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EXCEPTIONAL SETS FOR POISSON INTEGRALS OF POTENTIALS ON THE UNIT SPHERE IN $\mathbf{C}^n$, $p \le 1$

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In this article we show that the exceptional sets for Poisson-Szegő integrals of potentials of H^p functions in the unit ball in $\mathbf{C}^n$ have a certain Hausdorff measure zero, and that this result is sharp.

Let B^n denote the unit ball in $\mathbf{C}^n$ with boundary S, σ will denote the normalized Lebesgue measure on S. We let R denote the (holomorphic) radial derivative $R = \sum_{j=1}^n z_j \partial/\partial z_j$. A holomorphic function f belongs to $\mathscr{H}^p$ if $\sup_{0<r<1} \int_S |f(r\zeta)|^p \, d\sigma(\zeta) < \infty$. In **[2]** and **[5]** it was shown that if $R^k f \in \mathscr{H}^p$ where $0 < p \le 1$ and $n - kp > 0$ then the function f has an admissible limit on $S \backslash E$ where E has non-isotropic Hausdorff measure zero in dimension $m = n - kp$, and this result is sharp. For $p > 1$, the proper measure for the exceptional sets is a certain capacity; see **[4]**. In **[1]** D. Adams proved an analogous result for harmonic functions, see also **[2]**. For harmonic functions the result is the following: if u is a fractional integral of order β (i.e. Bessel potential) of an $H^p(R^n)$ distribution, $0 < p \le 1$, then the Poisson integral of u has non-tangential limits on $R^n \backslash E$ where E has Hausdorff measure zero in dimension $m = n - \beta p$. Again, for $p > 1$, the proper measure of the exceptional sets is capacity.

In this paper we prove an analogous result for certain non-isotropic potentials on S. If k is a positive integer, $k < n$, we let

$$I_k(z, \zeta) = |1 - \langle z, \zeta \rangle|^{k-n}, \qquad z, \zeta \in S.$$

For a function v on S let

$$(I_k v)(z) = \int_S I_k(z, \zeta) v(\zeta) \, d\sigma(\zeta).$$

The kernels I_k will play the role of the Bessel kernels in R^n. Indeed, I_1 is the fundamental solution for a certain sublaplacian on S, see **[9]**. In contrast to the cases mentioned above we can handle only the case where k is an integer. If

$$P(z, \zeta) = \frac{(1 - |z|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}}, \qquad z \in B^n, \ \zeta \in S,$$

is the Poisson-Szegő kernel we are interested in exceptional sets of functions

$$P[I_k v](z) = \int_S P(z, \zeta)(I_k v)(\zeta)\, d\sigma(\zeta)$$

where v is a distribution in the atomic Hardy space $H^p(S)$, $0 < p \le 1$, of Garnett and Latter [7]. We will show that the set where such a function fails to have an admissible limit has non-isotropic Hausdorff measure zero in dimension $m = n - kp$. The method of [2] shows the following: if u is a continuous function in B^n whose admissible maximal function $Mu \in L^p(d\sigma)$, $0 < p \le 1$, and if

$$F(z) = \int_0^1 \left(\log \frac{1}{t}\right)^{k-1} u(tz)\, dt$$

where $n - kp > 0$ then the admissible maximal function $MF \in L^p(d\nu)$ for any measure ν on S that satisfies $\nu(B(\zeta, \delta)) \le \delta^{n-kp}$ for all $B(\zeta, \delta) = \{y \in S\colon |1 - \langle \zeta, n\rangle| < \delta\}$. If we knew this to be true for all $F = P[I_k v]$, $v \in H^p$, then it would follow in a standard way that all such $P[I_k v]$ have admissible limits on the complement of a set whose non-isotropic Hausdorff measure is zero in dimension $n - kp$, see [2] and [5]. Assuming this, our problem reduces to the following: Given $v \in H^p$, $0 < p \le 1$, show that there is a u with $Mu \in L^p(d\sigma)$ so that

$$P[I_k v](z) = \int_0^1 \left(\log \frac{1}{t}\right)^{k-1} u(tz)\, dt. \tag{0.1}$$

Now it is an elementary exercise in integration by parts to show that (0.1) holds if

$$u(z) = \left(r\frac{\partial}{\partial r} + Id\right)^k P[I_k v](rz) = (R + \overline{R} + Id)^k P[I_k v](z),$$

where $\overline{R} = \sum_{j=1}^n \overline{z}_j \partial/\partial \overline{z}_j$. In other words we want to show that if $F = P[I_k v]$, $v \in H^p$, $0 < p \le 1$, then $(R + \overline{R} + Id)^k F$ has its admissible maximal function in $L^p(d\sigma)$. This is the content of this paper.

The main problem we face is that even though F is a Poisson-Szegő integral its derivatives may not be. However, the results of D. Geller give us a way around this difficulty. In [8], Geller introduces a family of differential operators

$$\Delta_{\alpha\beta} = (1 - |z|^2)\left\{\sum_{i,j}(\delta_{ij} - z_i\overline{z}_j)\frac{\partial^2}{\partial z_i \partial \overline{z}_j} + \alpha R + \beta \overline{R} - \alpha\beta\right\}$$

and a family of kernels

$$P_{\alpha\beta}(z\,,\,\zeta) = C_{\alpha,\beta}\frac{(1-|z|^2)^{n+\alpha+\beta}}{(1-\langle z\,,\,\zeta\rangle)^{n+\alpha}(1-\langle \zeta\,,\,z\rangle)^{n+\beta}}\,, \qquad z\in B^n\,,\ \zeta\in S.$$

Here $\alpha\,,\,\beta\in\mathbf{C}$ and $C_{\alpha\beta}$ is an appropriate constant. Note that Δ_{00} is the invariant Laplacian of **[11]**, and $P = P_{00}$ is the Poisson-Szegő kernel above. It is a straightforward calculation that $\Delta_{\alpha\beta}P_{\alpha\beta}\equiv 0$ (the differentiations being with respect to z) and that $P_{\alpha\beta}$ is an approximate identity as long as $\mathrm{Re}(n+\alpha+\beta)>0$, and hence for such values of α and β

$$U(z) = \int_S P_{\alpha\beta}(z\,,\,\zeta)u(\zeta)\,d\sigma(\zeta) = P_{\alpha\beta}[u](z)$$

solves the Dirichlet problem $\Delta_{\alpha\beta}U=0$, $U=u$ on S. The relevance of all this is that if $\Delta_{00}U = 0$ then certain derivatives DU satisfy $\Delta_{\alpha\beta}DU = 0$ for appropriate α and β. Returning to our original problem we have $F = P[I_k v]$, $v\in H^p$, $0<p\le 1$. We show that $(R+\overline{R}+Id)^kF$ can be written in the form $\sum S_{\alpha\beta}(R\,,\,\overline{R})F$, where $\alpha\,,\,\beta$ are non-positive integers, $|\alpha|+|\beta|\le k$, $S_{\alpha\beta}(R\,,\,\overline{R})$ has degree $|\beta|$ in R and $|\alpha|$ in $\overline{R}$ and $\Delta_{\alpha\beta}S_{\alpha\beta}(R\,,\,\overline{R})F = 0$. That is we write $(R+R+Id)^kF$ as a sum of solutions to the equations $\Delta_{\alpha\beta}U=0$. After establishing a unicity theorem for the Dirichlet problem for certain values of $\alpha\,,\,\beta$ (a unicity theorem that is already implicit in the work of C. R. Graham **[10]** in the case of the Heisenberg group) we see that for each $\alpha\,,\,\beta$ we have

$$S_{\alpha,\beta}(R\,,\,\overline{R})F(z) = P_{\alpha\beta}[S_{\alpha\beta}(R\,,\,\overline{R})F](z).$$

Now we want to get into a position to apply standard techniques from harmonic analysis; singular integrals and approximate identities. For our range of α and β, $P_{\alpha\beta}$ is a smooth approximate identity and hence if $S_{\alpha\beta}(R\,,\,\overline{R})F$ were in H^p it would follow that $P_{\alpha\beta}[S_{\alpha\beta}(R\,,\,\overline{R})F]$ would have its admissible maximal function in L^p, which is what we want. So what we want to show is that if $F = P[I_kv]$, $v\in H^p$, then if $j+l\le k$ $R^j\overline{R}^lF|_s$ lies in H^p. What we mean, of course, is that the map $v\to R^j\overline{R}^lP[I_kv]|_s$, originally defined for smooth functions, can be realized as a standard singular integral on S and hence maps H^p to H^p. We do this by exploiting an idea of R. Graham **[10]** who showed that certain radial derivatives of $U = P[u]$, when restricted to the boundary, are actually tangential. What we show is this: if u is sufficiently restricted then for each

α, β, $|\alpha| + |\beta| \leq k$, there is a polynomial $Q_{\alpha\beta}$ in two variables, of total degree at most $|\alpha| + |\beta|$ such that

$$S_{\alpha\beta}(R, \overline{R})P[u]|_s = Q_{\alpha\beta}(L, \overline{L})u$$

on S. Here $L, \overline{L}$ are certain tangential derivatives on S. Then it remains only to show that the map $v \to Q_{\alpha\beta}(L, \overline{L})I_k v$ can be realized as a standard singular integral and hence maps H^p to H^p.

We end the introduction with a few more definitions: for $i < j$,

$$T_{ij} = \overline{z}_i \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial z_i}$$

and

$$\overline{T}_{ij} = z_i \frac{\partial}{\partial \overline{z}_j} - z_j \frac{\partial}{\partial \overline{z}_i}.$$

Then we define

$$L = \sum_{i<j} \overline{T}_{ij} T_{ij}$$

and

$$\overline{L} = \sum_{i<j} T_{ij} \overline{T}_{ij}$$

and

$$\mathscr{L}_0 = -\frac{1}{2}(L + \overline{L}).$$

In [8], Geller gives the following "radial-tangential" form for $\Delta_{\alpha\beta}$:

$$\Delta_{\alpha\beta} = (1 - |z|^2)\left\{ \frac{1}{|z|^2}\left((1 - |z|^2)R\overline{R} - \mathscr{L}_0 + \frac{n-1}{2}(R + \overline{R}) \right) + \alpha R + \beta \overline{R} - \alpha\beta \right\}.$$

For the definition of admissible limit we need the admissible approach region

$$D_\alpha(\zeta) = \left\{ z \in B^n \colon |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}.$$

f has an admissible limit at ζ if

$$\lim_{z \to \zeta,\, z \in D_\alpha(\zeta)} f(z)$$

exists for all $\alpha > 0$ and the admissible maximal function $M_\alpha f(\zeta)$ is defined as

$$\sup_{z \in D_\alpha(\zeta)} |f(z)|.$$

For the definition of non-isotropic Hausdorff measure, see [4].

LEMMA 1.1. *If* $\Delta_{\alpha\beta} f = 0$ *then*

(i) $\Delta_{\alpha,\beta-1}(Rf - \beta f) = 0$,

(ii) $\Delta_{\alpha-1,\beta}(\overline{R} f - \alpha f) = 0$.

Proof. That something like this should hold is suggested by (1.3) of **[8]**. In fact a proof can be based on formulas (1.3) and (1.12) of **[8]**. If this line of reasoning is followed we see that, for example,

$$\Delta_{\alpha,\beta-1}\left(Rf - \beta f + \frac{\partial f}{\partial z_1}\right) = 0$$

and then we need to check directly that

$$\Delta_{\alpha,\beta-1}\left(\frac{\partial f}{\partial z_1}\right) = 0.$$

It seems just as easy to check the lemma directly. This is a straightforward calculation.

COROLLARY. *Suppose* $\Delta_{00} U = 0$ *in* B^n *and* j, l *are non-negative integers. Then there are polynomials* $F_{\alpha,\beta}(x, y)$, *with degree* $-\alpha$ *in* x *and* $-\beta$ *in* y *such that*

$$R^j \overline{R}^l U = \sum_{|\alpha|+|\beta| \le j+l} F_{\alpha\beta}(R, \overline{R}) U$$

and

$$\Delta_{\alpha\beta} F_{\alpha\beta}(R, \overline{R}) U = 0$$

in B^n.

Proof. The proof follows by induction on $j + l$, using the lemma.

In **[8]**, Geller introduces the kernels $P_{\alpha\beta}$ which solve the Dirichlet problem for the operator $\Delta_{\alpha\beta}$. We will need to know that, at least for certain values of α, β, this solution is unique. This uniqueness is implicit in the work of Graham **[10]**. However, since there is no proof in print we will provide one here. To that end we need the following lemma which gives the relation between the operators $\Delta_{\alpha,\beta}$ and certain automorphisms of the ball. The automorphisms are the φ_a given on page 25 of **[11]**. $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_a^{-1} = \varphi_a$, among other properties. Given $a \in B$ and α, β define

$$h_a^{\alpha,\beta}(z) = (1 - \langle a, z\rangle)^\alpha (1 - \langle z, a\rangle)^\beta.$$

LEMMA 1.2.

$$\Delta_{\alpha,\beta}[h_a^{\alpha,\beta}(U\circ\varphi_a)] = h_a^{\alpha,\beta}[(\Delta_{\alpha\beta}u)\circ\varphi_a].$$

(Just to be very clear, on neither side of the equation is $h_a^{\alpha,\beta}$ *composed with* φ_a*.)*

Proof. First we need the following: fix $0 < r < 1$ and let

$$\varphi(z) = \left(\frac{z_1 - r}{1 - rz_1}, \frac{sz_2}{1 - rz_1}, \dots, \frac{sz_n}{1 - rz_1}\right)$$

where $s = \sqrt{1 - r^2}$. Let $h(z) = (1 - r\bar{z}_1)^\alpha(1 - rz_1)^\beta$. We need to know that

$$\Delta_{\alpha\beta}(h \cdot u \circ \varphi) = h \cdot (\Delta_{\alpha\beta}u) \circ \varphi. \tag{1.1}$$

This can be done by appealing to formula (1.12) of [**8**] and using the dilation invariance of analogous operators $\Delta_{\alpha\beta}^H$ defined in the Siegel upper half space. Or it can be proved by a rather lengthy direct calculation which we omit. We will now assume (1.1) holds. Let U be defined by $U(z) = -z$, and apply (1.1) to $u \circ U$ and we have the conclusion of the lemma for $a = (r, 0, \dots, 0)$, if we take into account the fact that $\Delta_{\alpha,\beta}$ commutes with any unitary matrix. Now if we use the formula $U\varphi_a = \varphi_{Ua}U$, which is easily verified for any unitary U, we have the result of the lemma.

In [**8**], it is shown that if $\Delta_{\alpha\beta}f = 0$ in B^n then for every $0 < r < 1$, we have

$$F(-\alpha, -\beta; n; r^2)f(0) = \int_S f(r\zeta)\, d\sigma(\zeta). \tag{1.2}$$

Here $F(a, b; c; x)$ denotes the usual hypergeometric function. Now supposing that $\Delta_{\alpha\beta}u = 0$, and $w \in B^n$ we may apply (1.2) to $u = h_w^{\alpha\beta}(u \circ \varphi_w)$ to obtain

$$g_{\alpha\beta}(r)u(w) = \int_S h_w^{\alpha\beta}(r\zeta)u(\varphi_w(r\zeta))\, d\sigma(\zeta), \tag{1.3}$$

where we let $g_{\alpha\beta}(r) = F(-\alpha, -\beta; n; r^2)$. We will use (1.3) to draw some conclusions about boundary behaviour and uniqueness of solutions of $\Delta_{\alpha\beta}u = 0$.

LEMMA 1.3. *Fix* $\alpha, \beta \in \mathbf{C}$.

(i) *There is a bounded* u, $u \not\equiv 0$ *such that* $\Delta_{\alpha\beta}u \equiv 0$ *if and only if* $g_{\alpha\beta}$ *is bounded.*

(ii) *There is a function* u *continuous on* $\overline{B}^n$, $u \not\equiv 0$, *such that* $\Delta_{\alpha\beta} u \equiv 0$ *in* B^n, *if and only if* $\lim_{r\to 1} g_{\alpha\beta}(r)$ *exists.*

(iii) *There is a function* u *continuous on* $\overline{B}^n$, $u \equiv 0$ *on* $\partial\overline{B}^n$, $u \not\equiv 0$, *and* $\Delta_{\alpha\beta} u \equiv 0$ *in* B^n *if and only if* $\lim_{r\to 1} g_{\alpha\beta}(r)$ *exists and is zero.*

Proof. The proof follows immediately from (1.3) and the fact that if we define $G(z) = g(|z|)$ then $\Delta_{\alpha\beta} G \equiv 0$ in B^n, a fact which is clear from the discussion on page 369 of **[8]**. Part (iii) tells us that if $\lim_{r\to 1} g_{\alpha,\beta}(r)$ exists and is not zero then we have uniqueness for the Dirichlet problem for $\Delta_{\alpha\beta}$, i.e. if $u_1, u_2 \in C(\overline{B}^n)$ and $\Delta_{\alpha\beta} u_1 \equiv \Delta_{\alpha\beta} u_2 \equiv 0$ in B^n and $u_1 \equiv u_2$ on ∂B^n then $u_1 \equiv u_2$ in B^n. Note that this is the case when α, β are real and $n + \alpha + \beta > 0$.

Now assuming that α, β are non-positive integers and $n + \alpha + \beta > 0$, then the Dirichlet problem $\Delta_{\alpha\beta} u = 0$, $u = f$ on ∂B^n has a unique solution u, for any continuous f, given by $u(z) = \int P_{\alpha,\beta}(z, \zeta) f(\zeta) d\sigma(\zeta)$. We want to see what this solution looks like when $f \in H(p, q)$, the space of harmonic homogeneous polynomials of bidegree (p, q). As in **[6]**, we look for a solution of the form $u(r\zeta) = h(r^2) f(r\zeta)$. We conclude that the function h is a solution of the hypergeometric equation

$$t(1-t)h''(t) + [(p+q+n) - (p-\alpha+q-\beta+1)t]h'(t) \\ - (p-\alpha)(q-\beta)h(t) = 0.$$

The only solutions of this equation which are smooth at 0 are multiples of the hypergeometric function $F(p-\alpha, q-\beta; p+q+n; t)$. It follows that

$$u(r\zeta) = \frac{F(p-\alpha, q-\beta; p+q+n; r^2)}{F(p-\alpha, q-\beta; p+q+n; 1)} f(\zeta).$$

From known properties of the hypergeometric series we have that

$$h(r) = f_1(r) + f_2(r)(1-r)^{n+\alpha+\beta} \log(1-r) \tag{1.4}$$

where f_1, f_2 are analytic at $r = 1$.

Our next result shows if $\Delta_{\alpha\beta} u = 0$ then, with appropriate restrictions on α and β, certain radial derivatives of u are actually tangential. This type of phenomenon was first studied by R. Graham, **[10]**.

LEMMA 1.4. *Suppose $\alpha, \beta \le 0$ and $n+\alpha+\beta \ge 2$. Take $u \in H(p, q)$ for some p, q and let $U = P_{\alpha\beta}[u]$, then we have*

$$\begin{aligned} RU|_s &= \frac{1}{\alpha+\beta+n-1}\left\{\frac{\beta}{n-1}\overline{L} - \left(\frac{\beta+n-1}{n-1}\right)L - \alpha\beta\right\}u \\ &= q_{\alpha\beta}(L, \overline{L})u, \\ \overline{R}U|_s &= \frac{1}{\alpha+\beta+n-1}\left\{\frac{\alpha}{n-1}L - \left(\frac{\alpha+n-1}{n-1}\right)\overline{L} - \alpha\beta\right\}u \\ &= q_{\alpha\beta}(L, \overline{L})u. \end{aligned}$$

Proof. Using the "radial tangential" form for $\Delta_{\alpha\beta}$ we see that

$$\frac{1}{|z|^2}\left\{(1-|z|^2)R\overline{R}U - \mathscr{L}_0 U + \frac{n-1}{2}(R+\overline{R})U\right\} + \alpha RU + \beta\overline{R}U - \alpha\beta U \equiv 0.$$

Since $n+\alpha+\beta \ge 2$, it follows from (1.4) that $R\overline{R}U = O(\log\frac{1}{1-r})$, as $r \to 1$, and hence that $(1-|z|^2)R\overline{R}U \to 0$ as $|z| \to 1$. Letting $|z| \to 1$ we have, since $\mathscr{L}_0 = -\frac{1}{2}(L+\overline{L})$,

$$\frac{1}{2}(L+\overline{L})U + \frac{n-1}{2}(R+\overline{R})U + \alpha RU + \beta\overline{R}U - \alpha\beta U \equiv 0$$

on S, or,

$$\left(\frac{n-1}{2}+\alpha\right)RU + \left(\frac{n-1}{2}+\beta\right)\overline{R}U = \alpha\beta U - \frac{1}{2}(L+\overline{L})U$$

on S. But we also have that

$$R - \overline{R} = \frac{1}{n-1}(\overline{L} - L)$$

as differential operators. If we solve these two equations for RU and $\overline{R}U$ on S we get the lemma.

COROLLARY. *Suppose $u \in H(p, q)$ for some p, q and $j+l < n$. Then there is a polynomial Q in 2 variables of total degree $\le j+l$ so that if $U = P_{00}[u]$ then $R^j\overline{R}^l U|_s = Q(L, \overline{L})u$.*

Proof. We do induction on $j+l$. From the corollary to Lemma 1.1 we have

$$R^{j-1}\overline{R}^l U = \sum_{|\alpha|+|\beta| \le j+l-1} F_{\alpha\beta}(R, \overline{R})U,$$

where $\Delta_{\alpha\beta}F_{\alpha\beta}(R, \overline{R})U = 0$. Hence $R^j\overline{R}^l U = \sum RF_{\alpha\beta}(R, \overline{R})U$. By Lemma 1.4

$$R(F_{\alpha\beta}(R, R)U)|_s = l(L, \overline{L})(F_{\alpha\beta}(R, \overline{R})U)|_s$$

where l is first degree. By induction $F_{\alpha\beta}(R, \overline{R})U|_s$ is a polynomial in L, $\overline{L}$ of degree at most $j+l-1$ acting on u.

If we now combine the corollaries to Lemmas 1.1, 1.4 we get the following.

THEOREM 1. *Suppose $j+l<n$. Then there are polynomials $Q_{\alpha\beta}$, $|\alpha|+|\beta|\le j+l$ such that if $u\in H(p, q)$ and $U = P_{00}[u]$ we have*

$$R^j\overline{R}^l U = \sum_{|\alpha|+|\beta|\le j+l} P_{\alpha\beta}[Q_{\alpha\beta}(L, \overline{L})u].$$

Proof. From the corollary to Lemma 1.1 we have

$$R^j\overline{R}^l U = \sum_{|\alpha|+|\beta|\le j+l} F_{\alpha\beta}(R, \overline{R})U$$

where $\Delta_{\alpha\beta}F_{\alpha\beta}(R, \overline{R})U = 0$. Since $j+l<n$, $F_{\alpha\beta}(R, \overline{R})U\in C^1(\overline{B}^n)$ and hence by uniqueness for the Dirichlet problem we have

$$F_{\alpha\beta}(R, \overline{R})U = P_{\alpha\beta}[F_{\alpha\beta}(R, \overline{R})U].$$

Now on S, $F_{\alpha\beta}(R, \overline{R})U = Q_{\alpha\beta}(L, \overline{L})u$ by the corollary to Lemma 1.4.

We have proved the theorem for $u\in H(p, q)$, we will need to extend it to the case $u = I_k v$ where $v\in L^2$, provided $j+l\le k$. We need to know how I_k acts on $H(p, q)$.

LEMMA 1.5. *For $v\in H(p, q)$,*

$$I_k v = \frac{\Gamma(k)}{\Gamma(\frac{n-k}{2})^2}\left\{\left(p+\frac{n-k}{2}\right)_k\left(q+\frac{n-k}{2}\right)_k\right\}^{-1} v.$$

Proof. It is easy to check that $I_k(v\circ U) = (I_k v)\circ U$ for any unitary U. Since $H(p, q)$ is minimal under the action of the unitary group, it is enough to prove the lemma in case $v(\zeta) = \zeta_1^p\overline{\zeta}_2^q$. We write

$$|1-\langle z, \zeta\rangle|^{k-n} = (1-\langle z, \zeta\rangle)^{-(\frac{n-k}{2})}(1-\langle \zeta, z\rangle)^{-(\frac{n-k}{2})}.$$

If we expand each factor in a binomial series and integrate term by term we arrive at

$$(I_k v)(z) = \frac{1}{\Gamma(\frac{n-k}{2})}\left(\sum_{j=0}^{\infty}\frac{\Gamma(\frac{n-k}{2}+p+j)\Gamma(\frac{n-k}{2}+q+j)}{\Gamma(p+q+n+j)j!}\right)z_1^p\overline{z}_2^q.$$

Recognizing the series as essentially

$$F\left(\frac{n-k}{2}+p, \frac{n-k}{2}+q; p+q+n; 1\right)$$

we arrive at the desired formula.

On the other hand if $u \in H(p, q)$ then $Lu = -p(q+n-1)u$ and $\overline{L}u = -q(p+n-1)u$, see [3]. Hence if $v \in H(p, q)$ and $\deg Q_{\alpha\beta}(L, \overline{L}) \leq j+l = k$ then

$$Q_{\alpha\beta}(L, \overline{L})I_k v = C(p, q)v$$

where

$$|C(p, q)| \leq \frac{C[2pq+(p+q)(n-1)]^k}{(p+\frac{n-k}{2})_k(q+\frac{n-k}{2})_k} \leq C$$

independent of p, q.

Hence the mapping

$$v \to Q_{\alpha\beta}(L, \overline{L})I_k v$$

extends to be a bounded map from L^2 to L^2. Moreover, when $v \in L^2$ then the differential operator $Q_{\alpha\beta}(L, \overline{L})$ applied to $I_k v$ in the sense of distributions is the same as the operator just described above. From this it follows that if $I_k v$ is C^∞ on some open set $\Omega \subseteq S$ then the $Q_{\alpha\beta}(L, \overline{L})I_k v$ just described and the function obtained by applying the differential operator $Q_{\alpha\beta}(L, \overline{L})$ to $I_k v$ agree on Ω. This will be used later.

We now summarize our results so far.

THEOREM 2. *Fix $k < n$, then there are polynomials $Q_{\alpha\beta}$ in 2 variables of total degree at most k so that for $v \in L^2$ we have*

$$(R+\overline{R}+I)^k P_{00}[I_k v] = \sum_{|\alpha|+|\beta|\leq k} P_{\alpha\beta}[Q_{\alpha\beta}(L, \overline{L})I_k v].$$

Proof. We just note that $(R+\overline{R}+I)^k$ is a sum of terms of the form $R^j\overline{R}^l$ with $j+l \leq k$. We just add and group like terms.

Next we want to show that the operators $Q_{\alpha\beta}(L, \overline{L})I_k$, which extend to be bounded in L^2 actually extend to be bounded in H^p, $0 < p \leq 1$.

Theorem 3. *Suppose $r+s \le k$ and let K be the operator defined by $Kv = L^r\overline{L}^s I_k v$, then K is bounded in H^p, $0 < p \le 1$.*

Proof. We consider the smooth approximations K_r where $I_k(z, \zeta)$ is replaced by $|1 - r\langle z, \zeta\rangle|^{k-n}$. K_r is a multiplier on each $H(p, q)$ and in fact if $Ku = C(p, q)u$ for $u \in H(p, q)$, then

$$K_r u = r^{p+q} C(p, q)u$$

and so it follows that $||K_r u - Ku||_{L^2} \to 0$ as $r \to 1$. If we can show that for every (p, ∞) atom a we have $||K_r a||_{H^p} \le C$ where C is independent of r and a, then the theorem will follow in a standard way. To establish this we note the following: we calculate that for $a, b > 0$ we have that

$$\overline{T}_{ij} T_{ij} (1 - r\langle \zeta, w\rangle)^{-a} (1 - r\langle w, \zeta\rangle)^{-b}$$

is a sum of two terms, one of the form

$$|\zeta_i w_j - w_j \zeta_i|^2 (1 - r\langle \zeta, w\rangle)^{-a-1} (1 - r\langle w, \zeta\rangle)^{-b-1}$$

and the other of the form

$$(\zeta_i \overline{w}_i + \zeta_j \overline{w}_j)(1 - r\langle \zeta, w\rangle)^{-a-1} (1 - r\langle w, \zeta\rangle)^{-b}.$$

If we add on $i < j$ and use the fact that $\sum_{i<j} |\zeta_i w_j - \zeta_j w_i|^2 = 1 - |\langle \zeta, w\rangle|^2$ we see that $L_\zeta (1 - r\langle \zeta, w\rangle)^{-a} (1 - r\langle w, \zeta\rangle)^{-b}$ is a sum of terms of the form

$$(1 - |\langle \zeta, w\rangle|^2)(1 - r\langle \zeta, w\rangle)^{-a-1} (1 - r\langle w, \zeta\rangle)^{-b-1}$$

and

$$\langle \zeta, w\rangle (1 - r\langle \zeta, w\rangle)^{-a-1} (1 - r\langle w, \zeta\rangle)^{-b}.$$

There is a similar expression for $\overline{L}_\zeta$. So if we apply $L_\zeta - \overline{L}_\zeta$ to

$$(1 - r\langle \zeta, w\rangle)^{\frac{k-n}{2}} (1 - r\langle w, \zeta\rangle)^{\frac{k-n}{2}}$$

we get a sum of terms of the form

$$(1 - |\langle \zeta, w\rangle|^2)^l (1 - r\langle \zeta, w\rangle)^{-a} (1 - r\langle w, \zeta\rangle)^b$$

where $a + b - l \le n - k + r + s$. Now if we let D_w denote any w derivative which has k T_{ij}'s and $(R - \overline{R})l$-times we see that we have

$$|D_w K_r(\zeta, w)| \le \frac{C}{|1 - \langle \zeta, w\rangle|^{n + \frac{k}{2} + l}}.$$

From this estimate it follows in a standard way that K_r is uniformly bounded on (p, ∞) atoms (see **[4]**).

Finally we point out that in our case, $(\alpha, \beta \leq 0, n+\alpha+\beta > 0)$ $P_{\alpha\beta}$ is a smooth approximate identity and hence we have

THEOREM. *For* $0 < p \leq 1$ *we have*

$$||MP_{\alpha\beta}f||_{L^p} \leq C||f||_{H^p}.$$

Putting all these results together, as indicated in the introduction we have our main theorem.

THEOREM 4. *Suppose* $0 < k < n$, k *is a positive integer and* $0 < p \leq 1$, *and* $n - kp > 0$. *Then there is a constant* C *such that if* ν *is a measure on* S *that satisfies* $\nu(B(\zeta, \delta)) \leq \delta^{n-kp}$, *then*

$$\int M_\alpha P[I_k \nu]^p \, d\nu \leq C||v||_{H^p}^p$$

for all $v \in H^p$, *all* $\alpha > 0$.

COROLLARY. *For each* $v \in H^p$, $0 < p \leq 1$, *there is a set* $E \subseteq S$ *with non-isotropic Hausdorff measure zero in dimension* $n - kp$ *such that* $F = P[I_k v]$ *has admissible limits on* $S \backslash E$.

Proof. The corollary follows from the theorem and results of W. Cohn [5].

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