$

We introduce a function $\eta \in C^2(\overline{\Omega})$ satisfying:

$$\text{(6.13)} \qquad c_1\,\text{dist}(x, \partial\Omega) \leq \eta(x) \leq c_2\,\text{dist}(x, \partial\Omega)$$

where c_1, c_2 are positive constants.

LEMMA 6.2. *For any $T > 0$,*

$$\sum_m \iint_{Q_T} \sum_{i,j,k,l} \frac{\partial^2 W_\varepsilon(p^\varepsilon)}{\partial p_{ki} \partial p_{lj}} \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_m} \frac{\partial^2 u^\varepsilon}{\partial x_j \partial x_m} \eta^2 + \iint_{Q_T} |\nabla^2 u^\varepsilon|^2 \eta^2 \le C \tag{6.14}$$

where $p^\varepsilon = \nabla u^\varepsilon$ and C is a constant independent of ε.

Proof. We shall prove that

$$\begin{aligned} &\sum_m \iint_{Q_T} (T-t) \sum_{i,j,k,l} \frac{\partial^2 W_\varepsilon(p^\varepsilon)}{\partial p_{ki} \partial p_{lj}} \frac{\partial^2 u_k^\varepsilon}{\partial x_i \partial x_m} \frac{\partial^2 u_l^\varepsilon}{\partial x_j \partial x_m} \eta^2 \\ &\quad + \iint_{Q_T} |\nabla u^\varepsilon|^2 \eta^2 \\ &\quad \le C \left\{ \sum_{0 \le t \le T} \int_{\Omega_t} |\nabla u^\varepsilon|^2 + \iint_{Q_T} |\nabla \dot{u}^\varepsilon|^2 + \iint_{Q_T} W_\varepsilon(p^\varepsilon) + C_1 \right\} \end{aligned} \tag{6.15}$$

where C_1 depends only on the initial data and on f_k. Since the right-hand side of (6.15) is bounded for every $T > 0$ (by (6.12)) and since (6.15) holds for any $T > 0$, the assertion (6.14) then follows.

To prove (6.15) we multiply (6.11) by $\eta^2(\partial^2 u_k^\varepsilon / \partial x_m^2)$ and integrate over Q_t, then sum over k, m, and integrate once more in t. We compute the separate terms, dropping usually the index ε. First,

$$\begin{aligned} \sum_{m,k} \iint_{Q_t} \ddot{u}_k \frac{\partial^2 u_k}{\partial x_m^2} \eta^2 = &- \int_{\Omega_t} \sum \frac{\partial \dot{u}_k}{\partial x_m} \eta^2 + \int_{\Omega_0} \sum \frac{\partial \dot{u}_k}{\partial x_m} \frac{\partial u_k}{\partial x_m} \eta^2 \\ &+ \iint_{Q_t} \sum \frac{\partial \dot{u}_k}{\partial x_m} \frac{\partial u_k}{\partial x_m} \eta^2 - 2 \int_{\Omega_t} \sum \dot{u}_k \frac{\partial u_k}{\partial x_m} \eta \eta_{x_m} \\ &+ 2 \int_{\Omega_0} \sum \dot{u}_k \frac{\partial u_k}{\partial x_m} \eta \eta_{x_m} + 2 \iint_{Q_t} \dot{u}_k \frac{\partial \dot{u}_k}{\partial x_m} \eta \eta_{x_m}. \end{aligned}$$

Integrating over $(0, T)$ we find that

$$\int_0^T dt \sum_{m,k} \iint_{Q_t} \ddot{u}_k \frac{\partial^2 u_k}{\partial x_m^2} \eta^2 \text{ is bounded by the right-hand side of (6.15).} \tag{6.16}$$

Next,

$$-\sum_{m,k}\iint_{Q_t}\sum\frac{\partial}{\partial x_i}\left(\frac{\partial W}{\partial p_{ki}}\right)\frac{\partial^2 u_k}{\partial x_m^2}\eta^2$$

$$= -\iint_{Q_t}\sum\frac{\partial^2 W}{\partial p_{ki}\partial p_{lj}}\frac{\partial^2 u_k}{\partial x_i\partial x_m}\frac{\partial^2 u_l}{\partial x_j\partial x_m}\eta^2$$

$$-\iint_{Q_t}\sum\frac{\partial W}{\partial p_{ki}}\frac{\partial^2 u_k}{\partial x_i\partial x_m}\eta\eta_{x_m} + 2\iint_{Q_t}\sum\frac{\partial W}{\partial p_{ki}}\frac{\partial^2 u_k}{\partial x_m^2}\eta\eta_{x_i}.$$

In view of (6.8), (6.9), each of the last two integrals is bounded by

$$C\left\{\iint_{Q_t}\Gamma_\varepsilon(p)|\nabla^2 u|^2\eta^2\right\}^{1/2}\left\{\iint_{Q_t}W_\varepsilon\right\}^{1/2}$$

$$\le \mu\iint_{Q_t}\sum\frac{\partial^2 W}{\partial p_{ki}\partial p_{lj}}\frac{\partial^2 u_k}{\partial x_i\partial x_m}\frac{\partial^2 u_l}{\partial x_j\partial x_m}\eta^2 + C_\mu\iint_{Q_t}W$$

for any small positive constant μ. Hence

$$(6.17)\quad -\int_0^T dt\sum_{m,k}\iint_{Q_t}\sum\frac{\partial}{\partial x_i}\left(\frac{\partial W}{\partial p_{ki}}\right)\frac{\partial^2 u_k}{\partial x_m^2}\eta^2$$

$$= -(1-\tilde{\mu})\iint_{Q_T}(T-t)\sum\frac{\partial^2 W}{\partial p_{ki}\partial p_{lj}}\frac{\partial^2 u_k}{\partial x_i\, dx_m}\frac{\partial^2 u_l}{\partial x_j\partial x_m}$$

$$+\tilde{C}\iint_{Q_T}W$$

where $|\tilde{\mu}| \le 1/2$ and $\tilde{C}$ is a constant bounded independently of ε.

Finally,

$$-\sum_{m,k}\iint_{Q_t}\Delta\dot{u}_k\frac{\partial^2 u_k}{\partial x_m^2}\eta^2$$

$$= \iint_{Q_t}\sum\nabla\dot{u}_k\nabla\left(\frac{\partial^2 u_k}{\partial x_m^2}\right)\eta^2 + 2\iint_{Q_t}\sum\nabla\dot{u}_k\frac{\partial^2 u_k}{\partial x_m^2}\eta\nabla\eta$$

$$= -\frac{1}{2}\int_{\Omega_t}\sum\left|\nabla\frac{\partial u_k}{\partial x_m}\right|^2\eta^2 + \frac{1}{2}\int_{\Omega_0}\sum\left|\nabla\frac{\partial u_k}{\partial x_m}\right|^2\eta^2$$

$$-2\iint_{Q_t}\sum\nabla\dot{u}_k\nabla\left(\frac{\partial u_k}{\partial x_m}\right)\eta\eta_{x_m} + 2\iint_{Q_t}\sum\nabla\dot{u}_k\frac{\partial^2 u_k}{\partial x_m^2}\eta\nabla\eta.$$

Integrating over $(0, T)$ we obtain

$$(6.18)\quad -\int_0^T dt\sum_{m,k}\iint_{Q_t}\Delta\dot{u}_k\frac{\partial^2 u_k}{\partial x_m^2}\eta^2 = -\frac{1}{2}\iint_{Q_T}|\nabla^2 u|^2\eta^2 + I_\varepsilon,$$

where I_ε is bounded by the right-hand side of (6.15).

Combining (6.18) with (6.17), (6.16), the estimate (6.15) follows.

COROLLARY 6.3. *For any h, k, l, i, j, m, s*

$$\left\| \frac{\partial^2 W_\varepsilon}{\partial p_{ki} \partial p_{lj}} \frac{\partial^2 u_h^\varepsilon}{\partial x_m \partial x_s} \eta \right\|_{L^{r'}(Q_T)} \le C. \tag{6.19}$$

Indeed, by (6.6), (6.8) and Lemma 6.2, the left-hand side of (6.19) is bounded by

$$c \left\{ \iint_{Q_T} \Gamma_\varepsilon(p_\varepsilon) |\nabla^2 u^\varepsilon|^2 \eta^2 \right\}^{r'} \left\{ 1 + \iint_{Q_T} W_\varepsilon(p_\varepsilon) \right\}^{1-r'} \le C.$$

Consider the functions $v_k^\varepsilon = u_k^\varepsilon \eta$. Clearly

$$\ddot{v}_k^\varepsilon - \Delta \dot{v}_k^\varepsilon = f_k \eta - \sum \frac{\partial^2 W_\varepsilon(p_\varepsilon)}{\partial p_{ki} \partial p_{lj}} \frac{\partial^2 u_l^\varepsilon}{\partial x_i \partial x_j} \eta + g_k \equiv G_k \tag{6.20}$$

where

$$g_k = -(\Delta \eta) \dot{u}_k - 2 \nabla \eta \nabla \dot{u}_i. \tag{6.21}$$

By (6.12) and Corollary 6.3,

$$\|G_k\|_{L^{r'}(Q_T)} \le C, \tag{6.22}$$

and, by $L^{r'}$ parabolic estimates [9],

$$\begin{aligned} &\iint_{Q_T} |\ddot{v}_k^\varepsilon|^{r'} + \iint_{Q_T} |\nabla^2 \dot{v}_k^\varepsilon|^{r'} \\ &\quad \le C \left(\iint_{Q_T} |G_k|^{r'} + \int_{\Omega_0} |\nabla^2 \dot{v}_k^\varepsilon|^{r'} \right) \le C. \end{aligned} \tag{6.23}$$

Using the estimates (6.12), (6.14), (6.23), we can extract a sequence $\varepsilon = \varepsilon_m \to 0$ such that $\ddot{v}^\varepsilon \to \ddot{v}$, $\nabla^2 \dot{v}^\varepsilon \to \nabla^2 \dot{v}$ weakly in $L^{r'}(Q_T)$ for any $T > 0$ and, further, since

$$\|\nabla v^\varepsilon\|_{W^{1,r'}(Q_T)} \le C, \tag{6.24}$$

we may suppose that $\nabla v^\varepsilon \to \nabla v$ in $L^{r'}(Q_T)$ and a.e. But then we have

$$\frac{\partial W_\varepsilon(p_\varepsilon)}{\partial p_{ki}} \to \frac{\partial W(p)}{\partial p_{ki}} \quad \text{a.e.}$$

and strongly in $L^s(Q_T)$ for some $s > 1$ (by (6.8), (6.9) and (6.12)). Hence, by Corollary 6.3,

$$\frac{\partial}{\partial x_i} \frac{\partial W(p)}{\partial p_{ki}} \in L^{r'}_{\text{loc}}(Q_T). \tag{6.25}$$

Since, by (6.24), $p \equiv \nabla u \in W^{l,r'}_{\text{loc}}(Q_T)$, the distribution derivative in (6.25) can be identified with the function

$$\sum \frac{\partial^2 W(p)}{\partial p_{ki} \partial p_{lj}} \frac{\partial^2 u_l}{\partial x_i \partial x_j}$$

One can now easily check that u satisfies all the properties of a strong solution.

REMARK 6.1. All the results of §§1–5 extend to the case where $W = W(x, u, p), V = V(x, u, q)$; the results of §6 extend to the case where $\Delta \dot{u}$ is replaced by any linear elliptic operator $L\dot{u}$ with smooth coefficients.

REFERENCES

[1] G. Andrew, *On the existence of solutions to the equation* $u_{tt} = u_{xxt} + \sigma(u_x)_x$, J. Differential Equations, **35** (1980), 200–231.

[2] C. M. Dafermos, *The mixed initial-boundary value problem for the equations of nonlinear one-dimensional visco-elasticity*, J. Differential Equations, **6** (1969), 71–86.

[3] H. Engler, *Strong solutions for strongly damped quasilinear wave equations*, Contemporary Math., **64** (1987), 219–237.

[4] W. N. Findleky, J. S. Lai and K. Onaran, *Creep and Relaxation of Nonlinear Viscoelastic Materials*, North Holland, Amsterdam, New York, 1976.

[5] H. Gajewski, K. Groger and K. Zacharias, *Nonlinear Operator Equations and Operator Differential Equations*, Akademy Verlag, Berlin, 1974.

[6] J. M. Greenberg, R. C. MacCamy and V. J. Mizel, *On the existence, uniqueness and stability of the equation* $\sigma'(u_x)u_{xx} + \lambda u_{xxt} = \rho_0 u_{tt}$, J. Math. Mech., **17** (1968), 707–728.

[7] J. M. Greenberg, *On the existence, uniqueness and stability of the equation* $\rho_0 X_{tt} = E(X_x) + \lambda X_{xxt}$, J. Math. Anal. Appl., **25** (1969), 575–591.

[8] J. K. Knowles, *One finite antiplane shear for incomprehensible elastic materials*, J. Austral. Math. Soc. Ser. B, **19** (1975/76), 400–415.

[9] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Amer. Math. Soc., Providence, R.I., 1968.

[10] H. Petcher, *On global regular solution of third partial differential equations*, J. Math. Anal. Appl., **73** (1980), 278–299.

[11] M. Potier-Ferry, *On the mathematical foundation of elastic stability*, I. Archive Rat. Mech. Anal., **78** (1982), 55–72.

Received April 24, 1987. This work is partially supported by National Science Foundation Grants DMS-8420896 and DMS-8501397.

PURDUE UNIVERSITY
WEST LAFAYETTE, IN 47907