CONJUGACY CLASS STRUCTURE OF SMOOTH HYPERBOLIC SECTORS

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In this article we exhibit a continuum of explicit, cubic plus flat, nonconjugate hyperbolic sectors [Theorem 3.1]. We further show that, regardless of conjugacy class, any flat vector field with a hyperbolic sector can be locally C^∞ approximated by a conjugate of a linear model; however, there exist non-flat hyperbolic sectors which can be arbitrarily C^∞-approximated by linear conjugates, but are not conjugate to any of them [Theorem 5.2].

In addition, we address the following:

Problem. Which smooth maps can be *realized* as the pass-by or "sojourn time" map of some hyperbolic sector?

Such sojourn time maps must go to ∞ at zero; the asymptotic behavior at zero determines the sector's conjugacy class. We prove that if $\tau\colon (0, 1) \to \mathbf{R}^+$ is a smooth map such that $1/\tau$ is smoothly extendible to zero, then there is a smooth hyperbolic sector with τ as sojourn time map [Theorem 3.1]. In other results, the variation in successive oscillations of τ provide sufficient conditions for the realization of τ as a sojourn time map [Theorems 2.1 and 4.2].

These results greatly extend previous work. F. Dumortier gave sufficient conditions for local conjugacy (via a time preserving homeomorphism) of two smooth flows with hyperbolic sectors [D1]. Later in [SSW], a first example appears of a smooth flow with hyperbolic sector which is not conjugate to a linear model. This construction involves meticulous surgery on the sector domain.

We make two related conjectures: Call the conjugates of linear models the *central class*. Then the Hartman-Grobman Theorem [H] guarantees that the central class has interior in the C^1 topology. Vector fields C^r-interior to the central class are C^r-structurally stable under time conjugacy. Our evidence supports the converse in the C^∞-topology:

Conjecture A. The collection of time conjugacy stable hyperbolic sectors comprises, in the C^∞ topology, the interior of the central class.

Finally, do the C^∞-flat sectors represent all the conjugacy classes?

Conjecture B. Every time conjugacy class of C^∞ hyperbolic sectors contains a vector field which is C^∞-flat at the singularity.

1. Statement of results. Assume Σ_1 and Σ_2 are two fixed C^∞-smooth transversals to a C^∞-smooth flow ϕ_t, having a hyperbolic sector (Hartman [H]) at the singularity p (see §5 for more precise definitions). The flow ingresses the sector on Σ_1 and egresses through Σ_2. Let $W^s(p)$ denote the stable manifold of p. The *sojourn map* $\tau\colon \Sigma_1 \backslash W^s(p) \to \mathbf{R}^+$ is defined by requiring $\phi_{\tau(q)}(q) \in \Sigma_2$ and $\phi_t(q) \notin \Sigma_2$ for all $0 \le t < \tau(q)$ (see Figure 1).
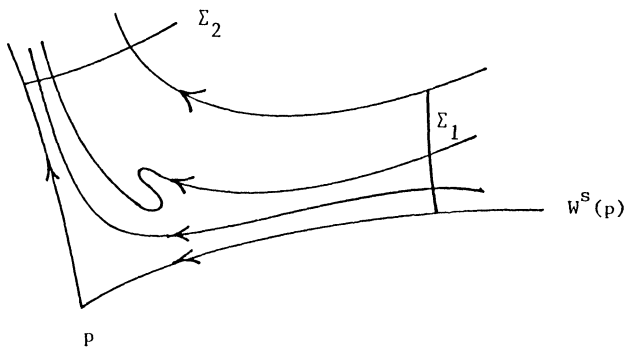

FIGURE 1

Our results concerning the realization problem depend on the sequence of time-dips and rises $\{(a_i, b_i)\}$ in the candidate τ for a sojourn map and on the degree of smoothness of $1/\tau$ near 0.

For now, we shall confine our attention to C^∞-maps $\tau\colon (0, 1) \to \mathbf{R}^+$ having a countable collection of ordered critical points $\{c_i\}_1^\infty$ such that the following hold:

(1) $\lim c_i = 0$ as $i \to \infty$.

(2) all c_i's occur at local maxima or at local minima.

(3) $\tau(c_1) > \tau(c_0)$.

Let $a_i = \tau(c_{2i-1}) - \tau(c_{2i})$ and also $b_i = \tau(c_{2i-1}) - \tau(c_{2i-2})$. Thus, for each i, $\tau' \ge 0$ on $[c_{2i}, c_{2i-1}]$ while $\tau' \le 0$ on $[c_{2i-1}, c_{2i-2}]$.

The resulting sequences $\{a_i\}$ and $\{b_i\}$ will be called, respectively, the *time-dips* and *time-rises* in $\tau(s)$ as $s \to 0+$ (see Figure 2).

If the map τ actually coincides with the sojourn map of a hyperbolic sector, then any cofinite collection of pairs $\{(a_i, b_i)\}_{i \ge n}$ determines that sector's time-conjugacy class ([SSW]). Also the collection corresponds to the standard conjugacy class unless the sequence $\{a_i\}$ does *not* tend to zero as $i \to \infty$ [SSW].
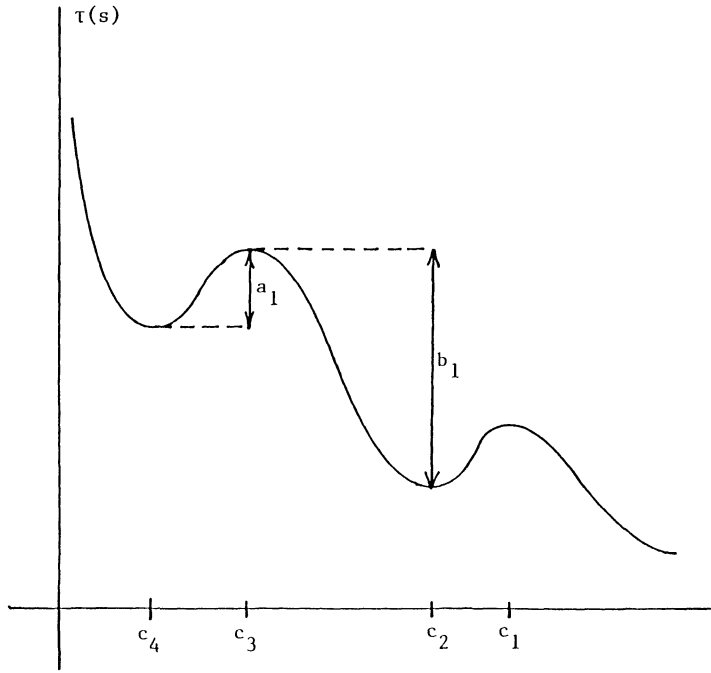
FIGURE 2

THEOREM 2.1. *Given any sequence $\{a_i\}$ with $a_i \geq 0$ there exists an explicitly constructed C^∞-smooth hyperbolic sector whose sojourn map τ realizes these $\{a_i\}$ as its time dips.*

THEOREM 3.1. *Let $\tau\colon (0, a) \to \mathbf{R}^+$ be a C^∞-map such that*
(1) $\lim \tau(s) = \infty$ as $s \to 0+$, and
(2) $1/\tau$ is C^∞-extendible to $[0, 1)$.
Then there exists a C^∞-flow on a hyperbolic sector whose sojourn map is given by τ.

THEOREM 4.2. *Suppose $\{a_i\}$ and $\{b_i\}$ are sequences of positive real numbers such that, for some constant K,*

(1) $$K + \sum_{i=1}^{n}(b_i - a_i) > 0, \quad \text{for all } n \text{ and}$$

(2) $$\sum_{n=1}^{\infty}\left\{K + \sum_{i=1}^{n}(b_i - a_i)\right\}^{-1/p} < \infty,$$

for all positive integers p. Then there is a C^∞-hyperbolic sector whose sojourn map τ has the time-dips and rises given by the sequences $\{a_i\}$ and $\{b_i\}$.

Conjecture C. *If the vector field X is C^∞-flat and generates a hyperbolic sector, then the inverted sojourn map $1/\tau$ is C^∞-flat at zero.*

Conjecture D. *The induced sequences $\{a_i\}$ and $\{b_i\}$ of time-dips and rises in the sojourn map τ of a C^∞-hyperbolic sector satisfy conditions (1) and (2) of Theorem 5.*

Conjectures A, C, and D, if true, would imply that every conjugacy class contains a C^∞-flat representative that is explicitly constructible. That conclusion combined with Theorem 5.2 below would prove that every conjugacy class meets the boundary ∂I_0 of the central class.

Our main contribution to the CCCP is that the C^∞-flat hyperbolic sectors lie on the boundary of the central class containing all the linear hyperbolic sectors.

THEOREM 5.2. *If the generating vector field X of a C^∞-smooth hyperbolic sector is C^∞-flat at the singularity, and the stable and unstable manifolds are transverse, then that sector lies on the boundary ∂I_0 of the central class. The boundary ∂I_0 also contains hyperbolic sectors that are not in I_0 and not C^∞-flat.*

REMARKS. We also know that if a sector is "straight," i.e., the orbits have at most one transversal intersection with a radial line from the origin, then that sector lies in the closure of the central class. It follows that an element in the exterior of I_0—a candidate for a new conjugacy-stable sector—cannot be either C^∞-flat or straight. Indeed such a hyperbolic sector cannot be polynomial or satisfy any Lojasiewicz inequality [D2].

2. Proof of Theorem 2.1. In a previous paper ([SSW]), the authors constructed a C^∞-smooth vector field, denoted by X_1, having the following properties:

(i) X_1 has a hyperbolic sector.

(ii) $X_1 = F \cdot X_0$, that is X_1 is a rescaling by a C^∞ positive function $F(x, y)$ of the C^∞-flat vector field

$$X_0: \begin{array}{l} x' = -x^3 e^{-1/x} \\ y' = y^3 e^{-1/y} \end{array}.$$

(iii) The rescaling function has the form $F = \sum_{i=1}^\infty (1 + F_i)$, such that the sequence of F_i's have disjoint supports P_i (as in Figure 3).

(iv) The sequence $\{(a_i, b_i)\}$, as defined before, corresponding to the sojourn map τ_1 of X_1 is such that $a_i \geq 1$ for each $i \geq 1$.
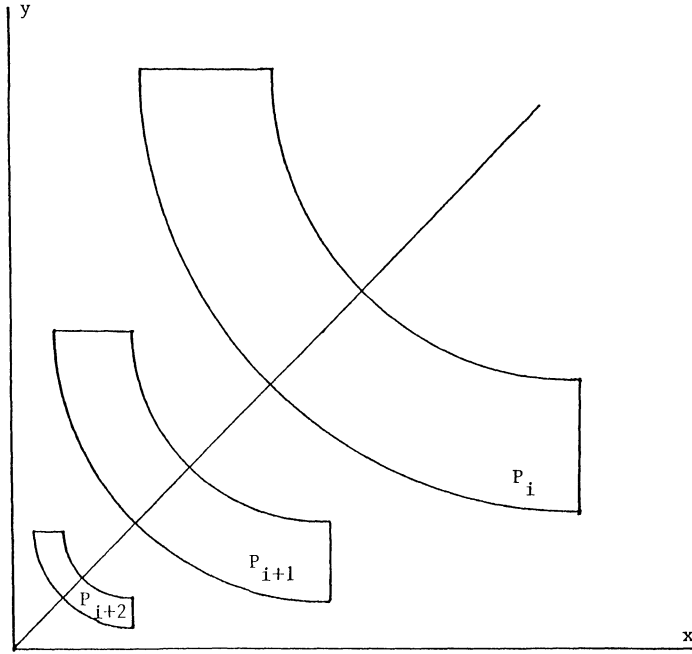
FIGURE 3

By rescaling the bump functions F_i, if necessary, employed in the construction of X_1 it is easy to create an example such that $a_i = 1$ for all $i \geq 1$.

Note that we may further adjust any particular F_i to equal zero without sacrificing smoothness.

Let $\{a_i^*\}$ denote the desired sequence of "dips" we need to attain in a sojourn map deriving from some rescaling X_* of X_0, as above.

The restriction $X_1 | P_j$ has the form $(1 + F_j)X_0$, for each j. The sojourn times of the flow within P_j tend to ∞ as j increases. Thus, given a_k^*, there exist m_{jk}, for all j sufficiently large, such that $(1 + m_{jk}F_j)X_0$ gives rise to a dip of size precisely a_k^*. Since a_k^* corresponds to an increasingly smaller proportion of the sojourn time in P_j, as j increases, the sequence m_{jk} is bounded (in fact tends to zero). Suppose $m_{jk} < M_k$ for all j.

Since X_1 is C^∞-flat at the origin, we may choose $j(k)$ such that

$$\|F_{j(k)}X_0\|_k \leq 1/kM_k$$

for all $k \geq 1$ where $\| \cdot \|_k$ denotes the uniform C^k-norm on vector fields. Putting $m_k = m_{j(k),k}$, the vector field

$$X_* = \left(1 + \sum_{k=1}^{\infty} M_k F_{j(k)}\right) X_0$$

has the desired properties.   □

REMARK. For further details of this construction consult [SSW]. This result will also be obtained as an easy corollary (Corollary 4.3) of the main theorem of §4.

3. Sojourn map realization: The Proof of 3.1. Let $a \neq 0$ be a constant, and suppose

$$x' = -x\left[a^2 + \left(\frac{x-y}{2}\right)^2\right], \qquad y' = y\left[a^2 + \left(\frac{x-y}{2}\right)^2\right].$$

Then the sojourn map time to pass from the (ingress) point $(1, s)$ to the (egress) point $(s, 1)$ is given by

$$\text{(1)} \qquad \tau(s) = \int_s^1 \frac{dx}{x[a^2 + (x - s/x)^2/4]}$$

since $xy = s$ is constant on solution curves. Given the substitution $u = (x - y)/2 = (x - s/x)/2$, the integral in (1) becomes

$$\text{(2)} \qquad \tau(s) = \int_{(s-1)/2}^{(1-s)/2} \frac{du}{\left(\sqrt{u^2 + s}\right)(u^2 + a^2)}.$$

Now replace a^2 by $sg^2(s)$ where $g^2(s) < 1$. Trigonometric substitutions now yield the result

$$\text{(3)} \qquad \tau(s) = \left(\frac{2}{s}\right) \frac{\text{Tan}^{-1}\left(\frac{1}{g}\sqrt{(1 - g^2)(1 - s)/(1 + s)}\right)}{g\sqrt{1 - g^2}}.$$

Our aim now is to implicitly solve for g as a C^∞ function of s. Toward this end, we can recast (3) in the form

$$\text{(4)} \quad g\sqrt{1 - g^2} - (2/s\tau(s))\,\text{Tan}^{-1}\left(\frac{1}{g}\sqrt{(1 - g^2)(1 - s)/(1 + s)}\right) = 0.$$

Suppose the left-hand side defines a function $T(s, g)$ for $0 < s < 1$ and $0 < g < 1$. Our hypotheses imply that $1/s\tau$ is C^∞-extendible to $[0, 1)$ as $s \to 0+$. Also, if the inverse tangent is defined to be $\pi/2$ when $g = 0$, then $T(s, g)$ is C^∞-extendible to $[0, 1) \times [0, 1)$. Now extend

$T(s, g)$ to $(-1, 1) \times (-1, 1)$ in an arbitrary, but C^∞, manner. This is possible by a theorem due to Borel (e.g. see [GG], p. 98). Suppose that $\lim_{s \to 0}(2/s\tau(s)) = a_0 \geq 0$. If $a_0 < 1$, the intermediate value theorem shows there exists a number $0 \leq g_0 < 1$ such that $T(0, g_0) = 0$. A simple computation shows that $(\partial T)/\partial g(0, g_0) \neq 0$.

The implicit function theorem implies that there is a $\delta > 0$ and a C^∞ function $g(s)$ defined for $|s| < \delta$ such that

$$(5) \qquad\qquad\qquad T(s, g(s)) = 0$$

for $|s| < \delta$. It also follows that $g(s) < 1$ for $|s| < \delta$. Thus, the system

$$X_g: \begin{array}{l} x' = -x(sg^2(s) + ((s-y)/2)^2) \\ y' = y(sg^2(s) + ((x-y)/2)^2) \end{array}$$

has a sojourn map τ_g which equals τ for $|s| < \delta$.

If $a_0 \geq 1$, then the map $\sigma = 2a_0\tau$ can be realized in the above fashion as a sojourn map of some X_g since

$$\lim_{s \to 0^+} \left(\frac{2}{s\sigma(s)} \right) = \frac{1}{2} < 1.$$

To realize $\tau(s)$ itself, use the vector field $(1/2a_0)X_g$. Again, this is valid for $s < \delta$.

To realize $\tau(s)$ on $\delta \leq s < 1$, first extend $g(s)$ in a C^∞-manner so that $g(s) \neq 0$ on $\delta \leq s < 1$. This smoothly extends the vector field X_g, still denoted by X_g. Let τ_g denote the sojourn map on $(0, 1)$ for X_g. The function $\tau_g(s)/\tau(s)$ is C^∞ on $[0, 1)$ and is identically 1 for $s < \delta$. Define $Y = (\tau_g(xy)/\tau(xy)) \cdot X_g$. Since $xy = s$ is constant, the vector field Y has sojourn map $\tau_Y = (\tau/\tau_g) \cdot \tau_g = \tau$ for $0 < s < 1$.

4. Realizing rises and dips in the τ-map.

LEMMA 4.1. *Let $f: [0, 1] \to \mathbf{R}^+ \cup \{0\}$ be a C^∞ function with $f(0) = 0$ and $f(x) > 0$ for $x > 0$. Suppose that: f has local extrema at $c_0 = 1 > c_1 > c_2 > \cdots$; $f(c_{2i})$ is a local max, $f(c_{2i+1})$ is a local min, and f is monotone on $[c_{i+1}, c_i]$ for $i = 0, 1, 2, \ldots$; and $c_i \to 0$ as $i \to \infty$. Let $\bar{a}_i = f(c_{2i}) - f(c_{2i-1})$ and $\bar{b}_i = f(c_{2i-2}) - f(c_{2i-1})$ for $i = 1, 2, \ldots$. Then:*

(i) *$\sum_{i=1}^\infty (\bar{b}_i - \bar{a}_i) > 0$, $\sum_{i=1}^\infty (\bar{b}_i - \bar{a}_i) > \bar{b}_{n+1} + \sum_{i=1}^n (\bar{b}_i - \bar{a}_i)$ for all $n \geq 1$, and*

(ii) *$\sum_{i=1}^\infty (\bar{b}_i)^{1/p} < \infty$ and $\sum_{i=1}^\infty (\bar{a}_i)^{1/p} < \infty$ for all positive integers p.*

Conversely, if $\{a_i\}$ and $\{b_i\}$ are sequences of positive integers satisfying (i) *and* (ii) *then there is a function f and a sequence of points $c_0 = 1 > c_1 > \cdots > 0$ satisfying the above hypotheses such that $\bar{a}_i = f(c_{2i}) - f(c_{2i-1})$ and $\bar{b}_i = f(c_{2i-2}) - f(c_{2i-1})$.*

Proof. Let f be as hypothesized. Then (i) follows from the fact that $f(x) > 0$ for $x > 0$. Let $\Delta X_i = c_i - c_{i+1}$ for $i = 0, 1, 2, \ldots$.

We claim that for each $i \geq 0$ and positive integer p there is a $z_i^p \in [c_{i+2p-1}, c_i)$ such that $f^{(p)}(z_i^p) = 0$. This is clear for $p = 1$ since $f'(c_i) = 0$ for $i = 1, 2, \ldots$. Also, if $z_{i+2}^p - 1 \in [c_{i+2p}, c_{i+2p-1})$ and $z_i^p \in [c_{i+2p-1}, c_i)$ are such that $f^{(p)}(z_{i+2}^p - 1) = f^{(p)}(z_i^p) = 0$, then, by the mean value theorem, there is a $z_i^{p+1} \in [z_{i+2}^p - 1, z_i^p] \subseteq [c_{i+2}, c_i)$ such that $f^{(p+1)}(z_i^{p+1}) = 0$. Induction on p establishes the claim.

Now we claim that for each $i \geq 0$ and positive integer p there is a point $\eta_i^p \in [c_{i+2p} - 1, c_i]$ such that

$$\frac{|f(c_{i+1}) - f(c_i)|}{(\Delta X_i + \cdots + \Delta X_{i+2p} - 1)^p} \leq |f^{(p)}(\eta_i^p)|.$$

For $p = 1$ this is just the mean value theorem. Assume that the claim is true for some positive p for all $i \geq 0$. Now $|f^{(p)}(x)|$ takes on values $|f^{(p)}(\xi_i^p)| = 0$ and

$$|f^{(p)}(\eta_i^p)| \geq \frac{|f(c_{i+1}) - f(c_i)|}{(\Delta X_i + \cdots + \Delta X_{i+2p} - 1)}$$

on the interval $[c_{i+2p} - 1, c_i]$. Thus there is a point $\eta_i^{p+1} \in [c_{i+2p} - 1, c_i]$ such that

$$|f^{(p+1)}(\eta_i^{p+1})| \geq \frac{|f(c_{i+1}) - f(c_i)|}{(\Delta X_i + \cdots + \Delta X_{i+2p} - 1)^p(\Delta X_i + \cdots + \Delta X_{i+2p} - 1)}$$

$$\geq \frac{|f(c_{i+1}) - f(c_i)|}{(\Delta X_i + \cdots + \Delta X_{i+2p})^p}.$$

Since $z_i^p \to 0$ as $i \to \infty$ for each p and f is C^∞, $f^{(p)}(0) = 0$ for all p. Thus, $f^{(p)}(\eta_i^p) \to 0$ as $i \to \infty$ for each p and we have:

$$\frac{\bar{a}_i}{(\Delta X_{2i-1} + \cdots + \Delta X_{2i-1+2p} - 1)^p} \leq |f^{(p)}(\eta_{2i-1}^p)| \to 0$$

as $i \to \infty$ for each p. It follows that, for each positive integer p, there is a $K_p < \infty$ such that

$$(\bar{a}_i)^{1/p} \leq K_p(\Delta X_{2i-1} + \cdots + \Delta X_{2i-1+2p} - 1)$$

for all $i \geq 1$. Thus,

$$\sum_{i=1}^{\infty} (\overline{a}_i)^{1/p} \leq (K_p) 2^p \sum_{i=0}^{\infty} \Delta X_i = 2^p K_p < \infty.$$

Similarly, $\sum_{i=1}^{\infty} (\overline{b}_i)^{1/p} < \infty$ for all positive integers p and we have established (ii).

For the converse, let $\{\overline{a}_i\}$ and $\{\overline{b}_i\}$ be sequences of positive real numbers satisfying (i) and (ii). Let $g: \mathbf{R} \to \mathbf{R}$ have the properties: g is C^{∞} and $g(x) = 0$ for $|x| \geq 1$; $g(0) = 1$ and $g^{(p)}(0) = 0$ for all $p \geq 1$; g is monotone on $[-1, 0]$ and on $[0, 1]$. Also, let $L = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} a_n > 0$.

Given a sequence $c_0 = 1 > c_1 > c_2 > \cdots > 0$ with $c_i \to 0$ as $i \to \infty$, define $f: [0, 1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} L - \left(\sum_{i=1}^{n+1} \overline{b}_i - \sum_{i=1}^{n} \overline{a}_i \right) + \overline{b}_{n+1} g \left(\dfrac{x - c_{2n}}{c_{2n} - c_{2n+1}} \right), \\ \qquad \text{for } x \in [c_{2n+1}, c_{2n}], \ n = 0, 1, \ldots ; \\ L - \left(\sum_{i=1}^{n+1} \overline{b}_i - \sum_{i=1}^{n} \overline{a}_i \right) + \overline{a}_{n+1} g \left(\dfrac{x - c_{2n+2}}{c_{2n+1} - c_{2n+2}} \right), \\ \qquad \text{for } x \in [c_{2n+2}, c_{2n+1}], \ n = 0, 1, \ldots ; \\ 0, \quad \text{for } x = 0. \end{cases}$$

Then f is C^{∞} on $(0, 1]$, continuous at $x = 0$, and $f(x) = 0$ if and only if $x = 0$. Also, f has local maxima at c_{2i}, local minima at c_{2i-1}, and f is monotone on $[c_i, c_{i-1}]$ for $i = 1, 2, \ldots$. Thus we need only choose the c_i such that f is C^{∞} at $x = 0$.

Let $M_1 = 1$ and, for $p \geq 2$,

$$M_p = \max \left\{ 1 + M_{p-1}, \inf \left\{ m \geq 1 \left| \sum_{i=m}^{\infty} (\overline{b}_i)^{1/p} \leq 1/(p^2) \right. \right\} \right\}.$$

For $M_p < i < M_{p+1}$ define ΔX_{2i-1} by $\Delta X_{2i-1} = (1/K_1)(\overline{b}_i)^{1/p}$ where K_1 is chosen so that $\sum_{i=1}^{\infty} \Delta X_{2i-2} = 1/2$. Similarly, let $N_1 = 1$ and, for $p \geq 2$,

$$N_p = \max \left\{ 1 + N_{p-1}, \inf \left\{ m \geq 1 \left| \sum_{i=1}^{\infty} (a_i)^{1/p} \leq 1/(p^2) \right. \right\} \right\}.$$

For $N_p \leq i < N_{p+1}$ define ΔX_{2i} by $\Delta X_{2i} = (1/K_2)(a_i)^{1/p}$ where K_2 is chosen so that $\sum_{i=1}^{\infty} \Delta X_{2i} = 1/2$.

It is an easy matter to check that $\bar{a}_i/(\Delta X_{2i})^p \to 0$ and $\bar{b}_i/(\Delta X_{2i-1})^p \to 0$ as $i \to \infty$ for each positive integer p. Now let $c_0 = 1$ and $c_i = 1 - (\sum_{j=1}^{i} \Delta X_j)$ for $i \geq 1$. With this choice of the c_i we claim that f is C^∞ at 0. Indeed, let $m_p = \max |g^{(p)}(x)|$ for $p = 0, 1, 2, \ldots$. Then

$$|f^{(p)}(x)| \leq m_p \cdot \frac{\bar{b}_{n+1}}{(\Delta X_{2n+1})^p} \quad \text{for } x \in [c_{2n+1}, c_{2n}]$$

and

$$|f^{(p)}(x)| \leq m_p \cdot \frac{\bar{a}_{n+1}}{(\Delta X_{2n+2})^p} \quad \text{for } x \in [c_{2n+2}, c_{2n+1}].$$

Thus, $f^{(p)}(x) \to 0$ as $x \to 0$ for all p and f is C^∞ at $x = 0$. $\qquad \square$

THEOREM 4.2. *If $\{a_i\}$ and $\{b_i\}$ are sequences of positive real numbers such that, for some constant $K, K + \sum_{i=1}^{n}(b_i - a_i) > 0$ for all n and $\sum_{n=1}^{\infty}(K + \sum_{i=1}^{n}(b_i - a_i))^{-1/p} < \infty$ for all positive integers p then there is a C^∞ hyperbolic sector X whose sojourn map τ_X has dips and rises given by $\{a_i\}$ and $\{b_i\}$.*

Proof. Let

$$\bar{a}_n = \left(K + \sum_{i=1}^{n}(b_i - a_i)\right)^{-1} - \left(K + \sum_{i=1}^{n}(b_i - a_i) + a_n\right)^{-1}$$

and

$$\bar{b}_n = \left(K + \sum_{i=1}^{n-1}(b_i - a_i)\right)^{-1} - \left(K + \sum_{i=1}^{n}(b_i - a_i) + a_n\right)^{-1}.$$

Then $\{\bar{a}_n\}$ and $\{\bar{b}_n\}$ satisfy the conditions of Lemma 4.1. Let f be the corresponding C^∞ function given by that lemma and let $\tau = 1/f$. By Theorem 3.1 there is a C^∞ hyperbolic sector X such that $\tau_X = \tau$. The dips and rises of τ_X are then given by $\{a_i\}$ and $\{b_i\}$.

COROLLARY 4.3. *Let $\{a_i\}$ be any sequence of positive real numbers. Then there is a C^∞ hyperbolic sector X whose sojourn map τ_X realizes the sequence $\{a_i\}$ as its sequence of dips.*

Proof. In the preceding theorem, let $b_i = a_i + 2^i$.

5. C^∞-flat hyperbolic sectors. In this section we prove that those hyperbolic sectors having generating vector fields which are C^∞-flat at $(0, 0)$, lie on the *boundary* of the central class, I_0, as determined by the C^∞-uniform topology. This topology is now formalized.

Our results apply to a class of "standard" C^∞-flows, $\mathscr{H}_0$, on $\mathbf{R}^2$ having hyperbolic sectors and, in fact, to all diffeomorphic images, $\mathscr{H}$, of such flows. In either case, we first fix the location of a hyperbolic sector, then topologize the collection of C^∞-flows having sectors in the same locale. The reader may consult Hartman ([H]) for standard terminology.

Let $\mathscr{H}_0$ denote the collection of C^∞-flows ϕ_t on $\mathbf{R}^2$ having a singularity at $(0,0)$ and such that

(a) the stable manifold, $W^s(0,0) = \{(x,0)|x \geq 0\}$, and the unstable manifold, $W^u(0,0) = \{(0,y)|y \geq 0\}$,

(b) the segments,

$$\Sigma_1 = \{(2,y)|0 < y \leq 1\}, \qquad \Sigma_2 = \{(x,2)|0 < x \leq 1\}$$

are differentiably transverse to the generating vector field X for ϕ_t,

(c) there exists $a > 0$ such that corresponding to

$$\Sigma_1^a = \{(2,y)|0 < y \leq a\} \subseteq \Sigma_1,$$

there is a well defined sojourn map, $\tau\colon (0,a] \to \mathbf{R}$ with $\tau(y) > 0$, $\phi_{\tau(y)}(2,y) \in \Sigma_2$, and $\phi_t(2,y) \notin \Sigma_1 \cup \Sigma_2$ for all $0 < t < \tau(y)$ (see Figure 4).
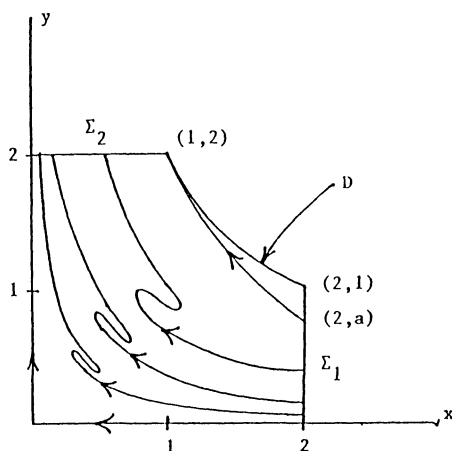

FIGURE 4

Let $D = \{(x,y)|0 \leq x \leq 2,\ 0 \leq y \leq 2,\ xy \leq 1\}$. The flows $\phi_t \in \mathscr{H}_0$ are topologized by the induced C^∞ uniform topology on their generating vector fields, X. That is, a neighborhood base is given by

$$\mathscr{B} = \{A_{k,\varepsilon}(\phi_t)\colon k \in N,\ \varepsilon > 0,\ \phi_t \in \mathscr{H}_0\}, \quad \text{where}$$
$$A_{k,\varepsilon}(\phi_t) = \{\eta_t \in \mathscr{H}_0\colon \|X - X_\eta\|_k < \varepsilon\}.$$

Here $\| \cdot \|_k$ denotes the uniform k-norm,

$$\|Y\|_k = \max_{\substack{0 \le j \le k \\ j = \alpha + \beta}} \left\{ \max_{p \in D} \left\| \frac{\partial^j Y}{\partial x^\alpha \, \partial y^\beta}(p) \right\| \right\}.$$

Now form the set $\mathscr{H}$ of all C^∞-flows, ϕ_t on $\mathbf{R}^2$, for which there exists $\phi_t \in \mathscr{H}_0$, and a C^∞ diffeomorphism $f \colon \mathbf{R}^2 \to \mathbf{R}^2$ with $\psi_t = f\phi_t f^{-1}$. Flows ψ_t in $\mathscr{H}$ can be C^∞-uniform topologized as have those of $\mathscr{H}_0$, by maximizing k-derivatives over an associated region, $f(D)$, comparing only those flows having a hyperbolic sector on $f(D)$.

The conjugacy class of a flow in $\mathscr{H}_0$ (or in $\mathscr{H}$), which is generated by a C^∞-flat vector field at $(0,0)$, can be altered either by gluing in a slow linear flow near $(0,0)$, or another C^∞-flat flow with dissimilar conjugacy class. For example, those C^∞-flat flows which lie outside the central conjugacy class, I_0, such as the example of [SSW], can be C^∞-closely approximated by a C^∞-flow which is linear near $(0,0)$. This new flow will have an asymptotically monotone sojourn map, and thus be an element of I_0. On the other hand, C^∞-flat flows within I_0 can be C^∞-closely approximated by a flow having the Shafer-Swanson-Walker example near $(0,0)$. In this way, C^∞-flat flows are shown to lie on ∂I_0. Lastly, a specific example of a non-C^∞-flat flow, lying on ∂I_0, will be constructed.

First, a vector field gluing lemma is proven. Let β denote the usual bump function such that for $x \le 0$ and $y \ge 1$, $\beta(x) = 0$, $\beta(y) = \beta(1)$, and $\beta(x) = \int_0^x e^{-1/t^2} e^{-1/(t-1)^2} \, dt$, for $0 < x < 1$. Next set

$$M_j = \max_{0 \le x \le 1} \|D^j \beta(x)\|,$$

and notice that, $M_j \le M_{j+1}$, for all $j \ge 0$. The bump function β may be rescaled to

$$\beta_R(x) = \beta(x/R - 1)/\beta(1), \qquad \text{for } R > 0.$$

Then, $\|D^j \beta_R\| \le M_j/R^j$, for all $j \ge 0$.

LEMMA 5.1. *Fix $k \in N$ and $\varepsilon > 0$. Let X be a C^∞ vector field which is C^∞-flat at $(0,0)$. Then there exists $R_0 = R_0(\varepsilon, k) > 0$ such that, in polar coordinates, for all $p = (r, \theta)$ with $r < 2R_0$,*

$$\|(1 - \beta_R)(r) \cdot X(p)\|_k \le \varepsilon.$$

Furthermore, there exists $\delta + \delta(\varepsilon, k) > 0$ such that, if L is a linear vector field for which $\|L\| \le \delta$, then

$$\|(1 - \beta_R)(r) \cdot L(p)\|_k \le \varepsilon.$$

Proof. Since the k-norm is the maximum over the first k derivatives, then for $0 \leq j \leq k$, and $0 \leq r \leq 2R$,

$$\|D^j \beta_R X\| \leq \|D^j \beta_R\| \cdot \|X\|_k$$
$$\leq \|D^k \beta_R\| \cdot \|X\|_k \leq M_k(\|X\|_k)/R^k.$$

But since X is C^∞-flat at $(0,0)$, we must have that $X = o(R^k)$. Thus there exists R_0 such that $\|X(p)\|_k \leq \varepsilon/2$ on $\{r \leq 2R_0\}$ and $\|X\|_k \leq (\varepsilon R_0^k)/2M_k$, verifying the first inequality. Similarly, since $\|L\|_k = \|L\| = \delta$, choose $\delta \leq \min(\varepsilon/2, (R_0^k)/M_k)$ and the Lemma follows.

Proof of Theorem 5.2. First assume that $\phi_t \in \mathcal{H}_0$ and is generated by a C^∞-flat vector field $X \notin I_0$. It does not suffice now to bump L off into X, using Lemma 5.1. This is because X may have an opposite sense from L at points of D: the newly formed vector field will thus not be an element of $\mathcal{H}_0$. Thus, a smooth bumped-off combination of L and X is formed, which respects sense, as follows: For $p \in (D \backslash W^s(0,0)) \backslash W^u(0,0)$, let $\overline{p} \in \Sigma_1$ and $t(p) \geq 0$ be such that $\phi_{t(p)}(\overline{p}) = p$ and $\phi_t(\overline{p}) \in D$ for $0 \leq t \leq t(p)$. Then define the "total rotation" in $Y \in \mathcal{H}_0$ by

$$\theta(Y)(p) = \int_0^{t(p)} \alpha'(Y(\phi_t(\overline{p}))) \, dt,$$

where

$$a(v) = \mathrm{Cos}^{-1}(v \cdot (1,0))/|v|.$$

Notice that $\theta(Y)$ is C^∞ on $(D \backslash W^s(0,0)) \backslash W^u(0,0)$ and can be C^∞ extended to $D \backslash (0,0)$ in the obvious way.

Now fix k and $\varepsilon > 0$, and let Z_ε be the vector field on D which has magnitude and sense prescribed by, $|Z_\varepsilon| = (1 - \beta_R)|L| + \beta_R|X|$ and, off $(0,0)$, $\alpha(Z_\varepsilon) = [(1 - \beta_R)\theta(L) + \beta_R\theta(X)] \bmod 2\pi$, where $R = R_0(\varepsilon/4, k)$ and $\|L\| = \delta(\varepsilon/4, k)$ as in Lemma 5.1. Then Z_ε is C^∞ and $Z_\varepsilon \in \mathcal{H}_0$ because only $(0,0)$ is a singularity and Z_ε ingresses and egresses properly on D.

As constructed, outside the ball R_{2R}, $\|Z_\varepsilon - X\|_k = 0$ since $Z_\varepsilon = X$. Inside B_{2R},

$$\|Z_\varepsilon - X\|_k \leq 2\|(1 - B_R)L - (1 - \beta_R)X\| \leq \varepsilon.$$

The factor of 2 is required since $(1 - \beta_R)L$ and $\beta_R X$ may be opposed. Thus, $\phi_t \in \mathrm{Cl}\, I_0$ since each Z_ε has a τ-map which is asymptotically monotonic near 0.

To show that, in fact, $\phi_t \in \partial I_0$, we select in place of L above a C^∞-flat element of I_0^c. The vector field X_1 of [SSW] has undamped

time oscillations and is C^∞-flat at $(0,0)$. For each k and $\varepsilon > 0$, a new Z_ε is formed which equals X_1 inside B_R. This is an element of I_0^c and $\|Z_\varepsilon - X\|_k < \varepsilon$. Thus $\phi_t \in \partial I_0$.

Lastly, a flow $\phi_t \in \partial I_0$ which is not C^∞-flat is constructed. Consider the system

$$(1) \qquad \begin{cases} x' = x(x^2 - y^2)^2 + xg(xy), \\ y' = -y(x^2 - y^2)^2 - yg(xy), \end{cases}$$

where

$$g(v) = \begin{cases} [\exp(-|v|^{-1})\sin^2(\pi v^{-1}) + \exp(-v^{-2})], & v \neq 0, \\ 0, & v = 0. \end{cases}$$

The first quadrant is a hyperbolic sector for (1). We will show that the flow of (1) is not topologically conjugate, in any neighborhood of the origin in the first quadrant, to the standard hyperbolic sector.

Let $u = x^2 - y^2$ and $v = xy$. Then

$$v' = 0 \quad \text{and} \quad u' = (2u^2 + 2g(v))(x^2 + y^2).$$

Thus, in $-1 \leq u \leq 1$, $0 < v \leq 1$, and along a solution curve $v = $ constant, we have:

$$(2) \qquad (2u^2 + 2g(v))2v \leq u' \leq (2u^2 + 2g(v))(3).$$

For fixed $v, 0 < v \leq 1$, let $\tau(v)$ be the time it takes the solution of (1) to pass from $u = x^2 - y^2 = -1$ to $u = 1$ along the curve $xy = v$. Integrating the inequality (2) we obtain:

$$(3) \qquad \tfrac{1}{3}(g(v))^{-1/2} \arctan(g(v))^{-1/2}$$
$$\leq \tau(v) \leq 2v(g(v))^{1/2} \arctan(g(v)^{1/2}).$$

Now let $v_n = 2/(n + 1)$, $n = 1, 2, \ldots$. From (3) and the definition of $g(v)$ we obtain

$$\tau(v_{2n-1}) \geq (\pi/12)\exp(n^2) \quad \text{and}$$
$$\tau(v_{2n}) \leq (((2n + 1)\pi)/8)\exp(n + (1/2)).$$

Thus, $\tau(1/n) - \tau(1/(n + (1/2))) \to \infty$ as $n \to \infty$. It follows from Proposition 2.2 of [SSW] that the flow of (1) is not topologically conjugate, in any neighborhood of the origin, to the standard hyperbolic sector.

Let X be the vector field given by (1). Then X is not in the standard class I_0 and X is not C^∞-flat. Also, if L is the linear vector field

$$\left. \begin{matrix} x' = x \\ y' = -y \end{matrix} \right\} \quad L.$$

Then $X + \varepsilon L$ is in I_0 for every $\varepsilon > 0$ by Hartman's Theorem. Clearly

$$X + \varepsilon L \to X$$

in the C^∞-uniform topology on any compact neighborhood of the origin as $\varepsilon \to 0$ so $X \in \partial I_0$.

Comment. The straight flows mentioned in §1 can be shown to lie in $\text{Cl}\, I_0$ simply by the addition of a small linear vector field.

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