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A REMARK ON THE SYMMETRY OF SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS

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This note gives a necessary and sufficient condition for solutions of second order elliptic equations to be radially symmetric.

1. Introduction.

1.1. In an elegant paper **[GNN]**, Gidas-Ni-Nirenberg proved that the positive solutions of

$$\begin{cases} \Delta u = f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ u \in C^2(\overline{B}), \end{cases} \tag{1}$$

must be radially symmetric. Here f is C^1 and B is the n-dimensional ball: $\{x \in R^n; |x| < 1\}$. Obviously a symmetric solution of (1) is not necessary to be positive. In this note, we give a necessary and sufficient condition for symmetric solutions of (1). The main result is the following

THEOREM 1. *Suppose $n \geq 2$. A solution u of (1) is radially symmetric if and only if its nodal set $\{x \in \overline{B}; u(x) = 0\}$ is radially symmetric.*

REMARK. It is interesting to note that Theorem 1 need not hold in case $n = 1$. For, $u = \sin x$ solves

$$u'' = -u \quad \text{in } [-\pi, \pi]$$

with the symmetric nodal set $\{0\} \cup \{-\pi, \pi\}$, but u is not radially symmetric since $\sin(-x) = -\sin x$.

It is clear that the result of **[GNN]** is a special case of Theorem 1 since the nodal set of a positive solution to (1) is the sphere ∂B.

In order to prove Theorem 1, we need the following two preliminary results.

THEOREM 2. *Let $u \in C^2(\overline{B})$ satisfy*

$$\Delta u = f(u) \quad \text{in } B. \tag{2}$$

If the nodal set of u consists of spheres with the center 0, then these spheres must be isolated unless $u \equiv 0$.

THEOREM 3. *Let $n \geq 2$ and $u \in C^2(\overline{B})$ satisfy*

$$\begin{cases} \Delta u = f(u) & \text{in } B, \\ u > 0 & \text{in } B\backslash\{0\}, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{3}$$

Then $u > 0$ in B.

REMARK. In case $n = 1$, Theorem 3 need not hold. For example, let $u(x) = \sin(x - \frac{\pi}{2}) + 1$ for $x \in [-2\pi, 2\pi]$, we have

$$\begin{cases} u'' = 1 - u & \text{in } (-2\pi, 2\pi), \\ u > 0 & \text{in } (-2\pi, 2\pi)\backslash\{0\}, \\ u = 0 & \text{at } x = 0, -2\pi, 2\pi. \end{cases}$$

1.2. The proof of Theorem 3 is based on Lemma 12.1 in **[GNN]**, we rewrite it in the form.

LEMMA A. *Let $p = (p^1, p^2, \ldots, p^n) \in \partial B$ with $p^1 > 0$. Assume for some $\varepsilon > 0$ that u is a C^2 function satisfying equation (2) in $\overline{\Omega}_\varepsilon$ where $\Omega_\varepsilon = B \cap \{x; |x - p| < \varepsilon\}$, $u > 0$ in $\overline{\Omega}_\varepsilon \backslash \partial B \cap \{x; |x-p| < \varepsilon\}$ and $u = 0$ on $\partial B \cap \{x; |x - p| < \varepsilon\}$. Then there exists $\delta > 0$ such that in $B \cap \{x; |x - p| < \delta\}$, $\frac{\partial u}{\partial x_1} < 0$.*

2. Proofs.

2.1. *Proof of Theorem* 2. We may assume that the nodal set of u is $\bigcup_{\lambda \in \Lambda} S(\lambda)$ where $\Lambda \subset [0, 1]$ and $S(\lambda) = \{x \in R^n; |x| = \lambda\}$. It needs to be proved that the set Λ contains only isolated points unless $u \equiv 0$. Suppose that there is a sequence $\{\lambda_i\} \subset \Lambda$ with $\lambda_i \to \overline{\lambda}$. Using the polar coordinates $x = r\xi$ where $\xi \in S^{n-1}$ and $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, we obtain that $u = \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} = 0$ for $r = \overline{\lambda}$, which implies that

$$u(0) = \frac{\partial u}{\partial x_i}(0) = \frac{\partial^2 u}{\partial x_l^2}(0) = 0 \qquad (l = 1, 2, \ldots, n)$$

when $\overline{\lambda} = 0$, and that $u = D_\xi u = D_\xi^2 u = 0$ on $S(\overline{\lambda})$ when $\overline{\lambda} > 0$. Thus, in both cases, $u = \Delta u = 0$ on $S(\overline{\lambda})$, and, from (2) we conclude that $f(0) = 0$. Set

$$c(x) = \int_0^1 f'(tu(x))\, dt.$$

In case $\bar{\lambda} > 0$, we have

$$\begin{cases} \Delta u - c(x)u = 0 & \text{in } \{x\,;\, |x| < \bar{\lambda}\}, \\ u = \dfrac{\partial u}{\partial r} = 0 & \text{on } S(\bar{\lambda}), \end{cases}$$

and obtain $u = 0$ in B by uniqueness of solutions to Cauchy's problem of linear elliptic equations. Now it remains to consider the case $\bar{\lambda} = 0$. Set

$$w(x) = \cos Nx_1 \cdot \cos Nx_2 \cdot \cdots \cdot \cos Nx_n,$$

where N is taken to be large enough so that

$$c(x) + N^2 \geq 0. \tag{4}$$

Put $u = w \cdot v$ for $|x| < \frac{\pi}{2N}$. It is easy to see that

$$\begin{cases} \Delta w = -N^2 w \\ & \text{in } \left\{x\,;\, |x| < \dfrac{\pi}{2N}\right\} \\ w > 0 \end{cases}$$

and $S(\lambda_i) \subset \{x\,;\, |x| < \frac{\pi}{2N}\}$ for i large enough since $\lambda_i \to 0$ as $i \to \infty$. On account of (2), it follows

$$\begin{cases} \Delta v + \frac{\nabla W}{W}\nabla v - (c(x) + N^2)v = 0 & \text{in } \{x\,;\, |x| < \lambda_i\}, \\ v = 0 & \text{on } S(\lambda_i). \end{cases}$$

Because of (4), a well-known maximum principle for second order linear elliptic equations can be applied, and that $v = 0$ is obtained, so $u = 0$ for $|x| < \lambda_i$, and in turn $u = 0$ in B. The proof is completed.

2.2. *Proof of Theorem* 3. Suppose for contradiction that $u(0) = 0$. Automatically $\nabla u(0) = 0$. For $0 \leq \lambda < 1$, denote $\Sigma_\lambda = \{x \in B\,;\, x_1 > \lambda\}$; $T_\lambda = \{x \in B\,;\, x_1 = \lambda\}$, and for $x \in \Sigma_\lambda$, denote by x^λ the reflexion of x with respect to T_λ, denote by Σ'_λ the reflexion of Σ_λ with respect to T_λ. Set

$$\Lambda = \left\{\lambda \in (0, 1)\,;\, u(x^\lambda) > u(x) \text{ in } \Sigma_\lambda,\, \frac{\partial u}{\partial x_1} < 0 \text{ on } T_\lambda\right\},$$

which is not empty by Lemma A and a similar argument to [GNN]. First of all we prove $\inf \Lambda \in \Lambda$. Indeed, there holds

$$\begin{cases} u(x^\alpha) \geq u(x) & \text{in } \Sigma_\alpha, \\ \dfrac{\partial u}{\partial x_1} \geq 0 & \text{on } T_\alpha \end{cases}$$

where $\alpha = \inf \Lambda$. Letting $w(x) = u(x^\alpha)$ for $x \in \Sigma_\alpha$ and

$$c(x) = \int_0^1 f'(u + t(w-u))\,dt,$$

we have

$$\begin{cases} \Delta(w-u) - c(x)(w-u) = 0, \\ (w-u) \geq 0 \quad \text{in } \Sigma_\alpha, \\ (w-u) = 0 \quad \text{on } T_\alpha. \end{cases}$$

Then for $K > 0$,

$$\Delta(w-u) - (K + c(x)) \cdot (w-u) = -K(w-u) \leq 0 \quad \text{in } \Sigma_\alpha.$$

Taking K large enough, we may apply the Hopf maximum principle to $(w-u)$ and obtain that either

$$(w-u) = 0 \quad \text{in } \Sigma_\alpha \tag{5}$$

or

$$\begin{cases} w(x) > u(x) \quad \text{in } \Sigma_\alpha, \\ \dfrac{\partial}{\partial \overline{n}}(w-u)(p) < 0, \end{cases} \tag{6}$$

where $p \in \partial\Sigma_\alpha$ such that $(w-u)(p) = 0$ and $\overline{n} = \overline{n}(p)$ is the outward normal vector of $\partial\Sigma_\alpha$ at p. Then (5) cannot hold since $n \geq 2$ and $u = 0$ on ∂B; $u > 0$ in $B \backslash \{0\}$. Now (6) holds, then $u(x^\alpha) > u(x)$ in Σ_α, and on T_α,

$$2\frac{\partial u}{\partial x_1} = \frac{\partial}{\partial(-x_1)}(w-u) < 0$$

since $(w-u) = 0$, which means $\alpha \in \Lambda$. Next it is easy to see that $\alpha \geq \frac{1}{2}$. If $\alpha = \frac{1}{2}$, let $p_0 = (1, 0, \ldots, 0) \in \partial B$, then $p_0^\alpha = 0$, and

$$(w-u)(p_0) = u(p_0^\alpha) - u(p_0) = 0.$$

By (6) we have

$$\frac{\partial}{\partial x_1}(w-u)(p_0) < 0, \qquad \text{i.e. } -\frac{\partial u}{\partial x_1}(0) - \frac{\partial u}{\partial x_1}(p_0) < 0.$$

Then we get

$$\frac{\partial u}{\partial x_1}(0) > -\frac{\partial u}{\partial x_1}(p_0) \geq 0,$$

a contradiction since $\nabla u(0) = 0$. Thus $\alpha > \frac{1}{2}$. In this case we claim that there exists $\alpha_0 < \alpha$ such that $\alpha_0 \in \Lambda$, which will contradict the assumption $\alpha = \inf \Lambda$ and our proof would then be completed. To this end, we assume again for contradiction that there exists a sequence $\{\alpha_i\}$ with $\alpha_i \to \alpha$ but $\alpha_i \notin \Lambda$ which means that either

$$u(a_i^{\alpha_i}) \le u(a_i) \quad \text{for some } a_i \in \Sigma_{\alpha_i} \tag{7}$$

or

$$\frac{\partial u}{\partial x_1}(b_i) \ge 0 \quad \text{for some } b_i \in T_{\alpha_i}. \tag{8}$$

The latter cannot always remain true for any subsequence of $\{i\}$ since, otherwise, it implies that $\frac{\partial u}{\partial x_1} \ge 0$ at some point on T_α when $\{b_i\}$ do not approach ∂B, contradicting $\alpha \in \Lambda$, and that there exists a point in any neighborhood of b such that $\frac{\partial u}{\partial x_1} \ge 0$ when $b_i \to b \in \partial B$, contradicting Lemma A since $b = (b^1, \dots, b^n)$ with $b^1 = \alpha > 0$. Now let $a_i \to \overline{a} \in \overline{\Sigma}_\alpha$. From (7) $u(\overline{a}^\alpha) \le u(\overline{a})$, and $\overline{a} \in \partial\Sigma_\alpha$ by $\alpha \in \Lambda$. But because $\alpha > \frac{1}{2}$, for $x \in \partial\Sigma_\alpha \backslash \overline{T}_\alpha \subset \partial B$, where $\overline{T}_\alpha$ is the closure of T_α, obviously $u(x^\alpha) > 0 = u(x)$. Thus we further have $\overline{a} \in \overline{T}_\alpha$. Let L_i be the segment joining $a_i^{\alpha_i}$ and a_i, having $(1, 0, \dots, 0)$ as the tangent vector. From (7) it is seen that there exists $y_i \in L_i$ such that $\frac{\partial u}{\partial x_1}(y_i) \ge 0$. Since $\overline{a} \in \overline{T}_\alpha$, y_i must also tend to $\overline{a}$. And automatically $\frac{\partial u}{\partial x_1}(\overline{a}) \ge 0$, which leads to a contradiction when $\overline{a} \in T_\alpha$. Then $\overline{a} \in \partial\overline{T}_\alpha \subset \partial B$. But we have seen that $\frac{\partial u}{\partial x_1}(y_i) \ge 0$ and $y_i \to \overline{a}$, which contradicts Lemma A. Thus we complete the proof.

2.3. *Proof of Theorem* 1. Denote $B(\lambda) = \{x \in R^n;\, |x| < \lambda\}$. The necessity is obvious. For sufficiency, by Theorem 2, the nodal set of u must be $\bigcup_{i=1}^k S(\lambda_i)$ where $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_k = 1$. We further prove $\lambda_1 > 0$.

Indeed suppose $\lambda_1 = 0$, i.e. $u(0) = 0$. We see that there are no nodal points of u in $B(\lambda_2)\backslash\{0\}$, which, together with the fact that $B(\lambda_2)\backslash\{0\}$ is path-connected (since $n \ge 2$), implies that u is positive (or negative) in $B(\lambda_2)\backslash\{0\}$. Then from Theorem 3 we have $u(0) > 0$ (or $u(0) < 0$) also. It contradicts $u(0) = 0$, which shows $\lambda_1 > 0$.

Now in $B(\lambda_1)$, u is positive (or negative). It allows us to apply the result of **[GNN]** to conclude that u is radially symmetric in $B(\lambda_1)$. It is clear that

$$\frac{\partial u}{\partial r} = \text{const.} \quad \text{on } S(\lambda_1). \tag{9}$$

Let $T\colon R^n \to R^n$ be any rotation transform. Since equation (2) is invariant under the transform T, $v = u(Tx)$ also solves (2). On

$S(\lambda_1)$, obviously $v = u$, and $\frac{\partial v}{\partial r} = \frac{\partial u}{\partial r}$ by (9). Then $(v - u)$ is a solution to the Cauchy problem

$$\Delta w = \left(\int_0^1 f'(tv + (1-t)u)\, dt \right) \cdot w \quad \text{in } B,$$
$$w = \frac{\partial w}{\partial r} = 0 \quad \text{on } S(\lambda_1)$$

and constantly equals 0 by the uniqueness of the Cauchy problem, i.e. $u(x) = u(Tx)$ in B for any rotation transforms T, which means u is radially symmetric in B. We finish the proof of our main theorem.

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