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SPECTRUM AND MULTIPLICITIES FOR RESTRICTIONS OF UNITARY REPRESENTATIONS IN NILPOTENT LIE GROUPS

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Let *G* be a connected, simply connected nilpotent Lie group, and let AT be a Lie subgroup. We consider the following question: for *n* <G G^{A}, how does one decompose *U\K* as a direct integral? In his pioneering paper on representations of nilpotent Lie groups, Kirillov gave a qualitative description; our answer here gives the multiplicities of the representations appearing in the direct integral, but is geometric in nature and very much in the spirit of the Kirillov orbit picture.

1. The problem considered here is the dual of the one investigated by us and G. Grelaud in [2]: give a formula for the direct integral decomposition of Ind^ *o*, *a* € K^{A}. The answer, too, can be regarded as the dual of the answer in [2]. Let g, *t* be the Lie algebras of *G*, *K* respectively, and let *g**, *t** be the respective (vector space) duals; *P*: *g** —• 6* denotes the natural projection. Given *n* €ˆ*G*, we want to write

$$n\backslash_{KC} \pm \overset{re}{{}_{\prime}I} \quad n\{a)odv(o);$$

we need to describe *n{o)* and *v*. To this end, we review some aspects of Kirillov theory. In [7], Pukanszky showed that *V* can be partitioned into "layers" U_e, each Ad*(AT)-stable, such that on U_e the *Ad*(K)*-orbits are parametrized by a Zariski-open subset ~L_e of an algebraic variety. (See also §2 of [2].) We can thus parametrize $\hat{K}$ by the union of the *L_e. Let $@_n$ c *g** be the Kirillov orbit corresponding to *n*. There is a unique *e* such that <$f_n nP\sim^{l}(U_e)$ is Zariski-open in <$?_n$. Let £* C S_e be the set of /' *e* $T._e$ such that $P\{@_n)$ meets *K* • /'. It turns out that 27 is a finite disjoint union of manifolds. Let *k** be the maximal dimension of these manifolds; define *v* to be A:*-dimensional measure on the manifolds of maximum dimension and 0 elsewhere. Then we will have

$$\pi|_K \simeq \int_{\Sigma^\pi}^{\oplus} n(l')\sigma_{l'}\, d\nu(l'),$$

where a> corresponds to /' G 27 via the Kirillov orbit picture.

It remains to describe «(/')• For / *e* @$_n$, define

$$\mathrm{To}(/) = \dim(G\text{- }/) + \dim(A'' \bullet\ PI) \text{ - } 26im\{K\bullet\ I),$$

where the action of *G, K* is the coadjoint action (so that *Gl* = *(f$_K$* and *K* · *PI* is the Kirillov orbit in *V* corresponding to *Pi).* This number is a constant, To, on a Zariski-open subset of @$_{\%}$, and we have

$$/!(/') = \mathrm{oo}, \quad v\text{-}a.e.\ /'\ e\ 27, \quad \text{if } \mathrm{T}_0 > 0.$$

When To = 0, we have

$$«(/') = \text{number of Ad*(\#)-orbits in } P\sim^{x(K\bullet\ l')n<?_n;}$$

moreover, this number is uniformly bounded a.e. on 27. This is the essential content of our Theorems 4.6 and 4.8. In fact, we note in Remark 4.7 that To > 0 whenever the number of AT-orbits in *P~*$^{l(l')}$ n <*f$_n$* is generically infinite.

It may be helpful to consider the simplest example of the theorems, where *K* is of codimension 1 in *G.* This situation was investigated in [4]. For / € @$_n$, let t/ be the radical of /. There are two cases to consider. If t/ ^ *t,* then *P* is a diffeomorphism of <?„ onto *K* • *PI*_C *V,* and *(f$_n$* = *K-1;* furthermore, *7\K* is irreducible, *n\x* = *o$_P$[.* Thus 27 reduces to a single point (corresponding to o>/), and, for /' e *P{<?*),* *P~*$^{l(K\text{-}1')}$ *n&$_n$* = *&$_n$,* so that *n(l')* = 1. It is easy to see that T$_O$(/) = 0, and that Theorem 4.8 says that *7\K* ≅ *O\,* (where /' € 2^{71} corresponds to *K* • *PI).* If t,_C *t,* then choose *X e g\t.* In this case, *PO$_n$* = U$_{rgR}$*K-Xf PI,* where x_t = exp *tX* (acting on *PI* by Ad*; note that *K* is normal) and the union is disjoint. Furthermore, *P~*$^{l(P<J_n)}$ = *&$_n$* (i.e., <*9$_n$* is P-saturated), and *P~\K* • x_t • *Pi)* = *K* • x_t • *I.* Thus

$$' \sim \int^{\oplus}_{JR} \circ_{x,\text{-}n}{}^{dt\text{-}}$$

Again, To = 0, and Theorem 4.8 gives this same decomposition. For in this case, 27 consists of representatives for the orbits <?$_t^K$ = *K-(x$_t$-Pl).* It is easy to see from the formula *P~*$^{l(K\text{-}x_rPl)}$ = *Kx$_t$l*.that *n(l')* = 1 for /' representing <*f$_t^K$*.

The proof in the general case is in essence an induction applied to this example. (In a sense, it is also dual to the proof in [2].) We construct a chain of subgroups from *K* to *G,* each of codimension 1 in the next, and restrict step by step. Keeping track of the geometry, however, soon becomes difficult. To keep matters straight, we introduce a fibration of most of *ff$_n$.* More precisely, we show that a Zariski-open set *U* ⊆ <?„ can be fibered into manifolds *U* = *\Ji$_{exf}$ fy,* such that all

points in the fiber *Nj* project to the same AT-orbit in *t*: P • Nj = K • PL* The *Ni* let us keep track of the way that the tangent space to a A>orbit grows as the Lie algebra grows from *t* to *g*. When To = 0, *N/* is (generically) the AT-orbit of /, but when To > 0, it is an infinite union of AT-orbits. Our construction of the *N[* is somewhat ad hoc, and we do not know if they have any further significance. (In some cases, they do depend on the chain of subgroups from *K* to *G.)*

Our first decomposition of *U\K* is as a direct integral over the iVj. We actually express it as a direct integral over the transversal *Xf.* This set is parametrized by a polynomial map *X:R^k* —> <*f_n*, where *2k* = dim *Gl*- - dim AT • *Pi* for generic / e <?_n; *X* is a diffeomorphism on a Zariski-open set *Af* c *R^k*, and *Xf* = *X(Af).* Then we prove that

(1) $$*\backslash K \cong \int_{JVL^k}^{\oplus} \sigma_{(P\circ\lambda)(u)}\, du;$$

where *du* is Euclidean measure. We also show that *Xf* and the iV) have the following properties:

(i) *leN,, =>l' e N_t* (the *N_t* partition *0_n)*
(ii) for generic /, dim TV/ = *r* + *k* (*r* = dim *K* • *PI);*
(iii) for *I eXf — X{Af),* TV/ and *Xf* are transverse;
(iv) for/€*/, *N_{lr}\Xf* = *{l};*
(v) \J_{leX} *N[* is an open dense subset of full measure in &„;
(vi) *P{Ni)_CKPl.*

This means that the direct integral in (1) can be taken over *Xf.* We show next that if To > 0, then *Af* fibers into manifolds of dimension ≥ 1 that are taken into the same Ad*(AT)-orbit by *P o X;* this gives the infinite multiplicity case. When To = 0, the *N/* are generically the orbits *K* • *I*, and the number of points in *P~x(V)* n *Xf* is the number of *Ni* in *P~x(V) C\&_n* this, plus some technical work, gives the finite multiplicity formula.

The integral (1) (our Theorem 3.5) is, of course, also a direct integral decomposition, though not a canonical one. It is useful, however, because it leads to a proof of the following results:

THEOREM 1.1. *Let G be a connected, simply connected complex nilpotent Lie group, and let K be a complex Lie subgroup. IfneĜ, then 7Z\K is ofuniform multiplicity.*

THEOREM 1.2. *Let Gbea connected, simply connected real nilpotent Lie group, and let K be a Lie subgroup. For n* e$\hat{G}$, *write*

$$n\backslash_K \cong \int^{\oplus} n\{o)odv(o).$$

Then either

$$\begin{array}{rl} & n\{o) \cong \text{oo}, \ v\text{-a.e.}, \\ or & n(a) \text{ is even}, \ v\text{-a.e.}, \\ or & n(a) \text{ is odd}, \ v\text{-a.e.} \end{array}$$

The proofs of these theorems are similar to the proofs of the corresponding theorems for induced representations, given in [1], and we shall not give further details here.

The duality between the results in [2] and those here is, of course, an aspect of Frobenius duality; in particular, the formula for *n(n)* in Ind^cr is the same as the formula for *n(a)* in *n\fc-* There are general results of this form; one is found in Mackey [5]. Mackey's theorem applies to almost all *n* and almost all *a,* while our results apply to all *n* € $\hat{G}$ and all *a* e $\hat{K}$ (except that, of course, *n(n)* and *n(o)* are defined only a.e.) Mackey's theorem also gives information on the measures in the direct integral decomposition. We hope to be able to say something about these measures on the exceptional set of representations not covered by Mackey's theorem, and about other aspects of Frobenius reciprocity; we defer these topics to future papers.

The outline of the rest of the paper is as follows: in §2, we construct the *Nfs* and describe various other algebraic constructions like those in §2 of [2], but somewhat more complicated. Section 3 is devoted to the proof of the noncanonical decomposition (1), and our main theorems are proved in §4. We give some examples in §5, including one of a tensor product decomposition. For a number of proofs, we rely heavily on results of [2]. We also use a number of results concerning semialgebraic sets; a sketch of the main facts about these sets is found in [2]. (See [9] for further details.)

2. Here we decompose *g** into sets *Us* adapted to both *G* and *K;* for each / € g*, we construct a set A7 with a number of useful properties analogous to those for the sets M/ constructed in §2 of [2]. Since the proofs closely follow proofs in [2], we will sometimes be quite sketchy about details.

Let *t* be a subalgebra of a nilpotent Lie algebra *g.* We fix a strong Malcev basis *{X\,...,X_P}* for 6 and extend it to a weak Malcev basis $\{X_{\{},\ldots,X_p,X_{p+i},\ldots,X_{p+m}\}$ for g. Let g, = R-span $\{X_{\{},\ldots,Xj\}$, and let { I$\mathfrak{p}^*$..., X^*_{p+m}} c 9* be the dual basis to the given basis for 0. Note that Gj = expg7 acts on both $Q*J$ and 0* by Ad*, and that these actions are intertwined by the canonical projection Pj• -. $Q*$ —• 0^. Also, K acts on each 0^*_j, and these actions commute with Pj because $X\backslash,..., X_p$ give a strong Malcev basis for t. We often write P for $Pp'.Q'$^t*.

Define dimension indices for / e 9* as follows:

ej(l) = dim Ad (K)Pj(1)* (= *dim ad* (t)Pj(I))* if $1 \le j \le p$;

dj{l) = dim Ad (Gj)Pj(l)* (= *dimad*(Qj)Pj(l))* *if j > p;*

$$e(l) = (e_1(l), \ldots, e_p(l)), \qquad d(l) = (d_{p+1}(l), \ldots, d_{p+m}(l));$$

$$\delta(l) = (e(l), d(l)) \subseteq \mathbf{Z}^{p+m};$$

Δ = *{Se* Z^{p+m}: 3/ € 0* with *5{l) = S};*

$$U_\delta = \{l \in \mathfrak{g}^* : \delta(l) = S\}$$ for<SeA.

(2.1) PROPOSITION. *Let* $\underline{K}$ *CG and a basis {X\... ,X*$_P$*,...,* X_{m+P}*} be given as above. Then:*

(a) *IfS = {S\,...,* d_{m+p}*)* e A, *then dj - Sj-i* = 0 *or* 1 *ifj* $\le$ *p and Sj - Sj-i =0or2 ifj > p (we set So = 0). Hence A is finite.*

(b) *There is an ordering ofA, A = {<5*$^{(1)}$ > £$^{(2)}$ > ...*}; such that for each 3* G A, *the set* $V_s = \bigcup_{S' \ge S} U_{S'}$*, is Zariski-open ing*.*

Proof, (a) For $j \le p$, this is clear, since the same group K acts on each $\mathfrak{g}^*_j$ and dim0^*_j increases by 1 at each step. For $j > p$, we have the coadjoint action of *Gj = exp(&j)* on 0^; orbits are even-dimensional and both *Gj,* 0^*_j increase in dimension by 1 at each step.

(b) Order the e's as in Theorem 1, (b), of [2]. For all *8 = (e,d)* with fixed *e,* further order the *d's* as in Proposition 2 of [2]. Now take the lexicographic order on A: *(e, d) > (e', d')* if *e > e'* or *e = e'* and *d > d'.* The proof of Proposition 2 of [2] is easily modified to show that this ordering has the desired properties. •

Now fix *6 = (e, d)* set

R'_2 = $R'_2(S)$ = $R'_2(e)$ = *{j:l*$\le$*j<p* and *ej* - *?,• _, # 0},

R'{ = *R*$_2$*(S)* = *R*$_2$*(d)* - *{j: p < j* $\le$ *p + m* and *dj - dj-i ?* 0}

(where $d_p = e_p$). Similarly, define

R\ = R\{8) = R\{e) = {j: $1 \le j \le p$ and *ej* = e_{j-1}*\,*

R'[= *R'((S) = R'l(d) = {j: p < j* $\le$ *p + m* and *dj = dj-i},* and let

$$\mathbf{R}_2 = R_2(S) = R'_2 u R_2'', \quad Ri = Ri(S) = R[\ L)R'(.$$

Define corresponding vector subspaces of g^*:

$$E\backslash = \mathbf{R}\text{-span}\ \{X_j^* : jeR\backslash\}, \qquad E'\{ = \mathbf{R}\text{-span}\ \{X_j^*: j\ e\ R_1''\},$$

$$E'_2 = \mathbf{R}\text{-span}\ \{jX^*:jeR'_2\}, \qquad E'_2{}' = \mathbf{R}\text{-span}\ \{XJ:jeR_2''\},$$

$$E_x = E[\circledR E'\{, \quad \bullet \quad E_2 = E'_2 \circledR E'\{.$$

Then $R\backslash$, R'_2 are complementary subsets of $\{1,2,\dots.\ p\}$, and $R\backslash$, $R2$ are complementary subsets of $\{1,2,\dots,m + p\}$. Hence we obtain splittings

$$Q^* = E_X \circledR E_2, \qquad V = E[\circledR E'_2.$$

If / € U_d and $R_2\{d) = R'_2 \cup R'_2{}^f = \{i_(< \bullet \bullet \bullet < i_r < \bullet \bullet \bullet < i_{r+k}\}$ (with $ir \leq P < {}^{\backprime}h+\backslash)$, as above, a set of vectors y— $\{Y\backslash,\dots, Y_{r+}\wedge\}$ $\subseteq 0$ is called an "action basis at /" if

(2) $\quad {}_ad^*(Y_j)P_{ij}(l) = P_{ij}(X^*_{i_j})$, and
$\quad YjEt$ if $1 \leq {}^{\cdot}\!\underline{l} < r$, $\quad YjZQ^\wedge \quad ifr\pm l<j\leq r + k$

(recall that $X_1^*,\dots,X^*i_n$ the dual basis in g^*). Note that the ij depend on 5. Given y at l, define a mapping $y/;: R^{r+k} \longrightarrow 5^*$ by

(3) $$y/_t(t) = (exp(tiYi)\text{---}exp(t_{r+k}Y_{r+k}))\text{-}l,$$

where $gl = Ad^*(g)l$, and set $N/ = N^\wedge y) = y/_t\{W^{+k})$. The next result shows that the $Ni\{y)$ are independent of the action basis y, partition Us, and can be chosen to vary rationally on $U\S$.

(2.2) PROPOSITION. *Fix notation as above andfixd* e A; *let* $Ri\{S) = R'_2UR_2'' = \{/,<\bullet \bullet \bullet < i_r < i_{r+l} < \dots < i_{r+k}\}$ *with* $i_r \leq p < i_r + \backslash$. *Then* Us *can be covered with a finite number of Zariski-open sets* $Z_a \subseteq 5^*$ *on which are defined rational nonsingular* Y_{ita}*:* $Z_a \to Q$ *such that*

$$\{Y\backslash_a(l), \bullet \bullet \bullet, Y_{r+k,a}\{l)\}\ \textit{is an action basis at I for every I eUsf)}\ Z_a.$$

IfI € Us andy -=$\{Y\backslash,\dots, Y_{r+k})$ is any action basis at I, then

(a) $N,(Y)CU_dnG\text{-}l$,

(b) *The* $N_{i_{dcl}}$ *are coherently defined:* If $l' \in N_l(y)$ *and* $y' = \{Y'_1,\dots, Y'_{r+k}\}$ *is any action basis at* l'*, then* $N_l(y') = N_l(y)$. *In particular,* $N_l(\mathscr{Y}) = N[$ *is independent ofy, and Us is partitioned by the sets N[.*

(c) $NiQK\text{-}l + t^{1\cdot}$

(d) //"Prj, Pr2 *are the projections of* g* = *E* © E_2 *onto E\, E_2 respectively, then* Vx_2 = *N/* —> R^{r+A}= E_2 *is a diffeomorphism. {In fact, t* H> Pr2 *y/i(t) is a diffeomorphism.)*

Proof. We use induction on dimg/6. If t = g, this is essentially the theorem in [7] on orbits applied to the unipotent action of *K* = expf on *V,* with *X\,..., X** as the Jordan-Holder basis. Then TV = *K* • *I* = Ad*(AT)/; (b) and (c) are thus trivial, (a) follows because the *Us* are always Ad*(AT)-invariant, and (d) is one part of Pukanszky's parametrization of orbits in *U$.*

If dimg/6 > 0, the proof is a nearby verbatim adaptation of the proof of Proposition 3 in [2]. •

The following observation about the properties of the action basis generating TV will be useful, and can be proved without going into details of the proof of Proposition 2.2.

(2.3) LEMMA. *Let ij e $R_2''(S)$, let I e U_s, and let Y* e g,; *satisfy*

$$\mathrm{ad}^*(Y)P_{i_j}(l) = P_{i_j}(X_{i_j}^*).$$

Then $YE_{0/>1}$.

Proof. Since we are projecting onto g(J, there is no loss of generality in assuming that *g^* = g, *ij* = *m* + *p,* and *j* = *r* + *k.* Writing r_2 = *r* + *k,* *n* = *m* + *p,* go for g_n-\, *P$* for P_n-\, etc., in what follows, we have

(4) *(ad* Y)l* = X^*_n, with y e g.

Obviously *Y* is determined modt/, the radical of /. Because the orbit dimension increases as we pass from *GQ* • *Po(l) to G* • *I, we* have

dimt/ = dimg - dim^f/ = dimg$_0$ - dim^/y - 1 = dimt/y - 1;

it follows easily that t/ ^ $t_{Po(i)}$ *Q* 00- Thus it suffices to show that Y ^ So and there exists some *Y* € go such that (4) holds. But if *Y* is any vector in $_xP_0(i)$ N •

l([Y,Oo\) = *(O),* /([7,0])^(O) (hence *l([Y,X_n]) ? 0).*

By scaling, we may assume that *l([X_n, Y])* = 1; this gives (4) with *Ye* 00. •

Next, we show that the partition of *Ug* into the TV respects the action of Ad*(AT). (It is easy to check that Ad*(AT) takes each *Us* to itself.)

(2.4) LEMMA. *If6 eAandle U_s, then Ad* (K)l* ⊆ N_h

Proof. Ad(K)* acts unipotently on g* and *X\,..., X^*_{p+m}* is a Jordan-Hölder basis for this action. Therefore, as in Pukanszky's parametrization theorem (see [6]), we may define dimension indices

c(/) = (fii (/),..., *„(/)), $n = p + m$, «,-(/) = dim *K-Pj(l),* the set e^K = *{e* e Z^n: *e = e(l)* for some / € g*}, layers *Uf,* and sets of "jump indices" $R_2^K(e)$. Here, e,· - £,_i = 0 or 1 and is 1 iff / € R_2^K(£)-of If / € *Uf* and $R_2^K(e)$ = *{j* < • • • < A}> tnen we can nn(1 "faction vectors" *yK{l)* = *{Yi(l),.... $Y_k(l)$}* c *t* such that

$$\mathrm{ad}^* Y_j(P_{i_j}(l)) = P_{i_j}(X_{i_j}^*);$$

moreover,

$$Ad^*(K)l = \{\mathrm{Ad}\hat{}\exp\hat{}F! \cdot\cdot -cxpt_kY_k)l:\ h,\dots,t_k \in \mathrm{R}\}.$$

(This last statement is proved on pp. 50-54 of [6].)

Now let / € U_s n *Uf.* It suffices to show that i?f(e) *Q Rii^),* since this will imply that the set *PK(1)* can be extended to an action basis at / for the action of *G.* Then (4) and Proposition 2.2 imply that *Ad*(K)l* ⊆ N_h as desired.

So choose / e i?f(e). If 1 ≤ / ≤ *p,* then

$$\dim(K \cdot P_{f\text{-}}(/)) - \dim(A'' \cdot P/\_i(/)) = 1,$$

and this implies that / *e R'2{S)* ⊆ i?2(<5)- If p + 1 < / < w + p, then there is an *X* € *i* with ad*(X)P((/) / 0 and ad*(X)P;_!(/) = 0. Therefore *X* € t^^/) and *X £ tp(i).* It follows from p. 149 of [6] that dim Ad* G,-(P/(~/)) = dim Ad* G)_I(P/_I(/))+2, or that i e /J^(<5) c $R_2(\delta)$. D

3. Here we give our first decomposition of *n\x* as a direct integral. In this section we let <*? = <fn.* Let 5 be the largest index in A such that *Us* meets @. Then *Us (~)<f is* Zariski-open in the R-irreducible variety *0.*

Let

$$R_2(d) = R_2 = R'_2 U R'_1 = \{ ;,< \cdots < j_r < j_{r+1} < \cdots < j_{r+k}\},$$

with $j_r < p <$ yV+i. Define ^: R^ x *(ffi* n f/j) -» ^ by

(5) <?(«,/) = Ad*(exp(w1X,v+1)---exp(M^+J)/,

and, for fixed *fe&nUs,* let *Xj = £(R^k, /)*. The set X^δ_f may extend outside of *Us',* to deal with this and with technical details of later arguments, we define a "Zariski-open subset" X_f as follows. Let A_f be the subset of $u \in R^k$ such that

(6) $\quad i(0,\dots,0,u_s,\dots,u_k,f) \in U_d, \qquad$ for each $s \le k$.

Now we define

(7) $$X_f = \xi(A_f, f), \quad \text{all } f \in U_\delta \cap \mathcal{O}.$$

Then A_f is a non-empty Zariski-open set in R^ because / G $U_\delta \cap \mathcal{O}$; U_s n (f is Zariski-open in <f, and £ is polynomial in w with range in ^f. Obviously X_f C_<^ n U_g; X_f will be the base space in our first decomposition of $n\backslash x$ into irreducibles.

(3.1) PROPOSITION. *Let* $@_K$ *be an orbit in* g*, *let* 8 G A *be the largest dimension index such that* U_s *meets* ... *as above, where* ... *and define* ... *fixed element of* $U_s n O_n$. *Then:*

(8) (a) $\xi(\cdot, f)$ is *injective from* A_f *to* X_f,
(b) *Each variety* N_l *in* U_s *meets* X_f *in at most one point.*

Proof. Consider two points $l, l' \in X_f$ of the form $l = \xi(u, f)$, $l' = \xi(v,f)$ with $u, v \in A_f$, such that $l' \in N_l \cap X_f$. If an action basis $y = \{Y_1,\dots, Y_{r+k}\}$ is specified at $l \in U_g$, we have $y/l(t) = l'$ for some $t \in R^{r+k}$. We will show that $u = v$ and $t = 0$. This clearly proves (b), and part (a) is the special case $l = l'$.

We use induction on dim g/t. When i = g, the result is trivial because $X_f = \{l\}$ and $N_l = K \cdot l = @_n$. Thus we assume the result for Q_{m+p-1} = go and prove it for g. Let $J(S)$ be d with the last index removed $(J(d) = (S_1,\dots, \delta_{m+p-1}))$; $J(d)$ is a dimension index for go-

There are two cases.

Case 1. $m + p \notin R_2(d)$. Then P_o: g* -> g£ maps $O_K = G \cdot f$ diffeomorphically to ^o = G_o-$P_o(f)$, and $P_O(U_s) \subseteq U_{J(s)}$. Thus y is an action basis at $P_0(l)$ in U_m, $P_0 y_l(t) = y/_{P_0(l)}(t)$, and $P_0(N_l) = N_{P_0(l)}$. The layer U_s is P_0-saturated: $P_0^{-1}P_0(U_s) = U_s$ (since $G \cdot X^*_{m+p} = X^*_{m+p}$). Therefore $P_Q(U_g \cap @_n)$ is topologically open and dense in Go • Po/- Thus if we define the dimension index set AQ for g^ using the basis $\{X_1,\dots,X_{m+p-1}\}$ in go, then $U_{J(s)}$ is the first layer to meet $GQ \cdot P_of$ = ^o> and $U_{J(6)}$ is Zariski-open in <%. That is, the vectors $\{X_j : j \in R_j(S)\}$ used to define ^ are precisely the ones needed to define £o: $R^k \times (\&b \cap U_j)) \to$ ^o- This map satisfies

(9) $$P_Q(\xi(u, l)) = \xi_0(u, P_0 f) \qquad (u \in R^k, f \in U_s \cap \mathcal{O}).$$

We can say more:

(10) $$Po\{u_{s}n^*) = u_{J(i)}\, n<r_{0}.$$

For if / *e* *Gf* and $P_0(l)$ *e* $U_m\backslash P_0(U_s)$, then $*,-(/) = e_t$ and $d_t(l) = d_t$ except that $d_{m+p}(l) = 2 + d_{m+p}$-\. However, the ordering of indices in (A, >) satisfies $8' \geq 8$ in A if $8\backslash \geq 8_t$ for all i. Then $3(1) > 8$ e A. But *8* is the largest index with *Us* meeting *G* • *f;* this contradiction proves (10).

We conclude from (9) that $A_f = A_{P(j(f)}$; thus $P_0(l)$, $PQ(I')$ lie in $Po\{Xf) = Xp_o(fy$ Now the claim that $u = v$ and / = 0 is immediate by induction, since *PQ* is a diffeomorphism on *G* • *f.*

Case 2. *m* + *p* € J?2(<5)- We write *u* = («', «o), with *UQ* = Mr+^, and use similar notation for *v* and *t*\ note that yr+^ = *m* + *p*. We have *if/t(t)* = *I'*, which means that

$$!<W)(0 =£(*>./).$$

or

$$(\exp(/iFi) \cdots \exp(t_{r+k}Y_{r+k})\exp(uiX_{Jr+l}) \cdots \exp(u_k\hat{\ }iX_{jr+k\_t})Qxp(u_0X_{m+p})) \cdot f$$
$$= (\exp(v_1X_{j_{r+1}}) \cdots \exp(v_{k-1}X_{j_{r+k-1}})\exp(v_0X_{m+p})) \cdot f.$$

Write this as $X\backslash \cdot f = X2 \cdot /$, and let $Rf = \exp(t\hat{\ })$. Then $X\backslash Rf \ast_2 R_f$. Since / € *Us,* we have $*p_o(f)$ 2 ty; thus ty c-g0 and $X\backslash GQ = x_2G_0$. From Lemma 2.3, we have *Yj* e g,-;,, Çg0 for all 7, so we get

$$\exp(u_0X_{m+p}) = \exp(\hat{\ }_0X_{m+p}) \quad \mathrm{mod} C?o,$$

or $u_0 = v_0$. Now let $f_x = \exp\{uoX_{m+p})f$ and /0 = Po(/i). Since *u* G *Ay,* it follows that *f\&Us',* hence /o G ^/(<j)-

We show next that /(J) is the first index J0 e Ao such that U_{6o} meets Go • /o- Since we are in Case 2, the set *GQ* • *f*\ is Po-saturated and *PQ*\ *GQ* • *f*\ —> Go • /o is a surjective open mapping. Hence *Uj*^ meets *&o* = *GQ* • /o in a nonempty open set. The first layer Us_0 to meet Go • /o intersects in a Zariski-open set; hence $8Q = J\{8)$. Therefore $\{Xi_j: r + 1 \leq 7 \leq r + A— 1\}$ is the set of vectors corresponding to $R^{\prime}_2(J(3))$, and these are the vectors used to define the map $£0: R^{k\sim}$ (^0 n $U_{J(s)})$) - ^0 (= $G_Q \cdot P_ofi = GQ \cdot /o$) and the variety $X_{fo} = X_{Pofr}$ Since *PQ* intertwines the actions of G on g* and $JQ, we have

$$P_0(\xi(u', «o;/)) = ZoW.\ Poifi)). \qquad \text{all } u' \in R^{\wedge 1}.$$

In particular, for our u, v we have

$$(11)\qquad P_0(l) = P_0\psi(u, f) = \xi_0(u', P_0(f_1)) = \xi_0(u', f_0);\qquad P_0(l') = \xi_0(v', f_0).$$

These lie in $\mathcal{O}_0 \cap U_{J(S)}$. If $\mathcal{B} = \{Y_1, \dots, Y_{r+k}\}$ is an action basis at $l \in U_S$, then $P_0\mathcal{B} = \{Y_1, \dots, Y_{r+k-1}\}$ is an action basis at $P_0(l) \in U_{J(S)}$, from the description of $J(S)$ given above. Moreover, $Y_{r+k} \cdot l = X^*_{m+p}$, since $m + p \in R'_-(S)$, and thus

$$\mathrm{Ad}^*(x_0 \cdot \exp(\mathbb{R}Y_{m+p}))l = x_0 \cdot l + \mathbb{R}X^*_{m+p}, \qquad \text{all } x_0 \in G_0.$$

It follows that $N_l(y) = P_0^{-1}N_{P_0(l)}(P_0 y)$, in particular, $P_0(l') \in N_{P_0(l)}$. The induction hypothesis applies once we show that $P_0(l)$, $P_0(l')$ are in the variety $X_{P_0 f} \subseteq \mathcal{O}_0 \cap U_{J(S)}$. From (11), this amounts to showing that $u', v' \in \Delta_0(f_0) \subseteq \mathbb{R}^{k-1}$. We give the proof for u'; that for v' is nearly identical. Since $u \in \Delta$, we have

$$\psi(0, \dots, 0, u_s, \dots, u_{k-1}, u_0; f) \in U_S, \qquad \text{all } s.$$

Hence

$$\xi_0(0, \dots, 0, u_s, \dots, u_{k-1}; P_0 f_1) = P_0\psi(0, \dots, 0, u_s, \dots, u_{k-1}, u_0; f) \in U_{J(S)}$$

for all s, and this means that $u' \in \Delta_{P_0}(f_0)$.

Since ad Y_{r+k} acts trivially mod ker P_0, we have

$$\psi^0_{\xi_0(u', f_0)}(t') = \xi_0(v', f_0).$$

By induction, $u' = v'$ and $t' = 0$. But now we have $u = v$, and

$$l = l' = \mathrm{Ad}^*(\exp t_0 Y_{r+k})l = l + t_0 X^*_{m+p}.$$

Therefore $t_0 = 0$, and we are done. ∎

(3.2) PROPOSITION. *Let $\mathcal{O} = \mathcal{O}_\pi$ be an orbit in $\mathfrak{g}^*$, let S be the largest index in Λ such that U_S meets $\mathcal{O}_\pi$, and fix a base point $f \in U_S \cap \mathcal{O}_\pi$. Define the varieties N_l, $l \in U_S$, as in Proposition 2.2, and for any set $S \subseteq U_S$ define its saturant $[S]$ to be $\bigcup\{N_l : l \in S\}$. Define $X_f \subseteq U_S \cap \mathcal{O}_\pi$ as in Proposition 3.1. Then $[X_f]$ is semialgebraic and is topologically dense in $\mathcal{O}_\pi$; hence it contains a dense open set in $\mathcal{O}_\pi$ and is co-null with respect to invariant measure on $\mathcal{O}_\pi$.*

Proof. Any semialgebraic set S has a stratification (see, e.g., [9]); that means, among other things, that S can be written as a finite disjoint union of manifolds that are also semialgebraic sets. Let $\dim S$

be the largest dimension of any manifold in the stratification; this is independent of the stratification. If T ç S is semialgebraic and dense, then necessarily dim(5' \ 7) < dim S; this follows from the fact that S has a stratification compatible with T. In particular, $S \setminus T$ is null with respect to (dim5')-dimensional measure on S. Thus the proposition will follow once we show that *[Xf]* is semialgebraic and dense in S.

Since *Xf* is the polynomial image of a Zariski-open set in $R^{k,}$ k = *R'{(S),* it is semialgebraic. We can cover *Ug* by finitely many Zariski-open sets Z_a C g* on which are defined rational nonsingular maps {*Yf(/)*,..., *Y*$_{r+k}$(l)}* that give an action basis at each *I e* $Z_a n U_s$ (Proposition 2.2). Let

y/Q(l, t) = exp(t$_{(}$ Yf (/)) • • • *cxp(t$_{r+k}$ Y?$_{+k}$(l))* *I,* *leZ$_{a}$,te* R^{r+k}.

Let S_a = $Z_a n$ $U_s r \setminus X_f$. Then *[S$_a$]* = *y/$_a$(S$_a$ R^{r+k})* is semialgebraic, and *[Xf],* the union of the S_a, is also semialgebraic.

To prove the density of *[Xf],* we work by induction on dim(g/t); the result is clear if g = *i.* In general we have two cases, as in previous proofs; the first, where *m + p ^ Ri(8),* is easy because the projection map *PQ* is a diffeomorphism for all the objects under consideration.

Thus we assume that *m + p* e *Ri{S).* We know that *Af* is Zariski-open in R^k and 0 e A_f. Hence *Si = {t e* R: (0,..., 0, /) € *A$_f$}* is nonempty and Zariski-open in R, and

$$t \in S_1 \Rightarrow f_t = \xi(0,\dots,t;f) = \mathbf{Ad}^*(\exp tX_{m+p})f \in U_\delta,$$

where ^: R^k x *(Us* n *(?)* —• *&* is as in (5). Also, *<9* is a disjoint union of Go-orbits in g*,

$$d? = {J \underset{t\in\mathbf{R}}{} Ad^*(G_o)ft \quad \text{(disjoint)};$$

see pp. 147-150 of [6]. For each *t, GQ • f* is P0-saturated, and *P$_o$: GQ-ft* —» Go • ^ (/ J) is surjective and intertwines the actions of Go. By the open mapping theorem for homogeneous spaces, this map is also open. The union of the Ad*(Go)/?, *t* e *S\,* is dense in *&.*

Fix / € *<9.* We want to show that *[Xf]* contains points arbitrarily close to /. Given *e* > 0, there is a *t* e *Si* such that dist(/, Go • *ft) <* e/2, where we take Euclidean distances on g*, g^ compatible with the projection P_o. Set^ = *Go-f, 0? = P$_0$((f$_t$) = G$_0$-P$_0$(ft).* Then *U$_s$r\setminus f$_t$* is Zariski-open in *<f$_t$* and is nonempty (because /, is in the intersection). An argument like the one in Proposition 3.1 now shows that *Uj($)* H^P is Zariski-open in *@f* and that *3(5)* is the largest index *8$* e Ao with $U_{\delta_0} \cap \mathscr{O}_t^0 \neq \varnothing$.

Write *Nfil* for the variety through *(p* G *Ui^s)* ^ 9> an(* *et ^ be the subset satisfying the condition analogous to (6) for $P_0(f_t) \subseteq R^{k\sim}$. *(t^f, t) € Af}*. Since 0 G *Bf*, B_{fi} is nonempty *Ui^s)* $\subseteq$ *QQ*. Let B_{fi} = *{t'* G $R^{k\sim l:}$
and Zariski-open; it is also easy to verify that

$$Bf, \wedge AP_0(f,y$$

Let

$$X_{P_0(f_t)} = \{\xi_0(t', P_0(f_t)) : t' \in A_{P_0(f_t)}\},$$

$$Y_{Po(fl)} = \{Pattf, t\text{-}J): f \text{ G } B_f\} = \{\&C. W/)) : t' e B_f\}.$$

where £o is defined as in (5), but on go- We have *Yp$_o$(f,) Q Xp$_o${f,)-* ^ we can show that *[Y$_{Po}$^]* is dense in ^° then we are done. For then, applying *PQ^{l,}* we have (since N_t = *P~$^{tN_{Po}}$^y*, see the second part of the proof of Proposition 3.1)

$$\{^{Ni}(t',t;f)'\text{-}\ t' € B_f\} \text{ dense in } G_o \bullet f_t.$$

Therefore there exists $(t^{l,} t)$ e *Af* and *I* e *N^'.rj)* with dist(*/, h) < e,* as required.

The induction hypothesis tells us that *[X$_{Pg}$^]* is dense in ^f. It suffices, therefore, to show that [1V$_O$(/,)] is dense in *[X$_{Po}$^]*. Suppose that *(p'* € N_n, *q>Q* G *X$_{Po}$(fy* Choose rationally varying maps on a Zariski-open set Z $\subseteq$ g* to get an action basis *{Y\{q>),..., Y$_{r+k}$_\{(p)}* on Z n $U_{J(S)}$, with *(p$_Q$* G Z, we may write *l,* we have *<p0* = *€o(t'$_o$,Po(ft)),* with *t'$_o$* G *A$_{Po}$(fy* Then for some *u* G $W^{+k\sim}$

$$\varphi' = \psi(u, \xi_0(t_0', P_0(f_t))).$$

Let *{t'$_n$}* be a sequence in.-Sy; converging to ^ such that *€o(t'$_n$,Po(ft))* is always in Z. Then {^(M,<^O(^.^O(/<)))} is a sequence in *{Y$_{Po(fi)}$}* converging to *cp'*, as desired. n

(3.3) THEOREM. *Let* g *be a nilpotent Lie algebra, t a subalgebra, G "2 K the corresponding simply connected Lie groups, and P:* g* —• *V the natural projection. Let n EĜ, and let & = &$_K$ be the corresponding orbit ing*. Fix a basis X\,..., X$_p$,..., X$_{m+P}$ through I as in Proposition* 2.2, *and define*

$$A, U_3, £\colon R^* \times (<? \cap U_s) \to (9 \quad \{k = \operatorname{card} R_2''(\delta)).$$

Fix any f G *&* n *U$, and define the sets A$_f$ ⊂ R^k, X$_f$ = £(Af, f) as in Proposition* 3.1. *Let dpi on Xf be Euclidean measure on Af (or R^k),* *transported via the map £. Then*

(12) $$*\backslash \mathcal{K} = \int_{JX,}^{r®} <rp\{i)d/i(l) \int_{JR^k}^{r®} a_{PmJ))}dt,$$

where a_9 E $\hat{K}$ is the representation corresponding to <p E t.*

Proof. We use induction on dimg/fc, the case t = g being trivial. As usual, let go = *9m+p-i>* an<i let *PQ:* g* —• g£ be the natural map. The inductive step divides into the usual two cases. Case 1, where *m + p $. R2(d),* is easy: *7\Go* is then irreducible, and *Xf* projects difFeomorphically to *Xp_o(f),* since *PQ{<9* n *U§) = PQ{@)* n *UJ^)* (see the proof of Case 1 of Proposition 3.1). In Case 2, m + *p* € *R2IS)* and we know (see, e.g., Lemma 6.3 of [4]) that

(13) $$\mathrm{TT}|_{GO} = \int_{JR}^{r®} n^{\circ}_{Po(fs)}ds, \quad f_s = Ad^*(expsX_{m+p})f.$$

Let k = cardR'$_2$'(^), let *P': Q^* —> *t** be the canonical projection, and let *Si = { t e R : (0,... ,0,0 e A°fj,* so that /, = *i(0,... ,0,t;f).* For each *t E Si, do = J{6)* is the largest index in AQ such that |K$, meets ^° = *Poi&t),* where *tf_t = Go-ft',* this was proved in the course of proving ^*(Uj(5)* n^ °) —> Proposition 3.2. The corresponding maps *£o,t'- R^{k~}* ff,^{0} are all denned in the same way, using the vectors *{X_{ij}:* 1 < *j* ≤ *k* - 1} corresponding to *R$_2$'(J(S)):*

$$\xi_{0,t}(u, \varphi) = \exp(u_1 X_{i_1}) \cdots \exp(u_{k-1} X_{4-1}) \cdot tp.$$

Thus for *u'* G $R^{k\sim l}$ and ? € 5$^{1)}$ we have

$$\xi_{0,t}(u', P_0 f_t) = P_0 \xi(u', t; f).$$

The inductive hypothesis says that for *t E Si, we* have

$$_nP_o(f,)\backslash K = \sim \int_{JR^{k\sim'}}^{/\cdot ©} L \quad ^\wedge P'ioAW.Pof,)^\wedge$$
$$= \int_{JR^{k-i}}^{r®} {}^{\circ}PZ\{u',ij)du',$$

since P'Po = *P-* Thus (13) (plus the fact that *Si* has full measure in R) gives

$$*\backslash K \cong \int_{JR}^{r®} \int_{JR^{k-<}}^{r®} Gpz(u',tj)^{dW} dt = \int_{JR^{l*}}^{r} a_{Pi(u,f)}du.$$

As Jy^ is Zariski-open in R^k, the rest of the theorem is clear. •

We note two important facts about our constructions. Fix / e $\&_n$ n *Us* and define *Xf* as above; cover *U$* with Zariski-open sets Z_a c g* equipped with rational maps {Ff (/),..., $Yf_{+k}\{l)\}$, *k+r* = card(i?$_2$(<*)), that provide an action basis at each / e *Ug* n Z_a and thus generate the variety JV/ through /. Recall our labeling of the jump indices: *j* \ < • • • < *jr* < • • • < j_{r+k}, where $j_r < p < j_{r+x}$.

(3.4) LEMMA. *For every I e* $X_f C\setminus Z_a$, *the vectors* {*Yf* (/),..., $Y^{\circ}{}_{+lc}(l)$, Xj_{r+l},..., Xj_{r+k}} *are linearly independent and span a complement to the radical* t/. *In particular, the map X{u) = £(«,/) has rank k at u = 0 and is a local diffeomorphism into* $@_n$ *near u = 0.*

Proof. If g = *t*, then *k* = 0 and we have Pukanszky's parametrization of $<?_a = K \bullet I = N_h$ so that the lemma holds. We proceed by induction; we have the usual two cases.

Case 1. *m* + *p* £ *R2(S).* Then $j_{r+k} < m + p$, and as in the discussion of this step in the proof of Proposition 3.1,.Po :0*-*9a maps <*f* = *Gf* diffeomorphically onto *GQ* • *Pof*, carrying *UsCX? onto Uj*^ n *G$* • *PQ/* and *Xf* onto X_{Pof}. We have $Af = A_{Pof} = A$ (say), and $£_0(u, Pof)$ = *P(i£{u,f),* all *u* € *A*. Since g \ go contains an element of t/ (t/ £ go because t/ ngo *Q* t/$_0$ and a computation gives dimt/ = dimt$_{/0}$ + 1), the inductive step is now easy.

Case 2. *m+p* e *R.2(S)*- Then $Xj_{r+k} = X_{m+P}$, t/ has codimension 1 in t$_{Po}$(/) C g$_0$, andt$_{Po/}$ = RF$_{r+}$^(/)©t$_/$. By induction, {*Y^l*),... ,Y_{r+k}^(*l*), Xj_{r+l},..., Xj_{r+k}} span a complement to r$_{Po/}$ in g$_0$. The first part of the lemma is now clear. At *u* = 0, A(0) = /; from the way that q is defined by the {X_{ji}, *r* + 1 ≤ *i* ≤ *r* + *k*} at /, we have rank(fiW)$_0$ = *k*. D

(3.5) REMARK. Let *X D Y* be semialgebraic sets in U_s. As the argument at the start of Proposition 3.2 shows, their saturants *[X], [Y]* are semialgebraic. Furthermore, if *Y* is dense in *X* (in the relative Euclidean topology), then

(i) *[Y]* is dense in *[X];*

(ii) dim[X] = dim[7] > *dim([X]\[Y]).*

In particular, the canonical measure classes for *[X], [Y]* are the same. (If dim[X] = *m*, the canonical measure class for *[X]* is m-dimensional measure on the submanifolds of dimension m in a stratification of *[*]•*)

4. In this section, we give the geometric interpretation of the direct integral decomposition in Theorem 3.3.

Let $\mathcal{O}_\pi = \mathcal{O}_{f_\pi}$ be the orbit in $\mathfrak{g}^*$ for $\pi \in \hat{G}$, and let $\mathfrak{k} \subseteq \mathfrak{g}$ be a subalgebra. Fix a basis $X_1, \ldots, X_p, \ldots, X_{m+p}$ for $\mathfrak{g}$ through $\mathfrak{k}$ as in §2, and define $\Lambda, d = (d,e)$, U_d, $k = \operatorname{card} R'_1(S)$, $r = \operatorname{card} R'_2(S)$, $\xi\colon \mathbb{R}^k \times (\mathcal{O}_\pi \cap U_S) \to \mathcal{O}_\pi$, etc., as in §3. Fix $f \in U_S \cap \mathcal{O}_\pi$ and let $\lambda\colon A_f \to X_f$ be given by $\lambda(u) = \xi(u, f)$. We need some information about X_f, which acts as the base space in the decomposition of Theorem 3.3. We already know that the varieties N_l $(l \in X_f)$ are transverse to X_f in the set-theoretic sense; we need a differentiable version of this fact.

(4.1) LEMMA. *In the above notation, there is a Zariski-open set $B_f \subseteq A_f$, containing 0, such that:*

(a) $\lambda\colon B_f \to Y_f = \xi(B_f, f)$ *is a bijective local diffeomorphism on*

(b) $\dim X_f \setminus Y_f < \dim Y_f = k$ *(thus X_f, Y_f have the same canonical measure classes);*

(c) *For all $l \in Y_f$, the following result holds between tangent spaces:*

$$T_l(\mathcal{O}_\pi) = T_l(Y_f) \oplus T_l(N_l).$$

Proof. From Proposition 2.2, the N_l are defined by rationally varying families $\{Y_1(l), \ldots, Y_{r+k}(l)\}$ defined on Zariski-open sets Z_α that cover U_S. Fix an index α such that $f \in Z_\alpha$. Lemma 3.4 says that for all $l \in Z_\alpha \cap X_f$, the vectors $\{Y_1(l), \ldots, Y_{r+k}(l), X_{j_{r+1}}, \ldots, X_{j_{r+k}}\}$ span a complement to $\mathfrak{r}_l$, and that $\operatorname{rank}(d\lambda)_0 = k = \dim X_f$. This maximal rank is achieved on a nonempty Zariski-open set $B_f^1 \subseteq A_f$ containing 0, since λ is polynomial. Thus $Y_f^1 = \lambda(B_f^1)$ is a dense open subset of X_f (in the relative Euclidean topology), and $\lambda\colon B_f^1 \to Y_f^1$ is a bijective local diffeomorphism. At $l = \lambda(0) \in Y_f^1$, the tangent space to Y_f^1 is $T_f(Y_f^1) = \mathbb{R}\text{-span}\{\operatorname{ad}^* X_{j_i}(f): r + 1 \le i \le r + k\}$, as one sees by direct calculation. (This need not hold elsewhere.) From the definition of the sets of jump indices $R'_2(S)$, $R'_1(S)$, we know that $r + 2k = \dim \mathcal{O}_\pi$; by the definition of the N_l we have $T_l(N_l) = \mathbb{R}\text{-span}\{Y_i(l): 1 \le i \le k + r\}$, all $l \in U_S \cap Z_\alpha$. Taking $l = f$, we have

$$T_f(\mathcal{O}_\pi) = T_f(Y_f^1) \oplus T_f(N_f),$$

by Lemma 3.4. But $\dim T_l(\mathcal{O}_\pi) = r + 2k$ everywhere on $\mathcal{O}_\pi$, while the subspaces $T_l(Y_f^1)$, $T_l(N_l)$ have respective dimensions k, $r+k$, and vary rationally on $Y_f^1 \cap Z_\alpha$. Since transversality is generic, there is a Zariski-open set $B_f \subseteq B_f^1 \cap \lambda^{-1}(Z_\alpha)$ such that $T_l(\mathcal{O}_\pi) = T_l(Y_f) \oplus T_l(N_l)$ for all $l = \lambda(u)$, $u \in B_f$. This proves (a) and (c), and (b) follows because Y_f

is dense in *Xf* and both are semialgebraic.(See the start of the proof of Proposition 3.2 for a similar argument.) •

We now consider the maps shown in Figure 1:

R* 2 A_f —i-> *XfCUgn^D_ [X_f]* ^— X_f x W^{+k}

P

TjK P_e^{-1} V w T/ *nT* . V ← T/

FIGURE 1

Here, *P*: g* —• f* maps (/$ into$_e$£/* (where d = *(e, d)); Uf* is a layer in 6* for the strong Malcev basis *{X\,..., X_p).* (Since *P~^{l}(Uf)* contains a Zariski-open subset of <?$_n$, *e* is the largest index in the ordering of layers in *V* such that *P~^{x}(Uf)* meets *&$_n$.)* The map *P~^{x}: Uf* -> £$_e$ x ^S(e)> ^e = *Uf^VT(e)>* is the inverse of the Pukanszky parametrization for this layer (see [7]), and *n$, nx* are the projections splitting *t** = *Vr(e)* © Ks(e)- Define

$$\varphi = \pi_T \circ P_e^{-1} \circ P : \mathscr{O}_\pi \cap P^{-1}(U_e^K) \to \Sigma_e;$$
$$\Phi = \varphi \circ \lambda : A_f \to \Sigma_e;$$
$$\tilde{\varphi} = \varphi|_{X_f} : X_f \to \Sigma_e.$$

Note that *(f$_n$DP'^{l} (Uf)* 2 ^r n *U_s;* both are Zariski-open in <?$_n$. These maps are rational and nonsingular. Fix a stratification & of X^ (it has dimension = dimX9 — *k),* and define

(14) *K* = max{rank(^)/: / € U_s D <?$_n$}
$$= \max\{\operatorname{rank}(d\varphi)_l : l \in P^{-1}(U_e^K) \cap \mathscr{O}_\pi\},$$
ko = max{rankc/(^|5)/: *I eS,S* G^.dim^ = *k},*
k = max{rank(dM>)$_M$: *u* G *Af}.*

As the maximal rank of *d(0\s)i* is attained on an open subset of *S* e *3^{s},* and as the pieces of maximal dimension in *3^{s}* are open in *Xf,* it follows that *ko,* does not depend on the stratification *3°.* Also, *d$>* attains rank *k* on a Zariski-open set in *R^k.* Since *S** = *\/{S* G *3^{d}:* dim *S* = *k}* is open in *Xf, A~^{l}(S*)* is open in *R^k,* and since A is a local diffeomorphism on the Zariski-open set *Bf,* we conclude that *ko* = *k\.* It is now easy to see that

ko = *ky* ≤ *k** and *k* ≤ *k* = dim *Xf.*

More is true, in fact.

(4.2) LEMMA. *In the above situation,* A* = k\ = k_0_< k = *dimXf.*

Proof. In view of the above remarks, we need only show that k^* = k\. Let 9° be a stratification of *Xf* compatible with *Yf,* as defined in Lemma 4.1. All *k*-dimensional pieces of & lie in *Yf,* since *dim(Xf* \ *Yf)* < *k.* From Proposition 2.2(c) and Lemma 2.4, *K* • *I*_c JV,_c *K* • *I* + e^x for any / € U_3. Thus P(JV,) = *K* • *PI* and <p is constant on each JV/ with / <E *UsC\(*f_n.

Consider a Zariski-open set Z 9 C g * containing / and such that the action bases *{Yx(l)*$_{t}$..., *Y^^l)}* are rationally defined on Z_ar\Us (see Proposition 2.2). Define

$$P(u,t) = y_a(i(u,f),t) = y/_a(A(u),t),$$
$$t \in \mathbf{R}^{r+k},\ u \in E = \xi^{-1}(Z_\alpha) \cap B_f,$$

where $y/_a(l,t)$ = *i//j(t),* as in (3); note that 0 e *E* and that *E* is Zariski-open in *Bf.* Clearly Range(/?) = *[*Z_a n *Yf],* since *X* is bijective on *R'Bf.* The set £ x W^{+k} contains (0,0) and is Zariski-open in R^k x *R'Bf.* Lemma 3.4 (plus an easy computation) shows that Rank(d/?)$_{(0,0)}$ = *r* + 2*k.* This rank is clearly maximal and is achieved on a Zariski-open set *S* C *E* x R^{r+fc}; furthermore, (0,0) € 5. Then S_x = *Sn(E* x {0}) is a Zariski-open set in R^ x {0} containing (0,0). The maximality of rank implies that /?: *S* —• &$_n$ is a local difFeomorphism and that /?(>S) is open in <f_n. Let (wi,0) e Si, and let JV = / x *J* C R^k x R^{r+fc} be a rectangular neighborhood on which *ft* is a diffeomorphism onto some open neighborhood of /j = *fi(ui,0)* in ^(S) _c [Z_a n Y_f*]* C ^. We have /j G Z_Q n Y_f.

As we remarked earlier, <p is constant on each JV/; thus *<p o fl is* constant on *{u}* x / for all *u e I.* Therefore *<po p*\$_W$ is determined by $\varphi \circ \beta|_{I\times\{0\}}$, **and**

(15) max{rankd(0>o/?)$_{(M?)}$: *(u,t) e* JV}

- max{rank(p o ySl$_{/x\{O\}}$})$_{(M,O)}$: « e /}

= max{rank c/(^ o A)$_M$: M G /}

= max{rank{(/(9» *oX)*$_u$*:* M6 5/ } = fci.

The penultimate equality holds because the maximum is achieved on a Zariski-open set and hence on any open set. As *<p{N)* is open in U_s n^r, (15) implies that

K = max{rank^($? ° *P)(*$_{u,t)'}$- *{u,t) e N}* = k_x,

as desired. •

The number *k*\ (the generic rank of *d{(poX)* on *Bf)* is an important constant for our geometric analysis of multiplicities. It is convenient to introduce the "defect index"

(16) To = dim^r - 2(generic dimension *{K• I: I* e *&n})*
+ generic dimension *{K• PI: I e tfn}.*

We will show that *k* = *k*\ o- To = 0.

The definitions of *r* and *k* show that dim <$n = *r+2k.* The generic (= maximal) dimension of *K• I, I €* @n, is achieved on a Zariski-open set; hence it equals the (constant) dimension *of K-1,1 e U§V\(9n.* Similarly, generic dim{AT· *PI: I e @n}* = generic *dim{K• PI: I eUsr\@n}-* Since *P~x(Uf)C\@n* is Zariski-open in *&n,* we have

(17) generic dimension *{K• PI: I e <?n}* = dim{AT• *(p: cp € Uf}* = *r.*

Since dim *N[* = *k* + *r* for generic / e @n, we have

(18) dim ^ + dim is: · P / - 2 dim AY = 0 for generic /e <fx n *Us.*

An immediate consequence is:

(4.3) LEMMA. *We have* r_Q = 0 *iff Nt = K• I for generic I e* @n n *Us.*

Proof. Formulas (16) and (18) show that To = 0 iff dim *N[* = *dim Kl* for generic /. From Lemma 2.4, *Kl* C TV/; since both of these varieties are graphs of polynomial maps, they have the same dimension iff they are equal as sets. •

We need another lemma to relate To and *k*\.

(4.4) TRANSVERSALITY LEMMA. *Let S"* = *{I* e [/jfK?,: **ank**$(d\varphi)_l$ *is maximal). Then ker(d<p)i* = ad*(g)/ n t£(/) *for all I € S", where*

$$\mathfrak{r}_{P(l)} = \{X \in \mathfrak{k} : \mathrm{ad}^*(X)Pl = 0\},$$

and the annihilator is taken in g*.

Proof. There are Zariski-open sets *Zp* C *t** covering *Uf,* plus rational nonsingular maps *Qp* defined on them, such that on *Zp* n *Uf,* *Qp* = *Pe~x* *(Pe* is the Pukanszky parametrizing map described earlier in this section). Let *Up* = *P~l(Zp).* Then the *Up* are Zariski-open sets in g* covering *P~l(Uf),* and *QpoP* = *Pe~l oP* on *Upf)P-1(Uf).* Hence <p = *nT* o *Qp* o *P* on *S"* n *Up.* (Since *S"* c *Cj* and *P(US)* C C/f, we have *S"* *QP~1(Uf)* automatically.)

Fix / e *S"*. Since rank(aty>) is constant on *S"*, a standard result (see Lemma 1.3 of [8]) shows that *S"* foliates into leaves on which *(p* is constant; at /, there is a rectangular coordinate neighborhood *N* = *I* x *J* in 5"' (with / a /^-dimensional cube and / a ^-dimensional cube, say), such that *(t)* x *J* is the intersection of a *<p-\eaf* with *N* and values of *<p* are distinct on each *(t)* x /, *t* e /. Since *(p\s>< = TIT* ° *Pj~** ° *P\s"* and

$$\{l' \in P^{-1}(U_e^K) \colon \pi_T \circ P_e^{-1} \circ P(l') = \pi_T \circ P_e^{-1} \circ P(l) = \varphi(l)\}$$

= {/ G *p-\Uf): K-PV* = *·KPl}* = *P~\KoPl),*

we see that the #>-leaf through / is contained in *P~^l(K • PI).* The p-leaf through / is obviously in *<f_n* = *G•* *I;* hence it is contained in *G-lnP~^l(K- PI).* The tangent space to *G•* *I* is ad*(g)/ = *xf,* and the tangent space to *K* · *Pi* is ad*(€)P/ = t^-* (= annihilator in *V* of *x_n);* thus the tangent space to *P~^l(K- PI)* is *P~^l(Xp)^⊥*

(19) *ker(d(p)i* = tangent space to $?-leaf through / ⊆ *xj-* n t^,.

On the other hand, if / e 5"', then we can find an index /? with *I € Up.* On t/jg n *S", (p* is the restriction of *%T* ° Q^ ° *P,* defined on *Up.* It is easy to see that

ker(d<p)i⊇D (tangent space to *S"* at /) n *ker d(nr °Qp° P)i-*

But *nj* o *Qp* o P is constant on t/^ fi P^{-1}*(K• PI),* and so

(20) ker(^),⊇t/-nr^.

Comparing (19) and (20) gives the lemma. •

(4.5) COROLLARY. *With notations as above, we have*

k-k_X= ½τ₀.

In particular, N[= K • I iffk = k\, i.e., generic rank {d<pi^m. I e *tf_x} = Card R'{(S).*

Proof. Lemma 4.4 says that for all generic /,

ker(d<p)i = xj- n t^ = *(t_l* + tp,)$^{⊥}$

Hence, for all such /,

$\mathbf{dim\,ker}(d\varphi)_l$ = dim9 - dimt/ - dimt/>/ + dim(t/ nr_w)

= $\mathbf{dim}\,\mathscr{O}_\pi$ + (dimt - dimr_P/) - (dimt - dim(t/ ∩ t_Pl))

= $\mathbf{dim}\,\mathscr{O}_\pi$ + *dim(K• PI) - dim(K• I),* and

k_x=k*= generic rank{ker(ofy>)/: / e @$_n$}
= *dimtfx* - generic dim{ker(cfy>): / e &$_n$)
= dim$K$ • I - dim$K$ • PI (for generic le@$_n$).

Since k = ^(dim^f/ - *dim(K* • *PI))* for generic /, see (17), we see that To = *2(k - k\).* The final claim now follows from Lemma 4.3. D

We now deal with the case To > 0; this corresponds to the case of infinite multiplicity, as we will see. Regard <p = $HJ^{0P^{\lambda x}}$ P as defined on $P_\sim^{x}$*{Uf),* and not just on &$_n$ n *Cj* as above. Let

(21)
$$I,^{Jt} = \varphi(\mathscr{O}_\pi \cap P^{-1}(U_e^K))$$
$$\Sigma^\delta = \varphi(\mathscr{O}_\pi \cap U_\delta)$$
$$2/ = 9\{X_f).$$

These are semialgebraic sets with 27 $\underline{D}$ U^5 D 2/; hence $Y7^l$ has a stratification <?> compatible with ~L^s and *1/.* Notice that dimE* = $\dim \Sigma^\delta$ = k\ = k* = generic rank{(flty>)/: / G <f_n}.

(4.6) THEOREM. *Let g be a nilpotent Lie algebra, i a subalgebra; let {Xi,..., X_p,..., X_{m+P}} be a basis ofg through t as in* §3. *Let n* € $\hat{G}$ *and let* @$_n$ *be its coadjoint orbit. Define d = (e, d), as in* §2, *to be the largest index with Us meeting* @$_n$, *and let P: g* → 1/ as in (21). Let [v] be the natural map; define* T_0 *as in* (16), *and* $I.^{71}$, *I.* I^s, *canonical measure class on* 57. *Then:*

(a) 27, "L^{6},*1/ differ by sets having lower dimension than* 27, *so that they all determine the same measure class [v].*

(b) *If* T_0 > 0, *then*

$$\pi|_K \cong \int_{\Sigma^\pi}^{\oplus} \infty \cdot \sigma_l \, d\nu(l).$$

Proof. The discussion so far applies to any base point / *e (*f_n n *U$.* Fix such an /. We have seen that *P{Us) Q Uf.* Theorem 3.3 gives us a decomposition

$$\pi|_K = \int^{\oplus} \sigma_{\varphi\circ\lambda(u)} \, dm(u),$$

where *k(u)* = *£(u,f)* (see (5)) and *m* is Lebesgue measure on R^, k as above. We know that k* = generic rank{d?(^ o $X)_u$: *u* e *Af}* and that this rank is achieved on some Zariski-open set *E** $\underline{C}$ *Af.* Let

Z* = *((pok)(E*)* ⊆ Z-^; clearly dimZ* = *K.* The map *<poX* corresponds to a foliation of *E** with *g> o X* constant on each leaf; in fact, for any *u* G *E** there is a centered coordinate patch $W_u \simeq I \times J$ ($I \subseteq R^{k-}$, *J* ⊆ R*-fc-) such that *q>* o A is constant on fibers (?) x 7 and has distinct values on the transversal / x (0)—see Lemma 1.3 of [8]. Hence if *U* ⊆ *E** is open, then *(p* o *X(u)* contains a *k**-dimensional manifold.

Stratify Z*, letting Zp be the union of the A:*-dimensional pieces and Z_s^* the rest. Call this stratification *3°*. Let E_r^* = *[<p o X)-*^{1}*(L*)* n £*, E_s^* = £• * n *{(p o* A)$^{-1}$(Z_s^*). These sets are semialgebraic and partition *E**\ further, E_r^* is open in *E** because Zp is open in Z* and poA is continuous. In addition, E_s^* cannot contain a A:-dimensional piece, since such a piece would be open in *E** and hence contain a coordinate patch *W* ≃ *I x J* like the one above. But then dim(^ o *X{W))* would be *k*,* contradicting the definition of ZJ. Thus *dim(E^)* < *k* and E_r^* has full measure in *Af.*

Let *S\,..., S_p e S^6* be the &*-dimensional pieces in Z*, so that the pullbacks *Ej* = *(cp* o *X)$^{-1}$ (Si)* n E_r^* are disjoint open sets filling E_r^*. Take rectangular patches *Wj* ≃ *Ij* x *Jj* covering E_r^*, each lying in a single pullback £,. We may assume that *(p o X* is a diffeomorphism of /,· x {0}. Therefore *Fi* = *cp o X(I{* x {0}) = *(p* o A(W,) is open in Z_r^*, and dirnF; = dim/;. Lebesgue measure ^«i x *du2* on ^ = /; x /,· is equivalent to *m* on W_h and c'Mi is transferred under *<poX* to a measure on F; equivalent to *v* there. So

$$\int_{JWi}^{\oplus} G<po\backslash(u)du \cong \int_{JI,xJ,}^{\oplus} O<poX(uuUi)dU\backslash XdU2$$

$$\cong \int^{\oplus} \simeq \int$$

The sets $G_($ = F/AdJ^/ ^j) partition 2£; the sets M, = n(^oA)-^{1} ((?,-) are disjoint in E_r^* and have the form M,· = AT; x /,-, where K_t ⊆ /; is such that *Gj* = *((poX)(Ki* x {0}) = *(poX{M_t)*. Hence

$$\int_{M_i}^{\oplus} \sigma_{\varphi\circ\lambda(u)}\, du \cong \int^{\oplus} \infty \cdot \ldots\, du_1 \cong \int^{\oplus} \infty \cdot \sigma_{l'}\, d\nu(l'),$$

and hence (writing ^i ≥ *ni* to indicate that *712* is equivalent to a subrepresentation of *n\)* we get

$$\pi|_K \cong \int_{E_r^*}^{\oplus} \sigma_{\varphi\circ\lambda(u)}\, du \geq \bigoplus_{JZi} \int_{M_i}^{\oplus} \sigma_{\varphi\circ\lambda(u)}\, du$$

$$\cong \bigoplus_{i=1} \int_{G_i} \infty \cdot \sigma_{l'}\, d\nu(l') \cong \int_{\Sigma_r^*}^{\oplus} \infty \cdot \sigma_{l'}\, d\nu(l').$$

On the other hand, if *(X, fi)* is a measure space and *X* = *\jf=l Xj* *(Xj* measurable, but not necessarily disjoint), then we can easily show, by partitioning *X* compatibly with the *Xj,* that

$$\int_{X} \infty \cdot n_x \, du \simeq \bigoplus^{N} \int_{X_j}^{\oplus} \infty \cdot n_x \, du.$$

Hence

$$\int_{\bigcup_{i=1}^{N} W_i}^{\oplus} \infty \sigma_{\varphi \circ \lambda(u)} \, dU \simeq \bigoplus_{i=1}^{N} \int_{W_i}^{\oplus} \infty \sigma_{\varphi \circ \lambda(u)} \, du$$

$$\ldots \nu \, d\nu(l') \quad (\text{since } \infty \cdot \infty = \infty)$$

$$\leq \int_{J\text{-}L_j}^{\oplus} \infty \cdot \sigma_l \, d\nu(l').$$

Summing over *N,* we get

$$\ldots K \leq \bigoplus^{\infty} \int_{H}^{\oplus} \infty \sigma_{\varphi \circ \lambda(u)} \, du < \infty \int^{\oplus} \infty \sigma_{l'} \, d\nu(l') \simeq \int^{\oplus} \infty \sigma_{l'} \, d\nu(l')$$

$$\leq T \setminus K \qquad \text{(from above).}$$

The "Schröder-Bernstein Theorem for representations" says that these representations are equivalent.

We now show that S* and I[71] differ by sets of dimension < *k**, and so determine the same canonical measure: *[v/* = [1/]; this will complete the proof. (This part of our discussion works for any value of To.) Let

$$S_1 = [\lambda(E^*)] = \bigcup\{N_l : l \in \lambda(E^*)\},$$
$$S_2 = (\mathscr{O}_\pi \cap P^{-1}(U_e^K)) \setminus \varphi^{-1}(\varphi(S_1)),$$
$$\Sigma_1 = \varphi(S_1) = \varphi(\lambda(E^*)) = \Sigma^*, \quad \Sigma_2 = \varphi(S_2).$$

The set *k{E*)* is semialgebraic and dense in *Xf* = *k{Af).* From Remark 3.5, *S* = *[X(E*)]* satisfies *dim((fn \ Si)* < dim^ and contains a dense open subset of *&n.* Next, Zj, X2 partition Z*. Then maximal $\mathrm{rank}\{(d\varphi)_l$: / e *<fn}* = *k** is reached on an open set of *S\,* so that $\dim \Sigma_1 = K$. Stratify 27 compatibly with Z_1; Z_2. If £2 contains a piece of dimension ≥ *K,* this set is open in the relative topology of *Ul,* and the pullback of this set is open in *ffn* n *P~xl(Uf).* It is also disjoint from *S\.* This contradicts the fact that *S* is dense in @n. Therefore *k** > dim(Z2) = dim(S?r \ L*), as required. D

(4.7) REMARK. When To > 0, we have *dimKl* < dim TV/ for generic *I €(fn*. From Lemma 2.4, TV/ is a union of ^-orbits, so in this case TV/ contains infinitely many A^-orbits. Hence so does *@n* n *P~x(K • Pi)* = *(fn* n *(K • I +1-1),* for generic */ €<?n*. Thus the multiplicity of *oj,* in *n\x* is equal to the number of Ad*(^T)-orbits in *&n* n *P~l (K • I')* for *v-a.e. I'* € 27 (provided that we do not distinguish among infinities). This interpretation of multiplicity as the number of certain Ad*(/Q-orbits also holds in the finite multiplicity case, To = 0, as the next theorem shows.

(4.8) THEOREM. *Let g be a nilpotent Lie algebra, t a subalgebra, and G, K the corresponding (connected, simply connected) groups. Let {X\,...,Xp,...,Xm+p} be a basis for g through t, as in* §3. *For % €* $\hat{G}$*, let @n be its coadjoint orbit, and let e be the largest index for layers in V such that P~l(U*f) meets @n, where P:* g* —* *t* is the natural projection. Define the defect index* To *as in* (16), *and define* 27 = *(p(Pl(U^)r\(fn) with its canonical measure class [v] as in* (21). *Suppose that* To = 0, *and let*

(22) *n(l')* = *number ofK-orbits in P~l(K-1')* n *0K, l'* G 27.

Then for v-a. e. I' el71,

(a) *P~l(K • I')* n *<fn is a closed submanifold and its connected components are K-orbits;*

(b) *There is a common bound N such that n(l')* ≤ *N;*

(c) *We have*

$$n\vert_K \cong \int_{\Sigma^*}^{\oplus} n(l')\sigma_{l'}\, d\nu(l'),$$

where o\, 6 $\hat{K}$ *corresponds to K • l'*⊆t*.

Proof. The proof is fairly long, and we divide it into a number of steps. Fix / G *<fn* n *Us* and define *X: Af* -> *X*(*d*)*y*: *P~l(UJf)n&n* -+ 27 ⊆ *H* as before. We have *Af* ⊆ *Rk, k* = card R^(d)= dimX/; from Lemma 4.2 and Corollary 4.5, our assumption that To = 0 gives

k = *k** = generic rank{(^)/: / e *(fn}* = dim27

and

k = generic *v&nk{d((p* o *X)u*: *ue Af}*.

For any set *A* ⊆ *P~\Uf)C\@n*, we define its ^-saturant, *[A^*, by

$$[A]_\varphi = \varphi^{-1}(\varphi(A)) = \mathscr{O}_\pi \cap \bigcup\{K\cdot l + \mathfrak{k}^\perp : l \in A\}.$$

Note that *[l]9* = ^*niK-l+t-*$^{1)}$ = *WP~\K-Pl)* for / € *(f$_n$r\P-\Uf).*
The proof proceeds as follows:

Step 1. We construct a semialgebraic set *H_C KX$_f$* ∈ $(f_n \cap P^{-1}(U_e^K)$
with the following properties:

(23) (i) *H* is ^-saturated: *[H]$_v$* = *H.*

(ii) The complement of *H* is of measure 0 in @$_n$ n *P~^{x}(\Jf).*

(iii) ^»(i/) = I/^{7} is semialgebraic and of full measure in 27.

(iv) For *I e H, [l]9* is a union of AT-orbits, each of which is a connected component of *[l]9.*

(v) For / € //, *N,* = *K* • *I.*

(vi) The set *B*° = // n Xy is a semialgebraic set of full measure in *X$_f$,* and C° = *X~^{l}(B°)* c ^ has full measure in R$^{fc.}$

Once Step 1 is completed, part (a) of the theorem is proved; furthermore, it will suffice to prove (c) when the integral is over ~*L*^{H} instead of I*.

Step 2. For 1 < *j* < oo, define *Z*$^{H(j)}$ = {/' e S^: the number of isf-orbits in *P~^{l}(K- l')* n *H* is *j}.* The S^O) obviously partition *U*$^{l;}$ we show that they are semialgebraic and that they are empty once *j* is sufficiently large. This proves (b).

Step 3. Let C, = *{<pok)-\I.*$^{HU))}$. We show that

$$\int_{C_j}^{\oplus} \sigma_{\varphi\circ\lambda(u)}\, du = \dot{I} \qquad j\sigma_{l'}\, d\nu(l').$$

If /' € Ig), pick / e *P~\K* • /') n *ff$_K$.* Then *<p(l),* = /' G *<p(H),* or / G [#], = *H.* Hence P " ^{1} ^ • *l')n<f$_x$* = *P-^{x}(K -l')nH* and *j* = «(/').
Since

$$\hat{}IA: \simeq \int_{Jc^{\circ}}^{/\bullet} {}^{\copyright}{}_{a}q>oitu)\, du$$

(from Theorem 3.5 and (vi) of Step 1), this proves (c).

Proof{Step 1). Let

$$U = \{l \in P^{-1}(U_e^K) \cap \mathscr{O}_\pi : \operatorname{rank}(d\varphi)_l = k\},$$

$$x) = x_f \cap u \cap \text{f/j}.$$

All A^-orbits in t/j have dimension *r* + *k;* thus dim A' • *I* =• *r* + *k* for / G *UsH(f$_n$,* and r+A: is the generic dimension of Af-orbits in < *?„.* The set *U* is Zariski-open in *&$_n$,* and is Ad*(A")-invariant, since Ad*(A:), *k* G A", is a difFeomorphism of ^ that fixes A"-orbits and commutes with *(p.*

For all / e *UgC\U, N{* = *K* • *I,* since *N{* 2 *K* • *I,* both are connected, and their dimensions agree (Corollary 4.5); in particular, *Ug* n *U* = *[Us* n *U],* where *[A]* is the iV/-saturant defined in Proposition 3.2. The set *B* = *X~^l(X_f)* = *A_fr\{t (=R^k: X{t)* € *U_snU}* is Zariski-open in R^k and is nonempty because *[X^lf]* = *[X_f f)U_snU]* = *[X_f] D U_sf)U is* dense in *<f_n*. Hence *B* is dense in *Af* and *Xj* = *X(B)* is a dense open semialgebraic set in *Xf.*

For all / €E *U* n C/j, we have dim*K* • *I* = dim*N[* = *k* + *r,* dim*G-1* = dim^t = /+2A:, and rank(aty)/ = A; The map ^ foliates C/nt/,5; if L/ is the leaf through /, then *K-l* c *Lj,* and dimL/ = *dim(f_n* -rank(^)/ = dim^f• /. Since Ar• / and *L*\ are connected manifolds and *K*• *I* is closed in £/ n *Us,* we must have

(24) *Li* = *K-l* = *N_h* *aNeUnU_s.*

Moreover, if 7 € *[l]^ nUnU_s,* then the leaf *Lj* coincides locally with [/]p D *Un Us-* But this last set is a closed subset in *Un Us,* stable under *K.* Hence it is the union of the AT-orbits it meets, and these are open in the relative topology coming from *<f_n* because each AT-orbit is a leaf of the foliation. Thus the components of *[1]^,* n *U* n *Us* are AT-orbits. We conclude that (iv) and (v) hold provided that *H* C *U* n *Ug* and (i) holds.

Since *B* is Zariski-open, *X* = *X(B)* is semialgebraic; we noted above that it is dense in *Xf.* In particular, dim(Xy\X|) < *k* = *dimXf.* Define

$$F = (P^{-1}(U_e^K)\cap\mathscr{O}_\pi)\setminus K\cdot X_f^1,$$
$$H = \{P^{-x}\{U?)Kd?_n) \setminus \wedge^{-1}(\varphi(F)) = (P^{-1}(U_e^K)\cap\mathscr{O}_\pi)\setminus[F]_\varphi.$$

Then // clearly satisfies (i). Since *H* C *K* • *X_f C U nU_s,* (iv) and (v) hold as well; furthermore, *F* and *H* are easily seen to be semialgebraic. The key fact to prove is:

(25) *dim((p(F))* < dimS*.

For if (25) holds, then (iii) is immediate, since *YF* = *I,^71* \ *<p(F).* Furthermore, dimfF^ < dim^, and (ii) follows. (Otherwise, *[F],p* contains an open set in <?,,, and hence in *&_n* n *P~^l(Uf).* Since *dip* has maximal rankm on every open set, *<p(F)* = *<p[F]_v* would contain an open set in *IF,* and this contradicts (25).) Finally, *[Xf* n *H]* = *Hf\ [Xf]* is dense in <?*_n*. Now define *Bf* C *Af* as in Lemma 4.1. Then A: *Bf* —» Fj is a bijective local diffeomorphism. Fix fo £ #/, / = ^(^o);

taking a rationally varying action basis, define $F(u, t) = if/x(t)(^{u})$ as in (3) for t near t_0 and $u \in W^{+k}$. If V is a neighborhood of t_0, then $F(W^{+k}, V) = [X(V)]$ contains an open neighborhood of l_0 in $\mathfrak{g}^*_n$, by Lemma 4.1(c). Hence $[X(V)]$ meets $H = [H]$, so that A(K) meets H. Because X is bijective on A_f, V meets C° = $X^{-1}(H \cap X_f)$. Thus C° is dense in 5y and $H \cap X_f$ is dense in Xy. Since C° is semialgebraic, (vi) follows.

We thus need only prove (25) to complete Step 1. Let $\mathcal{S}$ be a stratification of U^l compatible with the sets $\varphi(X_f) = \varphi(K \cdot X_l)$ and $\varphi(F)$. We suppose that there is a piece $M_0 \subseteq \varphi(F)$ with dim Af^ = k and produce a contradiction. Let ^ be a stratification of ^ compatible with $P^{-1}(U?) \cap \#_n$, the Cj, n^(<5; e A), $\varphi^{-1}(\&>\sim) \cap d?_K$, F, and U. The set Af^ is covered by ^-images of pieces lying in F; on one of them (M_0, say), we have

$$\text{maximum rank}\{d(\varphi|M_0)_l : l \in M)\} = \dim M_0^{\sim}.$$

Hence A/b meets 17, and hence A/Q ^ C/. The tangent space $(TM_0)_l$, $l \in$ Afo, must thus contain subspaces of dimension k that are transverse to the leaves of the ^-foliation of U; therefore there is a submanifold A/ $\subseteq$ A/Q, dim Af = k, such that $\varphi|M$ is a diffeomorphism to an open set in $M_0^{\sim}$.

Let Si G A be the largest index such that Us, meets M. Then Us, n Af is nonempty and open, by Proposition 2.1 (b); we may assume that Af $\subseteq U_{Si}$. From Proposition 2.1 (a) and (c), $N_l \subseteq (K \cdot l + t^n^nUs$, for all $l \in$ Af; since φ is a diffeomorphism on Af and is constant on each N_l, M meets N_l only at l. We claim:

(26) The set $Y = \bigcup\{N_l : l \in$ Af$\} = [$Af$] \subseteq U_{Si} \cap$ ^ contains an open subset of $\mathfrak{g}^*_n \cap P^{-1}(U^)$.

Assume this for the moment. Since $\mathfrak{g}^*_n \cap P^{-1}(Us)$ is Zariski-open in $\mathfrak{g}^*_n$, we have $d_j = S$. Furthermore, $[X_l] = K \cdot X_l$ contains an open dense set of $\mathfrak{g}^*_n$, because X_l is dense and open in X_f (see Proposition 3.2 and Remark 3.7). Hence 7n[X l] contains an open subset $S \subseteq Ur\backslash U\$$. Since $K \cdot X_l$ contains every N_l meeting it, Af meets $K \cdot X_l$. But $M \subseteq F$ is disjoint from $K \cdot X_l$, and this contradiction gives (25).

We now prove (26). We have Af $\subseteq U_s$, $\cap P^{-1}(Uf) \cap UD(?_n$. We know that $\dim(K \cdot Pi) = r$ for all $l \in U_6,' \cap P^{-1}(Uf) \cap \mathfrak{g}^*_n$, and that $\dim(G \cdot l) = \dim \mathfrak{g}_n = 2k + r$. Since $l \in Us$, we also have

$$\dim K \cdot Pl = \text{Cardi?^,)}, \quad \dim G \cdot l = \text{Cardi?^^)} + 2\,\text{Card/?£((*,}\cdot\text{)},$$

from the definitions of $R'_2(di)$, $R'_2{}'(Si)$; it follows that

$$Card R'_2(Si) = r. \quad Card R'_2{}'(Si) = k,$$

and hence that dim $N[= r + k$ for all / € Ug_r In particular, this holds for all / e M. Parametrize M via a $C^{\circ\circ}$ diffeomorphism /?: Q —• M, where Q is open in R^. By perhaps shrinking M slightly, we may assume that these are rational maps $\{Y\backslash(l),..., Y_{r+k}(l)\}$ providing an action basis at each I e M. As in §2, we may define a nonsingular map

$$V_n(I,t) = \{txpt_lY_lV)---expt_{f+k}Y_{r+k}(l))\text{-}l, \qquad leM, \quad teR^{r+k},$$

which defines the N_h Let $h(s,t) = y_a(fi(s),t)$ for (s,t) e Q x R^{r+k}. Then Range/? = $[M]$ = Y. Since t H> $h(s, t)$ gives $N\text{^}_s)$, while s i> /z($5^{1,0}$) gives M, and since Ni is transverse to M, we see that

$$\mathbf{rank}(dh)_{(s,0)} = dimM + \dim A\text{^}_{(5)} = 2A; + r = \mathbf{dim}\,\mathscr{O}_\pi.$$

This proves (26) and completes Step 1.

Proof(Step 2). For / e H, we know that [/]p is a union of AT-orbits $Kl> = N_{,,}$ all $I'eUn$ U_s. But each iV$_{l;}$ / e [X}] D //, meets $X)$ in a single point. Thus for all I eH,

(27) $n(<p(l))$ (see (22)) = number of tf-orbits in $(K \bullet I + t^{L)}$ n ff_n
= number of ^-orbits in $(K \bullet I + 1^{1)}$ n H
$= \mathbf{Card}\{(K \cdot l + \mathfrak{k}^\perp) \cap X^1_f\}.$

Recall that $Xj. = k\{B)$ for some Zariski-open set $B \subseteq A_f \subseteq R^k$. The map P o A: R^ —> V is polynomial. We also have the rational nonsingular parametrizing map P_e: E_e x $F_{5(e)}$ -* Uf, such that r i-c $P_e(l',t)$ is polynomial for each /' € lL_e. Fix /' € $E^H \subseteq Z_f$; then $K \bullet l' \subseteq$ r is the range of -P_e(/', W), and the map of W Xo $K \bullet$ /' is a diffeomorphism. Consider the polynomial $R(s, t) = P_e(l', t) - (PoX)(s)$, defined on B x R^r. The roots of $R(s, t) = 0$ correspond precisely to the points in $P\sim^{l}(K$- /') n $Xj.$, and this intersection *is* $(K \bullet I + t^{\pm)} n$ Xj for any / € p'H'') n^t. Thus the number of roots of U(j,0 = 0 is j iff /' G $Z^{H(j)}$, 1 < 7- < oo. Since ^ o A is a local diffeomorphism on 5 when to = 0, the roots must be isolated; that is, there is no one-parameter family of roots in B x W. Now we use the following result.

LEMMA. *Let* $Z \subseteq R^n$ *be a Zariski-open set* $Z = \{x \in R^n: Q(x) \neq 0\}$, *and let* $P: R^n \to R^m$ *be a polynomial,* $P = (P_1, \ldots, P_m)$. *Then* *there is a number N, depending only on m, n,* degA,..., degf$_m$, *and* deg(2, *such that either P(x) =— 0 has a l-parameterfamily ofsolutions in Z or the number ofsolutions to P(x) = 0, x e Z, is bounded by N.*

We omit the proof, since this is essentially part of Theorem 4 of [2]. •

To complete Step 2, we need to show that the $\wedge^{H(j)}$ are semialgebraic. This proof is essentially the same as that for Theorem 4 (b) of [2]. For instance, /' e U/>2 $\wedge^{HU)}$ if /' € Z^H and the system

$$P_e(l', t_1) - (P \circ \lambda)(s_1) = 0,$$
$$P_e(l', t_2) - (P \circ X)(s_2) = 0,$$
$$|t_1 - t_2|^2 + |s_1 - s_2|^2 > 0$$

has a solution. By taking relative complements, one sees that the $\Sigma^H(j)$ are all semialgebraic. •

Proof (Step 3). Define $Cj = (y \circ X)^{-l}(I.^{HU)})>$ as before, and let $Hj = X(Cj)$. As noted earlier, we may integrate over C (the disjoint union of the *Cj)* instead of *Af* in the direct integral decomposition of Theorem 3.5. On *Cj,* the map *q> oX* is a`y-to-1 map onto $X^{H(j)}$, and (27) says that

$$\int_{\Sigma^H}^{\oplus} n(l')a,. \, du(l') = Q \int_J^{\oplus} ja_v du\{l').$$

To prove the theorem, therefore, it suffices to prove that

(28) $$\int^{\oplus} ja_v \, dv\{V) \simeq \cdot \int^{\oplus} a_{(9ok)(u)} \, du.$$

To do this, we follow the argument of Theorem 4 in [2]. Stratify $*L^{H}\{j)$, and let $S^{(1)}$, $S^{(2)}$ be respectively the $\wedge$-dimensional pieces and the pieces of lower dimension. Since $<p \circ X$ is a local diffeomorphism, $(cp \circ A)^{-1}(Z^{(2)})$ has dimension $< k$ in *Cj* and is therefore negligible. Recall that $(p \circ X$ is defined on $Af \subseteq R^4$, with image $T\wedge$ we have $\sim L^H \subseteq$ 1 / c_p , and these differ by sets of dimension k. Therefore we may work with $J.^{H.}$ If Z^a is a $\wedge$-dimensional piece in Z^H, it is open in $1/$; hence $(\wedge \circ A)(£^a)$ is open in *Af* and lies in *Cj.* Let {Cyj: /? e /} be the (open) connected components of this set. Since *Cj* is semialgebraic, $\mathcal{C}$ is finite. Furthermore, *poX* is a local diffeomorphism on $(q>oX)^{-x}(L^a)$ $\subseteq$ 5. Fix x € S^Q and define $mp(x) =$ card{« € *Cp: cpo X(u) = x}.* Then *mp{x)* is integer-valued, and *l.pmp(x) = j* on H^a. If *XQ* G $I.^a$ is fixed, then for each /? there is a neighborhood *Np* C I" of Xo on which *mp{x)* ≥ *mp{xo),* all *x* G A^. Let *N = f]p Np.* For *x* G iV, we have

$$j = \sum_{fi} m_\beta(x_0) \le \sum_{p} m_\beta(x) = j.$$

Thus the *mp{x)* are constant on *N*. In particular, each m_β is locally constant on E^Q, hence constant because H^a is connected. That is, *(p o A: Cp —* Z*^{a} is a covering map with uniform covering index *nip,* and so

$$/^{r®} \quad ovoX{u)du= \ \tilde{}^{r®} \quad mpoi,dv{l').$$

Summing over *fi* e /, we get

$$/_{J{(p^\circ X)\text{-}\backslash j>)} \quad o_{vo}x{u)d\tilde{u}= \cdot I_{JZ''} \quad jo_vdv{l'),$$

since *Zpmp = j.* Now summing over a gives (28). D

5. We give here some examples and miscellaneous results.

(5.1) LEMMA. *Suppose that K is a normal Lie subgroup of the connected, simply connected nilpotent Lie group G. Then for n* G *G,̂ T\K is either uniformly of multiplicity* 1 *or uniformly of multiplicity* oo.

Proof. We show that for any /' G 57, *<p~*^{x}{V) *T\@*$_n$ is connected. Let X = *<p~*^{l}{l')*r\&*$_n$\ pick / G *X,* such that *P{1) = /'*. Since *t* is an ideal, *G* acts on 6* by Ad*, and *P: Q* —> t** intertwines these actions of *G.* Let *S* = Stab$_G$(/') = *{X<EG: Ad*(x)l' = /'*}; S is connected, since the action of *G* on *V* is unipotent.

Now suppose that Ad*(*)/ *e X* for some *x eG.* Then *P(Ad* x)l* G *K*• *V* and therefore there exists *k* G *K* such that

$$\text{Ad*(fot)/'} = \text{-P(Ad*}\ kx)l = (\text{Ad*}\ k)P(Ad^*\ x)l = I^{l\cdot}$$

That is, *kx* G 5, or *x* G ATS (a subgroup, since AT is normal). Conversely,

$$y<EKS=> P(Ad^*\ y)l\ e <9_V => (\text{Ad*}\ y)l\ \text{G}\ X,$$

or X = Ad*(AT5")*/* is connected.

It follows that if $T_0 = 0$, then *n{l)* = 1 for all /. (If $T_0 > 0$, then the lemma is trivial.) •

(5.2) EXAMPLE. Let g be the 5-dimensional Lie algebra spanned by *X\, X$_2$, X3, X$_4$,* and *X$_5$,* with nonzero brackets *[X$_5$,X$_4$]* = X3,

$[X_5, X_3] = X2,$ and $[X_5, X2] = X\backslash\backslash$ G is the corresponding simply connected group. We considered g (with slightly different notation) in Example 4 of [2]; it turns out that the orbits in general position are parametrized by elements $/ = a\backslash l\backslash + 03/3 + 04/4,$ $a.\backslash$ / 0, where $l\backslash,\ldots, l_5$ is the dual basis in g* to $X\backslash,\ldots, X_5;$ moreover,

$$n = \left\{\alpha_1 l_1 + t l_2 + \left(\alpha_3 + \frac{t^2}{2\alpha_1}\right) h \right.$$
$$\left. + \left(a_4 + \frac{ta\text{-}x}{OL_x} + \frac{t^*}{6a f}\right) l_4 + u l_5 : \ t, u \ e\ \mathrm{R} \right\}.$$

Let t = R-span$\{Z_4\}$, K = expt. A calculation shows that for / = Ej=i $fijlj'$ Ad*(A")/ = / + R/5 if h ^ 0 and = / if fo = 0.

We have $t^* \cong$ R in the obvious way; P maps ^ to 1 and the other basis elements to 0. Each point in R is an Ad*(/T)-orbit.

Let n correspond to / = ai/i + 03/3 + a4/4, $a\backslash$ ^ 0, and let Xx ^ $\hat{K}$ correspond to X € R:

$$\chi_\lambda(\exp tX_4) = e^{2\pi i\lambda t}.$$

We have TQ = 0, since generically on $\&_n,$

$$\dim G/ = 2, \quad \dim A'\text{-}/=1, \quad \dim K \bullet Pi = 0.$$

Thus Theorem 4.8 gives

$$\pi|_K \cong \int_{R}^{\oplus} n(\lambda)\chi_\lambda\, d\lambda,$$

where

$$n_\lambda = \text{number of Ad}^*(K)\text{-orbits in } P^{-1}(X) \cap \mathcal{O}_n$$
$$= \text{number of real solutions to } \frac{t^3}{6a_1^2} + \frac{t\alpha_3}{\alpha_1} + a_4 = X.$$

(In this case, H excludes the points where $03 + t^2/2a1 = 0$; these are also the only points where there can be repeated roots.) Hence $n(X)$ = 3 on a set of positive measure and = 1 on a set of positive measure; that is, $n\backslash fc$ does not have uniform multiplicity.

(5.3) EXAMPLE. Let g be the Lie algebra with basis vectors Z, $Y,$ $X,$ W and nontrivial commutators

$$[W,X] = Y, \quad [W,Y] = Z,$$

and let G be the corresponding Lie group. We let Z*,..., W^* be the dual basis for g*. Write

$$(z, y, x, w) = \exp zZ \exp yY \exp xX \exp wW,$$
$$[a, p, y, d] = aZ^* + PY^* + yX^* + SW^*.$$

A direct calculation gives

(29) $Ad^*\{z,y,x,w)[a,p,y,d)$
$$= [a, p - wa,\ y - wfi + w^{2}a/2, 6 + xfl + (y - wx)a].$$

Thus the radical of [a, /?, y, $d]$ is

(30) $x[a,p,y,S] = R\text{-}span\{Z, aX\text{-}pY\}$ ifa^O;
$= \text{R-span}\{Z, Y\}$ if $a = 0$ ^ p.

The generic orbits are those having dimension indices given by e^ = (0,1,1,2), for which $U_{eW} = \{l{:}a\text{^}0\}$ and $Z_{e,.,} = \{[a,0,y,0]{:}\ a$ ^ 0, y e R}. From (29), a typical orbit in $U_{eW} = U^{\wedge l)}$ is

(31) $(?_{a,y} = G \bullet [a, 0, y, 0] = \{[a, s, y + s^{2}/2a, t]{:}\ s, t \in R\}.$

Denote by $7r_{Q,7}$ the corresponding representation of G.

The next layer consists of those elements having dimension indices given by e^ = (0,0,1,2); we have U_{em} = £/(2) = {/: $a = 0, p$ ^ 0}, $Z_{e(2,}$ = {[0, p, 0,0]: p / 0}. A typical orbit in U^ is

(32) $<f_{fi} = G \cdot [0, p, 0, 0] = \{[0, p, s, t]{:}\ s, t \in \mathrm{R}\},$

and we let n^ be the corresponding representation of G.

Now consider G x G, with Lie algebra g © g, and take Z\, Z_2,..., W\, Wi to be the basis of g © g (with the obvious brackets). Let K be the diagonal subgroup; its Lie algebra I has a basis

$$\overline{Z} = Z_1 + Z_2, \ldots, \quad \overline{W} = W_1 + W_2;$$

we have $[W, \overline{X}] = \overline{Y}$, $[W, \overline{Y}] = \overline{Z}$. The dual basisjn g^* © g* will be denoted by Z\, Z\,..., W\, $W\}$, and that in V by T,..., $\overline{W^*}$\ the projection P: $(g$ ffig)* —* t^* thus satisfies

$$P[a_1, a_2, \ldots, S_1, S_2] = (e^*i + a_2)T + \bullet \bullet \bullet + (\wedge + S_2)W^*.$$

By an obvious change in notation, (31) and (32) describe orbits in t^*; orbits in $(g$ © g)* are Cartesian products of orbits in g*.

We shall compute $n_{ait7i} \otimes n_{a2iy_2} = n_{ai>yi} \times 7i_{a2,72}\!\setminus_K = n\setminus_K$, say; recall that $a\setminus, a_2$ / 0. The orbit representative for n is $7a\setminus, a_2, 0, 0; 71, 72:$ 0,0] = /o, say; then

(33) $<?_x = (G \times G) \cdot l_0$

$$= \left\{ \left[a_{\cdot x}, a_{2}, s_{x}, s_{2}, y\setminus + \frac{s^2}{2\alpha_1}, \gamma_2 + \frac{s_2^2}{2\alpha_2}, t_1, t_2 \right] : stJi \quad \mathrm{e R} \right\}.$$

Assume first that $a\setminus$ + 02 / 0. Then P maps ^ into t/O, since every element of $P\{<?_n)$ is of the form $\{a\setminus + a_2)Z^* + \bullet \bullet \bullet$. We must thus take a typical orbit representative / = [a, 0, 7, 0] € $2^*_{e'}$, and compute <?* n P-'(AT-/). Notice first that

dirndl = 4, dimA:-/ = 2, dim ^ • / = 3 for generic / e $@_n$ (from (29));

thus To = 0.

From (31) and (33), we see that / € $(f_n n P\sim^{l}(K \bullet f)$ iff there exist s, t € R such that

(i) «i $+a_2 = a$,

(ii) ^ + s_2 = J,

(iii) 7, + 52/2a, + $y_2 + s\%/2a_2 = y + s^{2}/2a$,

(iv) fj + r_2 = f.

Condition (i) shows that we must have $a = a.\setminus + a_2$; (iv) shows that $t\setminus$, t_2 are free. From (i), (ii) and (iii) we get

$$\frac{s_1^2}{2\alpha_1} + \frac{s_2^2}{2\alpha_2} - \frac{(s_1 + s_2)^2}{2(\alpha_1 + a_2)} = \gamma - \gamma_1 - \gamma_2,$$

or

(34) $(aii_2 - a_{2}s_{(})^2 = 2\{y - y_{(} - y_2)a_{l}a_{2}(a_{l} + a_2),$

as a condition on $s\setminus$ and s_2. The solutions form a pair of lines when $\{y - y\setminus - y2)^{a}\setminus_{a2}(a\setminus + a_2) > 0$, a line when $y = y\setminus + y_2$, and the empty set otherwise. That is,

0_n n P"1 (AT • [a, 0, 7,0]) ~ union of 2 copies of R^3

if $(7 - 7i - yi)\{a\setminus + a_2)aia_2 > 0$

~ one copy of R^3 if $y\setminus + y_2 = 7$

~ 0 if $(7 - 7i - 72)(«i + a_2)a\setminus a_2 < 0$.

Thus we may take

!* = {/ = [«• 0, 7,0]: a = aj + a_2,

(7 - 7i - 72)(a 1 + $a_2)a\setminus a_2 > 0$} « a half-line,

^ = Lebesgue measure on the half line = dy, and we have

$$\pi_{\alpha_1,\gamma_1} \otimes \pi_{\alpha_2,\gamma_2} \cong \pi|_K \cong \int_{\Sigma^\pi}^{\oplus} \text{2ft}_{\text{ai+a}_2\text{,y}}\ dy.$$

If *an* + a2 = 0, then *P* maps @*n* onto a set containing *U*^ but missing *U*^\ For / = [0, /?, 0,0] e I^2), we have

dim^ = 4, dim*K* • *f* = 2, dim*K* • *I* = *3* for generic / € ^,

as before; thus to = 0 again. Furthermore, / e ^ n *P*~*l* (*K* • *f*) if there exist *S*\, *S2,s, t*\, *ti, t* G R such that

(i) *S*\ + 5₂ = /?,

(ii) [illegible]

(iii) ^+^2 = f.

From (i), 5^1! is free to vary, but s2 is then determined; (ii) then determines 5, and (iii) lets us vary *t*\ and *ti* arbitrarily. The intersection is thus ≅ R^3 for all /? + 0, and we find that

$$\Sigma^\pi = \{f = [0, \beta, 0, 0]: \beta \neq 0\}, \qquad d\nu = d\beta,$$

$$\pi_{\alpha_1,\gamma_1} \otimes \pi_{\alpha_2,\gamma_2} \cong \pi|_K \cong \int_{\Sigma^\pi} \pi_\beta\, d\beta.$$

(5.4) REMARK. For some groups *G*, one can have *n*\ ®7i2 irreducible even though *U*\ and ^2 are infinite-dimensional. This is implicit in some of the calculations in [3]. The simplest example is probably the case where g is the group of strictly upper triangular 5 x 5 matrices. Let *Xjj*, 1 ≤ / < *j* ≤ 5, be the obvious basis *(Xij* has a 1 as its *(i,j)* entry and zeroes elsewhere), and let *//,* be the dual basis for g*; a tedious calculation shows that

$$\pi_{l_{1,5}} \otimes \pi_{l_{2,4}} \cong \pi_{l_{1,5}+l_{2,4}}.$$

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