

JENSEN POLYNOMIALS AND THE TURÁN AND LAGUERRE INEQUALITIES

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In this paper we will (1) establish a relationship between the Turán inequalities and the Laguerre inequalities, (2) provide a complete characterization of functions in the Laguerre-Pólya class in terms of the Turán inequalities involving the Jensen polynomials and (3) show that certain Hankel determinants of functions in the Laguerre-Pólya class are nonpositive.

Introduction. A real entire function $\psi(x)$ is said to be in the *Laguerre-Pólya class* if $\psi(x)$ can be represented in the form

$$(1.1) \quad \psi(x) := cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where c, β, x_k are real, $\alpha \geq 0$, m is a nonnegative integer and $\sum x_k^{-2} < \infty$. Pólya and Schur [PS] termed an entire function $\varphi(x)$ as a function of *type I* in the Laguerre-Pólya class if $\varphi(x)$ or $\varphi(-x)$ can be represented in the form

$$(1.2) \quad \varphi(x) := cx^m e^{\sigma x} \prod_{k=1}^{\infty} (1 + x/x_k),$$

where $\sigma \geq 0$, c is real, m is a nonnegative integer, $x_k > 0$, and $\sum x_k^{-1} < \infty$. If $\varphi(x)$ is in the Laguerre-Pólya class, we will write $\varphi \in \mathcal{L}\text{-}\mathcal{P}$ or $\varphi \in \mathcal{L}\text{-}\mathcal{P}(\text{II})$. Also, if $\varphi(x)$ is a function of type I in $\mathcal{L}\text{-}\mathcal{P}$, we will write $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}(\text{I})$. Now it is clear that if $\varphi \in \mathcal{L}\text{-}\mathcal{P}(\text{I})$, then $\varphi \in \mathcal{L}\text{-}\mathcal{P}(\text{II})$.

For the various properties of functions in the Laguerre-Pólya class we refer the reader to [PS], [O], [CC] and the references contained therein. For the reader's convenience, we single out here a few known facts [PS] concerning functions in the Laguerre-Pólya class. For a function

$$(1.3) \quad \varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

in the Laguerre-Pólya class the following statements are valid.

(i) For each n , the Jensen polynomials

$$(1.4) \quad g_n(x) := \sum_{k=0}^n \binom{n}{k} \gamma_k x^k, \quad n = 0, 1, 2, \dots,$$

have only real zeros.

(ii) For each k , the Turán inequalities hold; that is,

$$(1.5) \quad T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \quad k = 1, 2, 3, \dots$$

(iii) For each p , the Laguerre inequalities hold [S]; that is,

$$(1.6) \quad (\varphi^{(p)}(x))^2 - \varphi^{(p-1)}(x)\varphi^{(p+1)}(x) \geq 0,$$

for all real x and $p = 1, 2, 3, \dots$

(iv) If $\gamma_k \geq 0$ for $k = 0, 1, 2, \dots$, then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}(\text{I})$.

In §2, after reviewing some of the known properties of Jensen polynomials, we shall relate the Turán inequalities for an arbitrary entire function to the Laguerre inequalities and to Turán-type inequalities of the Jensen polynomials on the positive real axis (Theorem 2.3). We then prove that the requirement that the inequalities hold on the *whole* real axis forces $\varphi(x)$ to be in $\mathcal{L}\text{-}\mathcal{P}$. This culminates in a complete characterization of $\mathcal{L}\text{-}\mathcal{P}(\text{II})$ in Theorem 2.7. The section ends with a proof of a related problem of Karlin.

In §3, we shall cite two open problems and give several examples which illustrate the necessity of the hypotheses in our results.

2. The main results. We begin this section with the terminology and notation we will use in the sequel. If

$$(2.1) \quad f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

is a real entire function, so that $\gamma_k \in \mathbf{R}$, $k = 0, 1, 2, \dots$, then the n th *Jensen polynomial* associated with $f(x)$ is defined by

$$(2.2) \quad g_n(t) := \sum_{k=0}^n \binom{n}{k} \gamma_k t^k \quad (n = 0, 1, 2, \dots).$$

The n th Jensen polynomial associated with $f^{(p)}(x)$, $p = 0, 1, 2, \dots$, will be denoted by

$$(2.3) \quad g_{n,p}(t) := \sum_{k=0}^n \binom{n}{k} \gamma_{k+p} t^k \quad (n, p = 0, 1, 2, \dots).$$

The n th Appell polynomial associated with $f(x)$, if $\gamma_0 \neq 0$, is defined by

$$(2.4) \quad P_n(t) := \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \gamma_k t^{n-k} \quad (n = 0, 1, 2, \dots).$$

If

$$h(x) := \sum_{k=0}^n a_k x^k, \quad a_n \neq 0,$$

is any polynomial, we define

$$h^*(x) := x^n h(x^{-1}), \quad (x \neq 0).$$

In particular if $\gamma_n \neq 0$, then with the above definition, (2.3) and (2.4) we have

$$(2.5) \quad \begin{cases} g_n^*(t) := t^n g_n(t^{-1}) & (n = 0, 1, 2, \dots, t \neq 0), \\ P_n(t) = \left(\frac{1}{n!}\right) g_n^*(t) & (n = 0, 1, 2, \dots). \end{cases}$$

Some of the properties of the polynomials are summarized in the following proposition (see, for example, Csordas and Varga [CV]).

PROPOSITION 2.1. *The following properties hold for the functions defined by (2.1)–(2.5).*

(i) If $\gamma_0 \neq 0$, $P'_n(t) = P_{n-1}(t)$ ($t \in \mathbf{R}$, $n = 1, 2, 3, \dots$).

(ii) The sequence $\{g_n(t)\}_{n=0}^\infty$ is generated by $e^x f(xt)$; that is,

$$(2.6) \quad e^x f(xt) = \sum_{n=0}^\infty g_n(t) \frac{x^n}{n!} \quad (x, t \in \mathbf{R}).$$

If $\gamma_0 \neq 0$, the sequence $\{n!P_n(t)\}_{n=0}^\infty$ is generated by $e^{xt} f(x)$; that is,

$$(2.7) \quad e^{xt} f(x) = \sum_{n=0}^\infty P_n(t) x^n = \sum_{n=0}^\infty g_n^*(t) \frac{x^n}{n!} \quad (x, t \in \mathbf{R}).$$

(iii)

$$(2.8) \quad n g_n(t) = n g_{n-1}(t) + t g'_n(t) \quad (t \in \mathbf{R}, n = 1, 2, 3, \dots).$$

(iv)

$$(2.9) \quad g_{n+1,p}(t) = g_{n,p}(t) + t g_{n,p+1}(t) \quad (t \in \mathbf{R}, n, p = 0, 1, 2, \dots).$$

(v) If

$$(2.10) \quad \Delta_{n,p}(t) := g_{n,p}^2(t) - g_{n-1,p}(t) g_{n+1,p}(t) \quad (n = 1, 2, 3, \dots, p = 0, 1, 2, \dots),$$

then

$$(2.11) \quad \Delta_{n,p}(t) = t^2[g_{n-1,p+1}^2(t) - g_{n-1,p}(t)g_{n-1,p+2}(t)],$$

$$(n = 1, 2, 3, \dots, p = 0, 1, 2, \dots).$$

Proof. Direct verification yields (i)–(iv) (for (ii) and (iii), see for example, Rainville [R, p. 133]). To prove (v), we use (2.9) in the form

$$(2.12) \quad g_{n,p}(t) = g_{n-1,p}(t) + tg_{n-1,p+1}(t).$$

Substituting (2.9) and (2.12) into (2.10) yields (v). \square

The Jensen polynomials associated with an arbitrary entire function form a “natural” sequence of approximating polynomials. This property will be needed below and is established in the following lemma.

LEMMA 2.2. *Let*

$$(2.13) \quad h(z) := \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \quad (a_k \in \mathbb{C}),$$

be an arbitrary entire function. For each fixed nonnegative integer p , let

$$(2.14) \quad g_{n,p}(z) := \sum_{k=0}^n \binom{n}{k} a_{k+p} z^k \quad (n = 0, 1, 2, \dots).$$

Then

$$(2.15) \quad \lim_{n \rightarrow \infty} g_{n,p}\left(\frac{z}{n}\right) = h^{(p)}(z) \quad (p = 0, 1, 2, \dots)$$

uniformly on compact subsets of \mathbb{C} .

Proof. For a fixed nonnegative integer p and for $n \geq 2$, we have

$$(2.16) \quad g_{n,p}\left(\frac{z}{n}\right) = a_p + a_{p+1}z$$

$$+ \sum_{k=2}^n \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{a_{k+p}}{k!} z^k.$$

Then, by the Cauchy inequalities for the Taylor coefficients of $h^{(p)}(z)$, we have

$$(2.17) \quad \left| \frac{a_{k+p}}{k!} \right| \leq \frac{M(R, h^{(p)})}{R^k} \quad (k = 1, 2, 3, \dots, R > 0).$$

where

$$(2.18) \quad M(R, h^{(p)}) = \max_{|z|=R} |h^{(p)}(z)|.$$

Thus, for $n > m + 1$ and for $0 < |z| \leq r < R = 2r$, we obtain

$$(2.19) \quad \left| \sum_{k=m+1}^n \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{a_{k+p}}{k!} z^k \right| \leq M(R, h^{(p)}) \frac{r^{m+1}}{R^m(R-r)},$$

and

$$(2.20) \quad \left| \sum_{k=m+1}^{\infty} \frac{a_{k+p}}{k!} z^k \right| \leq M(R, h^{(p)}) \frac{r^{m+1}}{R^m(R-r)}.$$

Therefore, with $R = 2r$ and for any $\varepsilon > 0$, there is a positive integer m_0 such that $r^{m+1}(R^m(R-r))^{-1} < \varepsilon$ for all $m \geq m_0$. Finally, there is a positive integer $N > m_0$ such that for all $n \geq N$ and $|z| \leq r$

$$(2.21) \quad \left| \sum_{k=2}^{m_0} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \frac{a_{k+p}}{k!} z^k - \sum_{k=2}^{m_0} \frac{a_{k+p}}{k!} z^k \right| < \varepsilon,$$

and hence (2.15) follows from (2.19), (2.20) and (2.21). \square

With the aid of the foregoing preliminary results we will now prove

THEOREM 2.3. *Suppose*

$$(2.22) \quad f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \quad \gamma_k > 0, \quad k = 0, 1, 2, \dots,$$

is an entire function. Then the following are equivalent.

$$(2.23) \quad T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0 \quad \text{for } k = 1, 2, 3, \dots$$

$$(2.24) \quad \Delta_{n,p}(t) := g_{n,p}^2(t) - g_{n-1,p}(t)g_{n+1,p}(t) \geq 0,$$

for all $t \geq 0$ and $n = 1, 2, 3, \dots$, $p = 0, 1, 2, \dots$.

$$(2.25) \quad L_{p+1}(f(t)) := (f^{(p+1)}(t))^2 - f^{(p)}(t)f^{(p+2)}(t) \geq 0$$

for all $t \geq 0$ and $p = 0, 1, 2, \dots$.

Proof. (2.23) \Rightarrow (2.24). Assuming (2.23) (with $\gamma_k > 0$, $k = 0, 1, 2, \dots$) we will establish (2.24) by induction. First, we set $t = 1$. Then, for

$n = 1$, we have by (2.23)

$$(2.26) \quad \begin{aligned} g_{1,p}^2(1) - g_{0,p}(1)g_{2,p}(1) \\ = \gamma_{p+1}^2 - \gamma_p\gamma_{p+2} \geq 0, \quad p = 0, 1, 2, \dots \end{aligned}$$

Next, we assume that

$$(2.27) \quad \Delta_{k,p}(1) \geq 0 \quad \text{for } k = 1, \dots, n \quad \text{and} \quad p = 0, 1, 2, \dots$$

Then by the induction assumption (2.27) and by Proposition 2.1(v) (cf. (2.11)) we have

$$(2.28) \quad \Delta_{n,p}(1) = g_{n-1,p+1}^2(1) - g_{n-1,p}(1)g_{n-1,p+2}(1) \geq 0, \\ p = 0, 1, 2, \dots,$$

and

$$(2.29) \quad \begin{aligned} \frac{g_{n-1,p+1}(1)}{g_{n-1,p}(1)} &\geq \frac{g_{n-1,p+2}(1)}{g_{n-1,p+1}(1)} \\ &\geq \frac{g_{n-1,p+3}(1)}{g_{n-1,p+2}(1)}, \quad p = 0, 1, 2, \dots \end{aligned}$$

Let p be a fixed, but arbitrary, nonnegative integer. Then by (2.11),

$$(2.30) \quad \Delta_{n+1,p}(1) = g_{n,p+1}^2(1) - g_{n,p}(1)g_{n,p+2}(1),$$

and by (2.9),

$$(2.31) \quad g_{n,p}(1) = g_{n-1,p}(1) + g_{n-1,p+1}(1).$$

Thus, if we apply (2.31) to (2.30), then after some simplifications, (2.30) becomes

$$(2.32) \quad \begin{aligned} \Delta_{n+1,p}(1) &= \Delta_{n,p}(1) + \Delta_{n,p+1}(1) + g_{n-1,p+1}(1)g_{n-1,p+2}(1) \\ &\quad - g_{n-1,p}(1)g_{n-1,p+3}(1). \end{aligned}$$

Hence, it follows from (2.28) and (2.29) that $\Delta_{n+1,p}(1) \geq 0$. Since p was an arbitrary nonnegative integer we conclude that $\Delta_{n+1,p}(1) \geq 0$ for $n, p = 0, 1, 2, \dots$.

Now if $t_0 \geq 0$, $\alpha_k \geq 0$, $k = 0, 1, 2, \dots$, and if $\alpha_k^2 - \alpha_{n-1}\alpha_{n+1} \geq 0$, then with $\gamma_k := t_0^k \alpha_k$, $k = 0, 1, 2, \dots$, it follows that

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} = t_0^{2k}(\alpha_k^2 - \alpha_{k-1}\alpha_{k+1}) \geq 0, \quad k = 1, 2, 3, \dots$$

In light of these considerations, we see that (2.24) holds for all $t \geq 0$.

(2.24) \Rightarrow (2.25). Suppose (2.24) holds. Then by (2.11) and (2.24),

$$(2.33) \quad 0 \leq \Delta_{n,p}(t) = t^2[g_{n-1,p+1}^2(t) - g_{n-1,p}(t)g_{n-1,p+2}(t)]$$

for all $t \geq 0$, $n = 1, 2, 3, \dots$, $p = 0, 1, 2, \dots$. Therefore, by Lemma 2.2 (cf. (2.15)) we see that

$$(2.34) \quad \lim_{n \rightarrow \infty} \left[g_{n-1,p+1}^2 \left(\frac{t}{n-1} \right) - g_{n-1,p} \left(\frac{t}{n-1} \right) g_{n-1,p+2} \left(\frac{t}{n-1} \right) \right] \\ = [f^{(p+1)}(t)]^2 - f^{(p)}(t)f^{(p+2)}(t) \geq 0,$$

for all $t \geq 0$ and for all $p = 0, 1, 2, \dots$.

Since the implication (2.25) \Rightarrow (2.23) is immediate, the proof of the theorem is complete. \square

REMARK 2.4. Inequalities (2.23) and (2.24) are called the *Turán inequalities*. A comprehensive treatment of these inequalities and their applications in the theory of orthogonal polynomials may be found in Karlin and Szegő [KS] and the references contained therein. The inequalities (2.25) are frequently termed the *Laguerre inequalities* (see, for example, Skovgaard [S] or Patrick [P]). The Laguerre inequalities play an important role in the theory of the distribution of zeros of several classes of entire functions. See, for example, the recent proof of the Pólya-Wiman conjecture, Craven, Csordas and Smith [CCS], where these inequalities were used in the following, perhaps more familiar form,

$$(2.35) \quad \frac{d^2}{dt^2} [\log f^{(p)}(t)] \leq 0.$$

Thus by (2.35) (and under the equivalent conditions of Theorem 2.3), $f^{(p)}(t)$ is logarithmically concave for $t \geq 0$, where $f(t)$ is defined by (2.22). Therefore, it follows that $\log(1/f(t))$ is convex for $t \geq 0$. In particular, Theorem 2.3 also generalizes a result of Vincze [V]. In his study of the characterization of the Poisson and gamma distributions, Vincze shows, in particular, that if $\gamma_k > 0$ and

$$\gamma_k^2 - \frac{k}{k+1} \gamma_{k-1} \gamma_{k+1} > 0, \quad k = 1, 2, 3, \dots,$$

then

$$(2.36) \quad \frac{d^2}{dt^2} \left[\frac{1}{f^{(p)}(t)} \right] = \frac{2[f^{(p+1)}(t)]^2 - f^{(p)}(t)f^{(p+2)}(t)}{[f^{(p)}(t)]^3} > 0$$

for $t > 0$, $p = 0, 1, 2, \dots$. (Note that the open problem of Rényi and Vincze, discussed in [V], has been recently solved by Miles and

Williamson [MW].) But because of the factor of “2” in (2.36), this inequality does *not* imply the Laguerre inequalities (2.25). On the other hand, we were able to extend Vincze’s ingenious argument to deduce (2.25) directly from (2.23). However, this argument is somewhat involved and since it does not yield (2.24) we will omit it here. Finally, it is evident that Theorem 2.3 remains valid, *mutatis mutandis*, if we merely assume that the function $f(x)$ in (2.22) is a real analytic function with positive Taylor coefficients.

Next we consider the case when (2.23) is replaced by the following requirements

$$(2.37) \quad \begin{cases} \Delta_n(t) := \Delta_{n,0}(t) > 0 & \text{for all } t \neq 0, \ n = 0, 1, 2, \dots \\ \text{and} \\ \gamma_{k+1}^2 - \gamma_k \gamma_{k+2} > 0, \ \gamma_k > 0 & \text{for } k = 0, 1, 2, \dots \end{cases}$$

Then (2.37) constitutes a very strong set of conditions (see, for example, Csordas and Williamson [CW1] and [CW2]). Indeed, if (2.37) holds, then all the Jensen polynomials associated with $f(x)$ of (2.22) have only simple real zeros. Therefore, the following theorem is of particular interest.

THEOREM 2.5. *Let*

$$(2.38) \quad \varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \quad \gamma_k \neq 0,$$

denote a real entire function and let $g_n(x)$, $n = 0, 1, 2, \dots$, denote the n th Jensen polynomial associated with $\varphi(x)$. Then the following conditions are equivalent for any $n \geq 1$.

- (i) $\Delta_n(t) > 0$ for all real $t \neq 0$ and $\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} > 0$.
- (ii) $\Delta_n^*(t) > 0$ for all real t .
- (iii) For each $k = 1, 2, 3, \dots, n$, and for real t_0 , if $g_k(t_0) = 0$, then $g_{k-1}(t_0)g_{k+1}(t_0) < 0$, and if $\gamma_n = 0$, then $\gamma_{n-1}\gamma_{n+1} < 0$.
- (iv) For each $k = 1, 2, 3, \dots, n$, and for real t_0 , if $g_k^*(t_0) = 0$, then $g_{k-1}^*(t_0)g_{k+1}^*(t_0) < 0$.
- (v) $g_{n+1}(t)$ has only simple real zeros.

Proof. For each fixed n , we will prove the following implications: (ii) \Leftrightarrow (i) \Rightarrow (iii) \Leftrightarrow (iv) \Rightarrow (v) \Rightarrow (ii). Note that (i) \Rightarrow (iii) is trivial. Since $\Delta_n^*(t) = t^{2n}\Delta_n(t^{-1})$ for $t \neq 0$, and $\Delta_n^*(0) = \gamma_n^2 - \gamma_{n-1}\gamma_{n+1}$, it is clear that (i) \Leftrightarrow (ii). Similarly, using the fact that $g_n^*(t) = t^n g_n(t^{-1})$ for $t \neq 0$, and $g_n^*(0) = \gamma_n$, we see that (iii) \Leftrightarrow (iv). Next, assume that (iv)

holds. Since $\gamma_0 \neq 0$ by assumption (cf. (2.38)), and $(d/dt)g_{n+1}^*(t) = (n+1)g_n^*(t)$ and since (iv) holds, the sequence $g_{n+1}^*(t), g_n^*(t), \dots, g_0^*(t) = \gamma_0$, is a Sturm sequence for the interval $(-\infty, \infty)$ (see, for example, Jacobson, [J, pp. 278–283] or Wilf [W, pp. 90–95]). Therefore by Sturm's theorem [J, p. 283], the number of *distinct* real zeros of g_{n+1}^* in $(-\infty, \infty)$ is equal to the number of sign changes in the sequence $\{(-1)^{n+1}\gamma_0, (-1)^n\gamma_0, \dots, \gamma_0\}$, minus the number of changes of sign in the sequence $\{\gamma_0, \gamma_0, \dots, \gamma_0\}$ of leading coefficients. Since this number is $n+1$, we see that $g_{n+1}^*(t)$, and hence $g_{n+1}(t)$, have only simple real zeros. This proves that (iv) \Rightarrow (v).

Finally, suppose that (v) holds. Since by definition (cf. (2.2)), $\deg g_{n+1}^* \leq n+1$ and by (v) g_{n+1}^* has $(n+1)$ distinct zeros, $\deg g_{n+1}^* = n+1$. Let

$$(2.39) \quad g_{n+1}^*(t) := \gamma_0 \prod_{j=1}^{n+1} (t - \alpha_j) \quad (\alpha_1 < \alpha_2 < \dots < \alpha_{n+1}).$$

Then for all real t , $t \neq \alpha_j$, $j = 1, \dots, n+1$,

$$(2.40) \quad \begin{aligned} \frac{d^2}{dt^2} \log g_{n+1}^*(t) &= \frac{d}{dt} \frac{(n+1)g_n^*(t)}{g_{n+1}^*(t)} = \frac{d}{dt} \sum_{j=1}^{n+1} \frac{1}{(t - \alpha_j)} \\ &= - \sum_{j=1}^{n+1} \frac{1}{(t - \alpha_j)^2} < 0. \end{aligned}$$

Since the zeros of $g_{n+1}^*(t)$ are all simple, the Cauchy-Schwarz inequality yields

$$\left(\sum_{j=1}^{n+1} \frac{1}{t - \alpha_j} \right)^2 < (n+1) \sum_{j=1}^{n+1} \frac{1}{(t - \alpha_j)^2}.$$

Thus, we have by (2.40), for real $t \neq \alpha_j$, $j = 1, \dots, n+1$,

$$(2.41) \quad \begin{aligned} (g_{n+1}'^*(t))^2 - g_{n+1}^*(t)g_{n+1}^{*''}(t) &= (g_{n+1}^*(t))^2 \sum_{j=1}^{n+1} \frac{1}{(t - \alpha_j)^2} \\ &> \frac{(g_{n+1}^*(t))^2}{n+1} \left(\sum_{j=1}^{n+1} \frac{1}{(t - \alpha_j)} \right)^2 = \frac{(g_{n+1}'^*(t))^2}{n+1}. \end{aligned}$$

Therefore,

$$\begin{aligned}
 0 &< \left(1 - \frac{1}{n+1}\right) (g_{n+1}^{*'}(t))^2 - g_{n+1}^*(t) g_{n+1}^{*''}(t) \\
 &= \frac{n}{n+1} (n+1)^2 (g_n^*(t))^2 - (n+1) \cdot n g_{n+1}^*(t) g_{n-1}^*(t) \\
 &= n(n+1) \Delta_n^*(t),
 \end{aligned}$$

and so we see that $\Delta_n(t) > 0$ for all $t \neq 0$ and $n = 1, 2, 3, \dots$. This completes the proof of the theorem. \square

Next we recall that by a classical result of Pólya and Schur [PS], a real entire function $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ if and only if the associated Jensen polynomials $g_n(x)$, $n = 1, 2, 3, \dots$, have only real zeros. Hence, the following corollary is an immediate consequence of Theorem 2.15.

COROLLARY 2.6. *Let $\varphi(x)$ be the real entire function defined by (2.38). If the equivalent conditions (i)–(v) of Theorem 2.5 hold for $n = 1, 2, 3, \dots$, then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}(\text{II})$.*

To obtain a complete characterization of the functions in $\mathcal{L}\text{-}\mathcal{P}(\text{II})$, we must allow the possibility that $\Delta_n(t)$ equals zero in Theorem 2.5, or equivalently, that the polynomial $g_n(t)$ has multiple zeros. This is done in the next theorem.

THEOREM 2.7. *Let $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^k / k!$ denote a real entire function and let $g_n(t)$, $n = 0, 1, 2, \dots$ and $\Delta_n(t)$, $n = 1, 2, 3, \dots$ be defined by (2.2) and (2.37) respectively. Then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}(\text{II})$ if and only if the following conditions hold:*

- (i) $\Delta_n(t) \geq 0$ for all real t and $n = 1, 2, 3, \dots$
- (ii) If $\gamma_0 \neq 0$ and $\gamma_1^2 - \gamma_0 \gamma_2 > 0$, then (a) $g_{n+1}(t_0) = 0$ whenever $\Delta_n(t_0) = 0$, $t_0 \neq 0$, and (b) $\gamma_{n+1} = 0$ whenever $\gamma_n^2 - \gamma_{n-1} \gamma_{n+1} = 0$.
- (iii) If $\gamma_0 \neq 0$ and $\gamma_1^2 - \gamma_0 \gamma_2 = 0$, then $\varphi(x) = \gamma_0 e^{\gamma_1 x / \gamma_0}$.
- (iv) If $\gamma_0 = 0$, then $\varphi(x) = x^r \psi(x)$ with $\psi(0) \neq 0$ where ψ satisfies (i), (ii) and (iii) for the appropriately redefined γ_n , g_n and Δ_n .

Note that condition (ii)(b) is essentially the case $t_0 = \infty$ in condition (ii)(a). We shall begin our proof of this theorem with two lemmas.

LEMMA 2.8. *With $\varphi(x)$, Δ_n and g_n as in the theorem, assume that $\Delta_n(t) \geq 0$ for all real t and $n = 1, 2, 3, \dots$. If $\Delta_n(t_0) = 0$, $t_0 \neq 0$, then $g_{n+1}(t_0) \Delta_{n-1}(t_0) = 0$.*

Proof. Set $u = t_0^{-1}$. Then we have

$$(2.42) \quad \Delta_n^*(u) = 0 = (g_n^*(u))^2 - g_{n-1}^*(u)g_{n+1}^*(u),$$

and u must be a multiple zero of Δ_n^* since $\Delta_n^* \geq 0$. Therefore, using (2.5) and Proposition 2.1(i),

$$(2.43) \quad 0 = \Delta_n^{*'}(u) = (n-1)[g_n^*(u)g_{n-1}^*(u) - g_{n+1}^*(u)g_{n-2}^*(u)].$$

If $g_n^*(u) \neq 0$, we can use (2.42) to deduce that the right hand side of (2.43) equals

$$\begin{aligned} & \frac{n-1}{g_n^*(u)} [g_n^{*2}(u)g_{n-1}^*(u) - g_n^*(u)g_{n+1}^*(u)g_{n-2}^*(u)] \\ &= \frac{n-1}{g_n^*(u)} g_{n+1}^*(u)\Delta_{n-1}^*(u), \end{aligned}$$

and thus $g_{n+1}(t_0)\Delta_{n-1}(t_0) = 0$. On the other hand, if $g_n^*(u) = 0$, then (2.43) implies either $g_{n+1}^*(u) = 0$ or, if $g_{n+1}^*(u) \neq 0$, then $g_{n-2}^*(u) = 0$ and $g_{n-1}^*(u) = 0$ (cf. (2.42)) so that $\Delta_{n-1}^*(u) = 0$. Again we have $g_{n+1}(t_0)\Delta_{n-1}(t_0) = 0$. \square

LEMMA 2.9. *Under the hypotheses of Theorem 2.7, assume that $\Delta_n(t) \geq 0$ for all real t and $n = 1, 2, 3, \dots$, and that $\gamma_1^2 - \gamma_0\gamma_2 \neq 0$. If for some n , $\Delta_n(t_0) = 0$, $t_0 \neq 0$ and $g_{n+1}(t_0) = 0$, then g_{n+1} has at least two nonreal zeros.*

Proof. By Lemma 2.8, we have $\Delta_{n-1}(t_0) = 0$. By hypothesis, $\Delta_1(t_0) = (\gamma_1^2 - \gamma_0\gamma_2)t_0^2 \neq 0$. If we let k be the largest integer less than n with $\Delta_k(t_0) \neq 0$, then Lemma 2.8, together with a simple induction gives $g_{k+1}(t_0) = \dots = g_n(t_0) = 0$. Writing $u = t_0^{-1}$, we have $g_{n+1}^{*'}(u) = (n+1)g_n^*(u) = 0$ and $g_{n+1}^{*''}(u) = n(n+1)g_{n-1}^*(u) = 0$. Since $g_{n+1}^*(u) \neq 0$, an elementary geometric argument shows that g_{n+1}^* , and hence g_{n+1} , have nonreal zeros. \square

Proof of Theorem 2.7. We first consider necessity. Assume that $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}(\text{II})$. Then, for any fixed t , we have $e^x\varphi(xt) \in \mathcal{L}\text{-}\mathcal{P}(\text{II})$, so that (i) is just the Turán inequalities for the coefficients in the series in (2.6). Condition (ii) holds by Lemma 2.8 and 2.9. Condition (iii) was proved in [CC, Proposition 4.5]. When $\varphi(x) = x^r\psi(x)$ with $\psi(0) \neq 0$, then $\psi(x) = \sum_{k=0}^{\infty} \alpha_k x^k / k!$ with $\alpha_k = \gamma_{k+r} / (k+r) \cdots (k+1)$ and

$$g_n(x) = n(n-1) \cdots (n-r+1)x^r \sum_{k=0}^{n-2} \binom{n-r}{k} \alpha_k x^k.$$

Since $\psi(x) \in \mathcal{L}\mathcal{P}(\text{II})$, the above argument shows that (i)–(iii) must hold for $\psi(x)$.

Conversely, assume conditions (i)–(iv) hold. By (iv), we may assume $\gamma_0 = \varphi(0) \neq 0$. Fix $n > 0$. As noted previously, Pólya and Schur [PS] have shown that it will suffice to show that g_n (or equivalently, g_n^*) has only real zeros. Consider the sequence of polynomials

$$(2.44) \quad g_0^*, g_1^*, \dots, g_n^*.$$

If $\Delta_k^*(t) > 0$ for all t , $k < n$, then this sequence is a Sturm sequence for g_n^* as in the proof of Theorem 2.5. And, as in that proof, the number of real zeros of g_n^* is n . Again by Theorem 2.5, this holds whenever we have strict inequality in (i) for $t \neq 0$ and $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} > 0$, $k = 1, \dots, n-1$. Next suppose that $\Delta_n^*(t) \geq 0$ for all real t . By (iii), we are done if $\gamma_1^2 - \gamma_0\gamma_2 = 0$, so we may assume $\gamma_1^2 - \gamma_0\gamma_2 > 0$ and thus the hypotheses of (ii) hold. Now let u be such that $\Delta_k^*(u) = 0$ for some $k \leq n$, where k is minimal for u . Note that there are only finitely many such points u since $\Delta_k \equiv 0$ implies $g_{k+1} \equiv 0$ by (ii)(a), which in turn implies $\gamma_0 = 0$, contrary to assumption. From (ii) and $\Delta_k^*(u) = 0$, we obtain $g_k^*(u) = g_{k+1}^*(u) = 0$, so that $\Delta_{k+1}^*(u) = 0$. (If $u = 0$, then $\Delta_k^*(u) = \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} = 0$ and (ii)(b) is used.)

An induction using (ii) gives us $g_k^*(u) = g_{k+1}^*(u) = \dots = g_n^*(u) = 0$, with $g_{k-1}^*(u) \neq 0$ (by the minimality of k). Since $(d/dx)g_j^*(x) = jg_{j-1}^*(x)$ for each j , we have a sequence of polynomials

$$(2.45) \quad g_0^*(x), \dots, g_{k-1}^*(x), g_k^*(x)/(x-u), \dots, g_n^*(x)/(x-u)^{n-k+1},$$

which are all nonzero at $x = u$. It follows that (2.45) forms a Sturm sequence for $g_n^*(x)/(x-u)^{n-k+1}$ for any interval containing u and excluding other multiple zeros of g_n^* : indeed, for x near u , but $x \neq u$, $g_j^*(x) = 0$ implies that $g_{j-1}^*(x)/(x-u)^l$ and $g_{j+1}^*(x)/(x-u)^{l+2}$ have opposite signs since $(x-u)^l(x-u)^{l+2} > 0$ and $g_{j-1}^*(x)g_{j+1}^*(x) < 0$ by (i). Eliminating each of the multiple zeros of g_n^* in this way gives a Sturm sequence, where the leading coefficient of all the polynomials is still γ_0 as in (2.44). It follows that the resulting quotient of g_n^* has only real zeros, and hence so does g_n^* . \square

We next consider an open problem of Karlin [K, pp. 389–390]. Let

$$(2.46) \quad \varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$$

be a real entire function and let $H_3(\varphi^{(p)}; x)$, $p = 0, 1, 2, \dots$, denote the 3×3 *Hankel determinant* associated with $\varphi^{(p)}(x)$:

$$(2.47) \quad H_3(\varphi^{(p)}; x) := \det \begin{bmatrix} \varphi^{(p)}(x) & \varphi^{(p+1)}(x) & \varphi^{(p+2)}(x) \\ \varphi^{(p+1)}(x) & \varphi^{(p+2)}(x) & \varphi^{(p+3)}(x) \\ \varphi^{(p+2)}(x) & \varphi^{(p+3)}(x) & \varphi^{(p+4)}(x) \end{bmatrix}.$$

We will prove below that if $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ and $\gamma_k > 0$, $k = 0, 1, 2, \dots$, then $H_3(\varphi^{(p)}; x) \leq 0$ for all $x \geq 0$. To this end we will need the following lemma.

LEMMA 2.10. (i) *If $f(x) := e^{\sigma x} q(x)$, where $\sigma \in \mathbf{R}$ and $q(x) \in \mathbf{R}[x]$, then*

$$(2.48) \quad H_3(f; x) = e^{3\sigma x} H_3(q; x), \quad x \in \mathbf{R}.$$

(ii) *Let $q(x) \in \mathcal{L}\text{-}\mathcal{P}$ and suppose that $q(x)$ has nonnegative Taylor coefficients. Let*

$$(2.49) \quad G(q; x) := q^2(x)q'''(x) - 3q(x)q'(x)q''(x) + 2(q'(x))^3, \quad x \in \mathbf{R}.$$

Then

$$(2.50) \quad G(q; x) \geq 0 \quad \text{for } x \geq 0.$$

Proof. Part (i) follows directly from a computation. To prove (ii), we induct on the degree of $q(x)$. Clearly (2.50) holds if $\deg q(x) = 1$. Set

$$(2.51) \quad Q(x) := (x + a)q(x),$$

where $a \geq 0$, and assume that (2.50) holds. Then a calculation shows that

$$(2.52) \quad G(Q; x) = (x + a)^3 G(q; x) + 2(q(x))^3.$$

Since $a \geq 0$ and $q(x) \geq 0$, it follows from the induction assumption that $G(Q; x) \geq 0$ for all $x \geq 0$. \square

In the proof of our next theorem we will use the following generalization of the Turán and Laguerre inequalities (cf. (2.23) and (2.25)) due to Jensen [Je]. For a detailed proof of the following theorem see [CV].

THEOREM 2.11. *With (2.46), set*

$$(2.53) \quad L_n(\varphi(x)) := \sum_{k=0}^{2n} \frac{(-1)^{k+n}}{(2n)!} \binom{2n}{k} \varphi^{(k)}(x) \varphi^{(2n-k)}(x) \\ (x \in \mathbf{R}, n = 0, 1, 2, \dots).$$

Then $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$ if and only if

$$(2.54) \quad L_n(\varphi(x)) \geq 0 \quad (x \in \mathbf{R}, n = 0, 1, 2, \dots).$$

THEOREM 2.12. *Let $q(x) \in \mathbf{R}[x]$ with positive leading coefficient and suppose that the zeros of $q(x)$ are all nonpositive. Then*

$$(2.55) \quad H_3(q; x) \leq 0 \quad \text{for all } x \geq 0.$$

Proof. We will prove the theorem by induction on the degree of $q(x)$. Clearly, if $\deg q \leq 1$, then (2.55) holds, since in this case $H_3(q; x) \equiv 0$. Now set

$$(2.56) \quad Q(x) := (x + a)q(x),$$

where $a \geq 0$ and $q(x)$ has only real nonpositive zeros. Next, we assume that $H_3(q; x) \leq 0$ for all $x \geq 0$. For $x \in \mathbf{R}$, set

$$(2.57) \quad I(Q; x) := -Q(x)Q^{(4)}(x) + 4Q'(x)Q'''(x) - 3(Q''(x))^2.$$

Then, using (2.53), a computation shows that

$$(2.58) \quad I(Q; x) = -12L_2(Q(x)).$$

Since $q, Q \in \mathcal{L}\text{-}\mathcal{P}$, we conclude, by Theorem 2.11, that

$$(2.59) \quad I(q; x) \leq 0 \quad \text{and} \quad I(Q; x) \leq 0 \quad \text{for all } x \in \mathbf{R}.$$

Now another computation yields that

$$(2.60) \quad H_3(Q; x) = H_3(q; x)a^3 + 3xH_3(q; x)a^2 \\ + (3x^2H_3(q; x) + q(x)I(q; x))a \\ + x^3H_3(q; x) + xq(x)I(q; x) - 4G(q; x).$$

Now by Lemma 2.10 (cf. (2.50)), $-G(q; x) \leq 0$ for $x \geq 0$ and by (2.59), $I(q; x) \leq 0$ for $x \geq 0$. Thus, using the induction assumption and the fact that $a \geq 0$, we conclude that $H_3(Q; x) \leq 0$ for $x \geq 0$. \square

We note that the use of Jensen's inequality (2.54) in the proof above could have been avoided by doing yet another induction to show that

$$L_2((x + a)q(x)) = (x + a)^2 L_2(q(x)) + ((q'(x))^2 - q(x)q''(x))$$

is always nonnegative.

THEOREM 2.13. *Let*

$$(2.61) \quad \varphi(x) := \sum_{k=1}^{\omega} \frac{\gamma_k}{k!} x^k, \quad \gamma_k \geq 0, \quad 0 \leq \omega \leq \infty,$$

be a function in the Laguerre-Pólya class. Then

$$(2.62) \quad H_3(\varphi^{(p)}; x) \leq 0 \quad \text{for all } x \geq 0, \quad p = 0, 1, 2, \dots,$$

where $H_3(\varphi^{(p)}; x)$ is defined by (2.47). In particular,

$$(2.63) \quad \det \begin{bmatrix} \gamma_p & \gamma_{p+1} & \gamma_{p+2} \\ \gamma_{p+1} & \gamma_{p+2} & \gamma_{p+3} \\ \gamma_{p+2} & \gamma_{p+3} & \gamma_{p+4} \end{bmatrix} \leq 0 \quad (p = 0, 1, 2, \dots).$$

Proof. Since $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, and $\gamma_k \geq 0$, $\varphi(x)$ can be expressed in the form

$$(2.64) \quad \varphi(x) = ce^{\sigma x} \prod_1^{\omega} (1 + x/x_n), \quad 0 \leq \omega \leq \infty,$$

where $c > 0$, $\sigma \geq 0$, $x_n > 0$ and $\sum^{\omega} x_n^{-1} < \infty$. First, we consider the case when $p = 0$. If $\omega = N < \infty$, set

$$(2.65) \quad \varphi_N(x) := ce^{\sigma x} \prod_{n=1}^N (1 + x/x_n).$$

Then by Lemma 2.10 (cf. (2.48)) and Theorem 2.12

$$(2.66) \quad H_3(\varphi_N; x) \leq 0 \quad \text{for all } x \geq 0.$$

If $\omega = \infty$, we know that $\varphi_N(x) \rightarrow \varphi(x)$ as $N \rightarrow \infty$, uniformly on compact subsets of \mathbb{C} . Thus, it follows that $H_3(\varphi_N; x) \rightarrow H_3(\varphi; x)$ as $N \rightarrow \infty$, for $x \geq 0$. Consequently, (2.62) holds with $p = 0$. Since $\mathcal{L}\text{-}\mathcal{P}$ is closed under differentiation [PS] and since $\gamma_k \geq 0$, it follows that for any $p = 0, 1, 2, \dots$, $\varphi^{(p)}(x)$ can also be expressed in the form (2.64), and thus (2.62) holds for $p = 0, 1, 2, \dots$. Since the assertion (2.63) follows from (2.62) by setting $x = 0$, the proof of the theorem is complete. \square

COROLLARY 2.14. *Let*

$$(2.67) \quad \varphi(x) := \sum_{k=1}^{\omega} \frac{\gamma_k}{k!} x^k, \quad \gamma_k \geq 0, \quad 0 \leq \omega \leq \infty,$$

be a function in the Laguerre-Pólya class. Let

$$(2.68) \quad T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1}, \quad k = 1, 2, 3, \dots$$

Then the following extended Turán inequalities hold:

$$(2.69) \quad E_k := T_k^2 - T_{k-1}T_{k+1} \geq 0, \quad k = 2, 3, 4, \dots$$

Proof. A computation shows that

$$(2.70) \quad \det \begin{bmatrix} \gamma_k & \gamma_{k+1} & \gamma_{k+2} \\ \gamma_{k+1} & \gamma_{k+2} & \gamma_{k+3} \\ \gamma_{k+2} & \gamma_{k+3} & \gamma_{k+4} \end{bmatrix} (-\gamma_{k+2}) = E_{k+2}, \quad k = 0, 1, 2, \dots$$

Since $(-\gamma_{k+2}) \leq 0$, (2.69) follows from Theorem 2.13 (cf. (2.63)). \square

REMARKS 2.15. (a) The inequality (2.69) provides a new, easily tested *necessary* condition for an entire function, with nonnegative Taylor coefficients, to have only real zeros.

(b) Let

$$(2.71) \quad \varphi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \quad \gamma_k > 0,$$

be an entire function and let $g_{n,p}^*(x)$ denote the n th Appell polynomial (cf. (2.3) and (2.5)) associated with $\varphi^{(p)}(x)$, $p = 0, 1, 2, \dots$. Then a computation shows that

$$\begin{aligned} A_p(t) &:= \det \begin{bmatrix} g_{0,p}^*(t) & g_{1,p}^*(t) & g_{2,p}^*(t) \\ g_{1,p}^*(t) & g_{2,p}^*(t) & g_{3,p}^*(t) \\ g_{2,p}^*(t) & g_{3,p}^*(t) & g_{4,p}^*(t) \end{bmatrix} \\ &= \det \begin{bmatrix} \gamma_p & \gamma_{p+1} & \gamma_{p+2} \\ \gamma_{p+1} & \gamma_{p+2} & \gamma_{p+3} \\ \gamma_{p+2} & \gamma_{p+3} & \gamma_{p+4} \end{bmatrix}. \end{aligned}$$

Hence, if $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}$, $\gamma_k \geq 0$, then $A_p(t) \leq 0$ for all $t \in \mathbf{R}$ and for $p = 0, 1, 2, \dots$.

3. Examples and open problems. The purpose of this section is to provide some examples which show that in a sense our results are best possible. In addition, we will cite here two open problems.

EXAMPLE 3.1. In reference to Theorem 2.3, consider

$$(3.1) \quad f_1(x) := e^{3x} - x - \frac{2}{3} = \frac{1}{3} + 2x + \sum_{k=2}^{\infty} \frac{3^k}{k!} x^k.$$

Thus, the Turán inequalities (2.23) all hold. But then, using the notation in (2.25), we find that

$$(3.2) \quad L_1(f_1(t)) = 9te^{3t} + 1.$$

Since $L_1(f_1(-1/3)) = 1 - 3e^{-1} < 0$, we see that, if (2.23) is valid, then, in general, the Laguerre inequalities (2.25) need not hold for all real t .

In addition, we note that if

$$(3.3) \quad f_2(x) := x + e^x = 1 + 2x + \sum_{k=2}^{\infty} \frac{x^k}{k!} := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k,$$

then $\gamma_2^2 - \gamma_1\gamma_3 = -1 < 0$, and $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$ for all $k \geq 1$, $k \neq 2$, so that exactly one of the inequalities (2.23) fails. In this case, with the notation in (2.25), $L_2(f_2(t)) = -e^t < 0$ for all real t .

EXAMPLE 3.2. In reference to Corollary 2.6 and Theorem 2.7, we now show that if

$$(3.4) \quad f_3(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \quad \gamma_0 \neq 0,$$

is a real entire function and if

$$(3.5) \quad \Delta_n(t) = (g_n(t))^2 - g_{n-1}(t)g_{n+1}(t) \geq 0, \\ \text{for all real } t \text{ and } n = 1, 2, 3, \dots,$$

where $g_n(t)$ is the n th Jensen polynomial associated with $f_3(x)$, then $f_3(x)$ need not belong to the Laguerre-Pólya class. Indeed, with $\gamma_0 := 1$, $\gamma_2 := \gamma_3 := 0$, $\gamma_4 := -1$, $\gamma_k := 0$, $k = 5, 6, 7, \dots$, we have that $f_3(x) = 1 - x^4/4!$, $g_n(t) = 1$, $0 \leq n \leq 3$ and $g_n(t) = 1 - \binom{n}{4}t^4$ for $n = 4, 5, 6, \dots$. Then a calculation shows that (3.5) is satisfied since

$$\Delta_n(t) := \frac{(n-1)(n-2)}{144} t^4 [(n-3)(n-2)(n-1)nt^4 + 72] \geq 0$$

for all $t \geq 0$, $n = 1, 2, 3, \dots$, but $f_3(x) = 1 - x^4/4!$ is not a function in the Laguerre-Pólya class.

EXAMPLE 3.3. In reference to Theorem 2.13, consider the Hankel determinant $H_3(\varphi; x)$ (see (2.47) for the notation), where $\varphi(x) := (x+1)(x+2)(x+3)$. Thus, $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}(\mathbf{I})$, but a calculation shows that

$$(3.6) \quad H_3(\varphi; x) = -36(x+2)(x^2 + 4x + 5).$$

Consequently, $H_3(\varphi; x) > 0$ for $x < -2$, and $H_3(\varphi; x) \leq 0$ for $x \geq -2$; that is, in general, $H_3(\varphi; x)$ cannot have an invariant sign as x ranges over the real axis. (This example also shows that there is a misprint in [K, p. 389, inequality (12.5) and p. 390, line 3].)

The results of Theorem 2.13 and Example 3.3 suggest the following open problem.

Open Problem 3.4 (Karlin [K]). Let $\varphi(x) \in \mathcal{L}\text{-}\mathcal{P}(\mathbb{I})$ and suppose $\varphi(x)$ has nonnegative Taylor coefficients. Let $H_n(\varphi^{(p)}; x)$ denote the $n \times n$ Hankel determinant

$$H_n(\varphi^{(p)}; x) := \det \begin{bmatrix} \varphi^{(p)}(x) & \varphi^{(p+1)}(x) & \dots & \varphi^{(p+n-1)}(x) \\ \varphi^{(p+1)}(x) & \varphi^{(p+2)}(x) & \dots & \varphi^{(p+n)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(p+n-1)}(x) & \varphi^{(p+n)}(x) & \dots & \varphi^{(p+2n-2)}(x) \end{bmatrix}$$

where $p = 0, 1, 2, \dots$. Then the conjecture is that

$$(3.7) \quad (-1)^{n(n-1)/2} H_n(\varphi^{(p)}; x) \geq 0 \quad \text{for } x \geq 0, \quad p = 0, 1, 2, 3, \dots, \\ \text{and for } n = 4, 5, 6, \dots$$

Open Problem 3.5. In reference to Corollary 2.14 consider the function $\varphi(x)$ in $\mathcal{L}\text{-}\mathcal{P}$ defined by (2.67). Then the Turán expressions (cf. (2.68))

$$(3.8) \quad T_k := \gamma_k^2 - \gamma_{k+1}\gamma_{k-1}, \quad k = 1, 2, 3, \dots,$$

are all nonnegative. But then elementary considerations show that the function

$$(3.9) \quad f(x) := \sum_{k=0}^{\infty} \frac{T_{k+1}}{k!} x^k,$$

where T_k is defined by (3.8) is also an entire function. The open problem then is to characterize those functions $\varphi(x)$ in $\mathcal{L}\text{-}\mathcal{P}$ for which the associated functions $f(x)$ (cf. (3.9)) are also in $\mathcal{L}\text{-}\mathcal{P}$ (see also the example below).

EXAMPLE 3.6. Let

$$(3.10) \quad \varphi_1(x) := (x^2 + 10x + 1)e^x,$$

so that $\varphi_1(x)$ is in $\mathcal{L}\text{-}\mathcal{P}(\mathbb{I})$. Then a calculation shows that $T_k = 78 + 18k + 2k^2$ and that the associated entire function

$$(3.11) \quad f_1(x) := \sum_{k=0}^{\infty} \frac{T_{k+1}}{k!} x^k = (98 + 24x + 2x^2)e^x.$$

Therefore, we see that $f_1(x)$ is *not* in $\mathcal{L}\text{-}\mathcal{P}$.

On the other hand, if

$$(3.12) \quad \varphi_2(x) := (x^2 + 2x + 1/2)e^x$$

then $T_k = 2k^2 + 2k - 1$, and the associated function is

$$f_2(x) := \sum_{k=0}^{\infty} \frac{T_{k+1}}{k!} x^k = (2x^2 + 8x + 3)e^x$$

is clearly in $\mathcal{L}\text{-}\mathcal{P}(\mathbf{I})$.

Other choices of $\varphi(x)$ for which $f(x)$, defined by (3.9), is again in $\mathcal{L}\text{-}\mathcal{P}(\mathbf{I})$ are $(x + 1)^n$ for any positive integer n and the function $\sum_{k=0}^{\infty} x^k / (k!)^2$.

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