PACIFIC JOURNAL OF MATHEMATICS
Vol. 154, No. 2, 1992

ON MEANS OF DISTANCES ON THE SURFACE OF A SPHERE. II (UPPER BOUNDS)

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Given N points $x_1, \dots, x_N$ on the unit sphere S in Euclidean d space $(d \geq 3)$, lower bounds for the deviation of the sum $\sum |x-x_j|^\alpha$, $\alpha > 1-d$; $x \in S$, from its mean value were established in terms of L^1-norms in the first part of this paper. In the present part it is shown that these bounds are best possible. Our main tool is a multidimensional quadrature formula with equal weights.

1. Introduction. On the surface $S = S^{d-1}$ of the unit sphere in d-dimensional Euclidean space E^d $(d \geq 3)$, we consider a certain class of distance functions and distance functionals, associated with a given N point set $\omega_N = \{x_1, x_2, \dots, x_N\}$ on S. Denote by $|x-y|$ the Euclidean distance between two points x and y in E^d. Let $x \in S^{d-1}$ be a variable point. For each value of a parameter α $(1-d < \alpha < \infty)$ consider the distance function $U_\alpha(x, \omega_N)$ which we define as follows:

$$U_\alpha(x, \omega_N) = \sum_{j=1}^{N} |x-x_j|^\alpha - N \cdot m(\alpha, d) \quad \text{for } \alpha \neq 0,$$

and

$$U_0(x, \omega_N) = \sum_{j=1}^{N} \log|x-x_j| - N \cdot m(0, d).$$

Here $m(\alpha, d)$ denotes the mean value of $|x-x_j|^\alpha$ on S^{d-1}, i.e.

$$m(\alpha, d) = \frac{1}{\sigma(S)} \int_S |x-x_j|^\alpha \, d\sigma(x) \quad \text{for } \alpha \neq 0,$$
$$m(0, d) = \frac{1}{\sigma(S)} \int_S \log|x-x_j| \, d\sigma(x),$$

where σ is the $(d-1)$-dimensional area measure on S^{d-1}.

In the first part [**4**] we proved certain lower bounds for the L^1-norms of the functions $U_\alpha(x, \omega_N)$ (see Theorem 1 in [**4**]). The existence of such lower bounds is due to the fact that uniform distribution on S^{d-1}

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can be approximated by an N point distribution to a certain degree of accuracy only. In this part we will show that the lower bounds obtained in **[4]** are best possible, apart from the values of certain constants. More precisely, we shall prove:

THEOREM A. *For any* $\alpha > 1-d$ *and some positive constant* $c = c(\alpha, d)$, *there exists, for each* $N \geq 1$, *an* N*-tuple* ω_N^0 *of points* S^{d-1} *(depending on* α*) such that the following relations hold*:

(a) $\max_{x\in S} |U_\alpha(x, \omega_N^0)| \leq c(\alpha, d) \cdot N^{-\alpha/(d-1)}$ *if* $0 < \alpha < \infty$; $\alpha \neq -2, 4, \dots$,
(b) $\min_{x\in S} U_\alpha(x, \omega_N^0) \leq c(\alpha, d) \cdot N^{-\alpha/(d-1)}$ *if* $1-d < \alpha < 0$,
(c) $\max_{x\in S} U_0(x, \omega_N^0) \leq (0, d)$ *if* $\alpha = 0$,
(d) $U_\alpha(x, \omega_N^0) \equiv 0$ *if* $\alpha \in \{2, 4, \dots\}$ *and* $N > N_0(\alpha, d)$.

In view of the relations $\int_S U_\alpha(x, \omega_N)\, d(\sigma(x)) = 0$, the bounds in (a)–(c) are also upper bounds for the L^1-norms

$$\frac{1}{\sigma(S)} \int_S > |U_\alpha(x, \omega_N^0)|\, d\sigma(x).$$

(The reader should compare Theorem A with Theorem 1 in **[4]**.)

Part (d) of the assertion describes an exceptional case: if α is a positive even integer, the function $U_\alpha(x, \omega_N)$ is a trigonometric polynomial in the spherical coordinates of S^{d-1}. Note that the logarithmic case $\alpha = 0$ for dimension $d = 3$ has already been treated in **[3]**. In **[4]** we also considered, for a given set $\omega_N = \{x_1, x_2, \dots, x_N\}$ of points of S^{d-1}, distance functionals $E_\alpha(\omega_N)$ defined by

$$E_\alpha(\omega_N) = \sum_{j=1}^{N}\sum_{k=1}^{N} (|x_j - x_k|^\alpha - m(\alpha, d)) \quad \text{for } 0 < \alpha < 2,$$

$$E_0(\omega_N) = \sum_{j\neq k}\sum (\log|x_j - x_k| - m(0, d)),$$

and

$$E_\alpha(\omega_N) = \sum_{j\neq k}\sum (|x_j|^\alpha - m(\alpha, d)) \quad \text{for } 1-d < \alpha < 0.$$

For $0 < \alpha < 2$ and $N \geq 2$, the sum $E_\alpha(\omega_N)$ is known to be negative (see Theorem 2 in **[4]**).

An application of Theorem A immediately yields

THEOREM B. *For any α with $0 < \alpha < 2$ and some positive constant $c(\alpha, d)$ there exists, for each $N \geq 2$, an N-tuple ω_N^0 of points on S^{d-1} (depending on α) such that the following inequality holds:*

$$E_\alpha(\omega_N^0) \geq -c_1(\alpha, d) \cdot N^{1-\alpha/(d-1)}.$$

Here $c_1(\alpha, d)$ is a positive constant, independent of N.

Theorem B shows that the inequality proved in [**4**] (Theorem 2(a)) is best possible, apart from the value of $c_1(\alpha, d)$. We remark that the special case $\alpha = 1$ has already been proved by K. B. Stolarsky [**2**].

The situation for the sums $E_\alpha(\omega_N)$ in the unbounded case $1-d < \alpha \leq 0$ is more complicated. The bounds obtained in [**4**] are thought to be best possible only for parameters α satisfying $1-d < \alpha \leq 3-d$. Unlike as in the preceding case, Theorem A can no longer be used to derive the existence of "good" point sets ω_N^0. Instead, we give a direct construction of such point sets, but only for spheres in three-dimensional space. We have the following proposition:

THEOREM C. *Let $d = 3$. For any α with $-2 < \alpha < 0$ and some positive constant $c_1(\alpha)$ there exists, for each $N \geq 2$, an N-tuple ω_N^0 of points on S^2 such that*

$$E_\alpha(\omega_N^0) \leq -c_1(\alpha) \cdot N^{1-\alpha/2}. \tag{1}$$

Similarly, for $\alpha = 0$ and $N \geq 2$, there exists an ω_N^0 such that

$$E_0(\omega_N^0) \geq \frac{N}{2} \cdot \log + O(N). \tag{2}$$

Note that the logarithmic case has already been handled in the author's paper [**3**]. There the construction of the set ω_N^0 is described completely, but the proof of relation (2), due to its highly computational nature, is only sketched. This unpleasant situation prevails even more in the case $-2 < \alpha < 0$, and so again we shall omit the computational details.

For a physical interpretation of results in the special case $\alpha = -1$, $d = 3$, we refer to the author's paper [**4**].

2. Proof of Theorems A and B. The construction of "good" point sets ω_N^0 for the proof of Theorems A and B depends on a result ("Main Lemma") on numerical integration with equal weights. As usual, the spherical coordinates on S^{d-1} are denoted by $\theta_1, \theta_2, \ldots,$ θ_{d-2} $(0 \leq \theta_\mu \leq \pi)$ and ϕ $(0 \leq \phi < 2\pi)$. Futhermore, we denote by Ω_r the set of trigonometric polynomials in the variables $\theta_1, \theta_2, \ldots, \theta_{d-2}$ and ϕ, of degree not exceeding r, i.e. polynomials of the form

$$p(\theta_1, \theta_2, \ldots, \theta_{d-2}, \phi) = \sum_{|j_\mu|\le r, |k|\le r} a(j_1, j_2, \ldots, j_{d-2}, k) \cdot \exp\left(i \cdot \left(\sum_{\mu=1}^{d-2} j_\mu\theta_\mu + k\cdot\phi\right)\right),$$

where j_μ $(\mu = 1, 2, \ldots, d-2)$ and k are integers, and the $a(j_1, \ldots, j_{d-2}, k)$ are arbitrary complex coefficients.

Main Lemma. *For all $d \ge 3$ and all $r \in N$ there exists an $n_0 = n_0(r, d)$ such that, for all domains $D \subseteq S^{d-1}$ of the form*

$$D = \{(\theta_1, \ldots, \theta_{d-2}, \phi) \colon \beta_{1\mu} \le \theta_\mu \le \beta_{2\mu}, \ \gamma_1 \le \phi \le \gamma_2\},$$

the following is true:

For each $(d-1)$-tuple of integers $(m_1, m_2, \ldots, m_{d-2}, n)$ satisfying $m_j \ge n_0$ $(j = 1, \ldots, d-2)$ and $n \ge n_0$, there is a set P of $n \cdot \prod_{j=1}^{d-2} m_j$ points $(\theta_{\mu_1}, \theta_{\mu_2}, \ldots, \theta_{\mu_{d-2}}, \phi_\nu)$ $(1 \le \mu_j \le m_j, \ 1 \le \nu < n)$ on D with the property that

$$\frac{\sigma(D)}{\operatorname{card} P}\sum_{u\in P} p(u) = \int_D p(u)\, d\sigma(u) \tag{3}$$

for each trigonometric polynomial $p(u) = p(\theta_1, \theta_2, \ldots, \theta_{d-2}, \phi) \in \Omega_R$.

Let us make a few remarks.

(1) The mere existence of the number $n_0(r, d)$ for a given fixed domain D follows from a general result of P.D. Seymour and T. Zaslavsky **[1]**. However, we need independence of the bound $n_0(r, d)$ from the special choice of the domain D. As the proofs given in **[1]** are not constructive, the results of these two authors cannot be used for our purpose.

(2) We may consider formula (3) as a quadrature formula with equal weights for the system of functions Ω_r. A classical negative result for ordinary polynomials on an interval (due to S. N. Bernstein) shows that we may not expect the bound $n_0(r, d)$ to be of an order as small as r^{d-1}.

(3) In order not to interrupt the line of the proof of Theorems A and B, we shall postpone the proof of the Main Lemma to the end of the paper.

The proof of Theorem A splits into several cases according to the value of the parameter α.

The case $0 < \alpha < 2$. Let α be fixed, and let $n_0 = n_0 \ (r = d, d)$ be the number the existence of which is guaranteed by the Main Lemma. Let N be sufficiently large, $N = k \cdot n_0^{d-1} + l$, where $0 \le l < n_0^{d-1}$ and $k = [N/n_0^{d-1}]$. By cutting the coordinate intervals $0 \le \theta_\mu \le \pi$ $(\mu = 1, 2, \dots, d-2)$ and $0 \le \phi < 2\pi$ into pieces appropriately, it is not difficult to see that we may divide the surface S^{d-1} into subdomains $D_1, D_2, \dots, D_t$, $t = t(N)$, which are "rectangles" in the system of spherical coordinates, and which possess the following basic properties:

(a) We have $\sigma(D_\tau) = N^{-1} \cdot n_0^{d-1} \cdot \sigma(S)$ for $\tau = 1, 2, \dots, t-1$, and for $\tau = t$ if $l = 0$, and $\sigma(D_t) = N^{-1} \cdot l \cdot (1 + n_0^{d-1})^{d-1} \cdot \sigma(S)$ if $0 < l < n_0^{d-1}$.

(b) Denoting by $|D_\tau| = \sup_{x, y \in D_\tau} |x - y|$ the diameter of D_τ, we have

$$|D_\tau| \le c_2 \cdot N^{-1/(d-1)} \qquad (\tau = 1, 2, \dots, t), \tag{4}$$

where c_2 is a positive constant depending on the dimension d only.

We apply the Main Lemma to each of the domains D_τ. We choose $m_1 = m_2 = \cdots = m_{d-2} = n = n_0$ for the domains $D_1, \dots, D_{t-1}$, and for D_t if $l = 0$, and $m_1 = m_2 = \cdots = m_{d-2} = (1 + n_0^{d+1})$, $n = l \cdot 1 + n_0^{d-1}$ for D_t if $l > 0$. The set of interpolation points, distributed on each D_τ according to the Main Lemma, will be denoted by P_τ, where card $P_\tau = n \cdot \prod_{j=1}^{d-2} m_j$, with m_j, n as defined above.

Let $z_\tau = (\frac{1}{2}(\beta_1 + \beta_1'), \dots, \frac{1}{2}(\beta_{d-2} + \beta_{d-2}'), \frac{1}{2}(\gamma + \gamma'))$ be the "midpoint" of the domain $D_\tau = \{\beta_\mu \le \theta_\mu \le \beta_\mu', \gamma \le \phi \le \gamma'\}$. (This choice of z_τ on D_τ is rather arbitrary.)

Fix $x \in S^{d-1}$, and denote by $\overline{D}_\tau$ the convex hull of D_τ in d-dimensional space E^d. By relation (4), there are at most $O(1)$ domains D_τ for which the inequality

$$|x - y| \le c_2 \cdot N^{-1/(d-1)} \tag{5}$$

holds for some point $y \in \overline{D}_\tau$. (For simiplicity, we use the same constant c_2 in (4) and (5).) On each of these $O(1)$ domains, the following inequality is true:

$$\text{(6)} \qquad \left| \frac{N}{\sigma(S)} \int_{D_\tau} |x-y|^\alpha \, d\sigma(y) - \sum_{u \in P_\tau} |x-u|^\alpha \right| \le \frac{N}{\sigma(S)} \cdot \sigma(D_\tau) \cdot (2c_2 \cdot N^{-1/(d-1)})^\alpha + \sum_{u \in P_\tau} (2c_2 \cdot N^{-1/(d-1)})^\alpha \ll N^{-\alpha/(d-1)} .$$

Let M_q $(q = 1, 2, \dots)$ be the class of domains D_τ such that

$$q \cdot c_2 \cdot N^{-1/(d-1)} \le \min_{y \in \overline{D}_\tau} |x-y| < (q+1) \cdot c_2 \cdot N^{-1/(d-1)} .$$

By (4), there are at most $\ll q^{d-2}$ domains D_τ in M_q. On each D_τ of M_q, consider the Taylor expansion

$$\text{(7)} \qquad |x-y|^\alpha = |x-z_\tau|^\alpha + \sum_{m=1}^{d} \frac{1}{m!} ((y-z_\tau) \operatorname{grad}_w)^m |x-w|^\alpha_{w=z_\tau} + R(x,y) = T_d(x,y) + R(x,y),$$

where

$$R(x,y) = \frac{1}{(d+1)!} ((y-z_\tau) \operatorname{grad}_w)^{d+1} |x-w|^\alpha_{w=z_\tau+\delta(y-z_\tau)} , \qquad 0 < \delta < 1 .$$

The remainder term can be estimated as

$$\text{(8)} \qquad |R(x,y)| \ll N^{-(d+1)/(d-1)} \cdot (q \cdot N^{-1/(d-1)})^{\alpha-d-1} \ll q^{\alpha-d-1} \cdot N^{-\alpha/(d-1)} .$$

The main term $T_d(x,y)$ is a polynomial in the cartesian coordinates of y of degree $\le d$ which, after introducing spherical coordinates, becomes a trigonometric polynomial in $\theta_1, \dots, \theta_{d-2}, \phi$ of the class Ω_d, again of degree $\le d$. By our choice of the point set P_τ, we have

$$\frac{N}{\sigma(S)} \int_{D_\tau} T_d(x,y) \, d\sigma(y) = \sum_{u \in P_\tau} T_d(x,u) .$$

Hence, for each D_τ in M_q, noting (8), we have the inequality

$$\left| \frac{N}{\sigma(S)} \int_{D_\tau} |x-y|^\alpha \, d\sigma(y) - \sum_{u \in P_\tau} |x-y|^\alpha \right| \ll q^{\alpha-d-1} \cdot N^{-\alpha/(d-1)} .$$

Here, as in the preceding inequalities, the constants implicit in the Vinogradov symbols $\ll$ may depend on α and d, but are independent of q, τ, x, and N. Summing over all classes M_q and noting (6), we finally obtain:

$$\left| \frac{N}{\sigma(S)} \int_S |x-y|^\alpha \, d\sigma(y) - \sum_{\tau=1}^{t} \sum_{u \in P_\tau} |x-u|^\alpha \right| \\ \ll N^{-\alpha/(d-1)} + \sum_{q=1}^{\infty} q^{d-2} \cdot q^{\alpha-d-1} \cdot N^{-\alpha/(d-1)} \ll N^{-\alpha/(d-1)}.$$

This proves Theorem A in the case $0 < \alpha < 2$, and Theorem B.

The case $2 \le \alpha < \infty$. In the cases $2 < \alpha < 4$, $4 < \alpha < 6, \ldots,$ we proceed as before, choosing successively $r = d+2$, $d+4, \ldots,$ and approximating $|x-y|^\alpha$ by a Taylor polynomial of degree $\le r$. In the case $\alpha = 2h$ $(h = 1, 2, \ldots)$, note that $|x-y|^\alpha$ is a trigonometric polynomial of degree $2h$ in the variables $\theta_1, \ldots, \theta_{d-2}, \phi$. Choosing $D = S$ and $r = 2h$ in the Main Lemma, the assertion follows.

The case $1-d < \alpha \le 0$. We proceed as in the case $0 < \alpha < 2$, choosing $r = d$ in the Main Lemma. The only difference in the argument concerns the derivation of the estimate (6), which has to be replaced in the following way: For fixed x on S, consider again those domains D_τ for which $|x-y| \le c_2 \cdot N^{-1/(d-1)}$ holds for some point y in the convex hull of D_τ. Then the following one-sided estimate is true for $\alpha < 0$:

$$(9) \quad \sum_{u \in P_\tau} |x-u|^\alpha - \frac{N}{\sigma(S)} \int_{D_\tau} |x-y|^\alpha \, d\sigma(y) \ge -c_3(\alpha, d) \cdot N^{-\alpha/(d-1)}.$$

In order to prove (9), we simply omit the sum and estimate the integral from above, using relation (4). In the logarithmic case $\alpha = 0$, the corresponding inequality is

$$(10) \qquad \sum_{u \in P_\tau} \log|x-u| - \frac{N}{\sigma(S)} \int_{D_\tau} \log|x-y| \, d\sigma(y) \le -c_4(d).$$

Here the sum cancels the logarithmic part of the integral, leaving a remainder which is bounded from above.

From (8), (9), and (10) the assertion follows. This finishes our proof of Theorem A.

3. Outline of a proof of Theorem C. The method of constructing "good" point sets ω_N^0 in the case $d = 3$, $-2 < \alpha < 0$, is of a similar type as the one given in §2. The verification of the inequality in Theorem C, however, requires careful direct estimation.

We begin by describing the construction.

Let α and $N \geq 2$ be fixed. Put $N = [\sqrt{N \cdot b}]$, where B is a positive constant to be determined later. Denoting the spherical coordinates on S^2 as usual by θ $(0 \leq \theta \leq \pi)$ and ϕ $(0 \leq \phi < 2\pi)$, we define angles $\theta_0, \dots, \theta_M$ by the conditions

$$0 = \theta_0 < \theta_1 < \cdots < \theta_M = \pi$$

such that $N_\mu := \frac{N}{2}(\cos\theta_{\mu-1} - \cos\theta_\mu)$ $(\mu = 1, 2, \dots, M)$ are positive integers, and such that

$$K_1/\sqrt{b \cdot N} \leq \theta_\mu - \theta_{\mu-1} \leq K_2\sqrt{b \cdot N}$$

holds for $b < \mu < M - b$ and certain numerical constants $0 < K_1 < K_2$. Each zone $D_\mu := \{(\theta, \phi)\colon \theta_{\mu-1} \leq \theta \leq \theta_\mu\}$ is divided into N_μ subdomains $D_{\mu j}$, where

$$\begin{aligned} D_{\mu j} = \Big\{(\theta, \phi)\colon \theta_{\mu-1} \leq \theta \leq \theta_\mu,\ & 2\pi \cdot N_\mu^{-1}\left(j - \frac{1}{2}\right) \\ & \leq \phi \leq 2\pi \cdot N_\mu^{-1}\left(j + \frac{1}{2}\right)\Big\} \\ & (\mu = 1, \dots, M;\ j = 0, \dots, N_\mu - 1). \end{aligned}$$

On each $D_{\mu j}$, we choose a point $x_{\mu j} = (\xi_\mu, \phi_{\mu j})$, where $\phi_{\mu j} = 2\pi j/N_\mu$ and $\cos\xi_\mu = \frac{1}{2}(\cos\theta_{\mu-1} + \cos\theta_\mu)$. Let $\omega_N^0 = \{x_{\mu j}\}$. By a heuristic argument we will try to explain why the set ω_N^0 can be expected to satisfy inequality (1).

For fixed $x_{\mu j}$, the term $|x_{\mu j} - x_{\nu k}|^\alpha$ is roughly equal to the integral

$$\frac{N}{4\pi}\int D_{\nu k}|x_{\mu j} - y|^\alpha\, d\sigma(y);$$

hence the whole sum

$$\sum\sum_{(\mu, j)\neq(\nu, k)} (|x_{\mu j} - x_{\nu k}|^\alpha - m(\alpha, 3)) \tag{11}$$

corresponds to the sum of integrals

$$-\sum_{(\mu, j)} \frac{N}{4\pi}\int_{D_{\mu, j}} (|x_{\mu j} - y|^\alpha - m(\alpha, 3))\, d\sigma(y). \tag{12}$$

This latter sum (12) is easily seen to be

$$\leq -c_5 \cdot b^{-1-(\alpha/2)} \cdot N^{1-(\alpha/2)}. \tag{13}$$

What remains to be shown is the fact that the error which we commit when replacing (11) by (12), is of smaller order than the bound (13). This turns out to be true if we choose b large enough, and if the numbers N_μ satisfy some additional condition of arithmetical nature. The proof, however, is too laborious to be presented here.

4. On quadrature formulas with equal weights. The Main Lemma will be derived from the following theorem which may be of independent interest in itself.

THEOREM. *Let $w(x) \geq 0$ be an integral weight function on the interval $[-1, 1]$, satisfying the relations $\int_{-1}^{1} w(x)\,dx = 1$ and*

$$L_1 \geq w(x) \geq L_2 \cdot (1-|x|)^\beta, \tag{14}$$

with constants $L_1 > 0$, $L_2 > 0$, and $\beta > 0$. Let $\Phi = \{\phi_1, \ldots, \phi_s\}$ be a system of three times continuously differentiable functions on $[-1, 1]$, with the additional property that the derivatives $\phi_1', \phi_2', \ldots, \phi_s'$ form an orthonormal system with respect to the weight function $w(x)$. Let

$$K_1 = \max_{[-1,1]} \max_{\mu=1,\ldots,s} (|\phi_\mu'|, |\phi_\mu''|, |\phi_\mu'''|). \tag{15}$$

Then there exists a number n_0, depending only on L_1, L_2, K_1, β, and s, such that for each $n \geq n_0$, there exist points $t_1, t_2, \ldots, t_n$ with $-1 < t_1 < t_2 < \cdots < t_n < 1$ and

$$\frac{1}{n}\sum_{j=1}^{n} \phi_\mu(t_j) = \int_{-1}^{1} \phi_\mu(x) w(x)\,dx \tag{16}$$

for all $\phi_\mu \in \Phi$ simultaneously.

Proof. 1. In the sequel we will have to deal with the functions ϕ_μ', $\phi_\mu'\phi_\nu'$ $(\mu, \nu = 1, 2, \ldots, s)$, and their derivatives up to the second order. By our assumption (15), all these functions are bounded in absolute value by

$$K := \max(K_1, 4K_1^2).$$

For the construction of the point set $\{t_j\}$, we use Newton's method. We begin by defining intervals $I_j = [x_{j-1}, x_j]$ by the relation

$$\int_{-1}^{x_j} w(x)\,dx = \frac{j}{n} \qquad (j = 0, 1, \ldots, n).$$

By assumption (14), we have ($|I_j| =$ length of I_j)

(17) $$(L_1 \cdot n)^{-1} \le |I_j| \le 2 \cdot (L_2 \cdot \gamma \cdot n)^{-\gamma},$$

where we write $\gamma := 1/(\beta + 1)$ for the sake of brevity.

In the interior of each interval I_j, choose the (uniquely determined) point ξ_j with the property that

(18) $$\int_{I_j} (x - \xi_j) w(x)\, dx = 0.$$

By the assumption $w(x) \le L_1$ in (14), the following inequality holds:

(19) $$\min(x_j - \xi_j, \xi_j - x_{j-1}) \ge (2L_1 \cdot n)^{-1}.$$

We use the point set $\{\xi_j\}$ as the starting point of a Newton iteration process. By changing the values of ξ_j successively, we obtain a sequence of n-point sets on $[-1, 1]$, converging to a set $-1 < t_1 < \cdots < t_n < 1$ with the desired property (16), provided that the number n is chosen large enough. We remark here that if not otherwise stated, all the constants that appear in the following parts of the proof are assumed to depend on L_1, L_2, K_1, β, s, but not on n.

2. Let f be any function on $[-1, 1]$, twice continuously differentiable and satisfying the relation

(20) $$\max_{[-1,1]} (|f'(x)|, |f''(x)|) \le K.$$

By Taylor's theorem, using (18), we have the following basic estimate:

$$\left| n \int_{I_j} f(x) w(x)\, dx - f(\xi_j) \right| = n \left| \int_{I_j} (f(x) - f(\xi_j)) w(x)\, dx \right|$$
$$= \frac{n}{2} \left| \int_{I_j} (x - \xi_j)^2 f''(\xi(x)) w(x)\, dx \right| \le \frac{K}{2} \cdot |I_j|^2.$$

Summing over all intervals I_j, and noting (17), we obtain:

(21) $$\left| \sum_{j=1}^{n} f(\xi_j) - n \int_{-1}^{1} f(x) w(x)\, dx \right| \le \frac{K}{2} \sum_{j=1}^{n} |I_j|^2 \le c_1 \cdot n^{-\gamma}.$$

Now assume that $-1 < \eta_1 < \cdots < \eta_n < 1$ is a new set of points, satisfying $|\xi_j - \eta_j| \le \delta$ for $j = 1, 2, \ldots, n$, and some real $\delta > 0$. By (20) and (21) we have the estimate

(22) $$\left| \sum_{j=1}^{n} f(\eta_j) - n \int_{-1}^{1} f(x) w(x)\, dx \right| \le c - 1 \cdot n^{-\gamma} + \delta \cdot n \cdot K.$$

3. Without loss of generality we may assume that $\int_{-1}^{1} \phi_\mu(x)w(x)\,dx = 0$ holds for all ϕ_μ in Φ. Suppose that after the rth step of the iteration procedure we arrive at a point set $-1 < \eta_1 < \cdots < \eta_n < 1$ ($r = 0$ describes the initial situation $\eta_j = \xi_j$) with the following two properties:

$$\left| \sum_{j=1}^{n} f(\eta_j) - n \int_{-1}^{1} f(x)w(x)\,dx \right| \le C_r \cdot n^{-\gamma} \tag{23}$$

for each f satisfying relation (20), and

$$\sum_{j=1}^{n} \phi_\mu(\eta_j) = \rho_\mu \qquad (\mu = 1, \ldots, s), \tag{24}$$

where $|\rho_\mu| \le \sigma_r$ for all values of μ, and C_r, σ_r are positive constants which may depend on r.

Put $\eta'_j = \eta_j - h_j$. Replacing η_j by η'_j in (24), and linearizing, we obtain the following linear system of equations for the corrections h_j:

$$\sum_{j=1}^{n} h_j \phi'_\mu(\eta_j) = \rho_\mu \qquad (\mu = 1, \ldots, s). \tag{25}$$

We are looking for a solution vector $(h_1, h_2, \ldots, h_n)$ of (25) with all the h_j being small. Here we make essential use of the orthogonormality of the derivatives ϕ'_μ with respect to $w(x)$. We interpret the system (25) as a set of hyperplanes in Euclidean n-space. By (23), we obtain the following estimates for the scalar products between their normal vectors $(\phi'_\mu(\eta_1), \ldots, \phi'_\mu(\eta_n))$:

$$\left| \sum_{j=1}^{n} \phi'_\mu(\eta_j)\phi'_\nu(\eta_j) \right| =: |A_{\mu\nu}| \le C_r \cdot n^{-\gamma} \qquad (\mu \ne \nu),$$

and

$$\sum_{j=1}^{n} \phi'^2_\nu(\eta_j) =: A_{\mu\mu} \ge n - C_r \cdot n^{-\gamma}. \tag{26}$$

An application of Lagrange's method (with multipliers λ_ν) to the expression

$$\sum_{j=1}^{n} h_j^2 - \sum_{\nu=1}^{s} \lambda_\nu \left(\sum_{j=1}^{n} h_j \phi'_\nu(\eta_j) - \rho_\nu \right)$$

leads to a minimal solution of (25). For the multipliers λ_ν we obtain the following linear system of equations:

$$(27)\qquad 2\cdot h_j=\sum_{\nu=1}^{s}\lambda_\nu\cdot\phi'_\nu(\eta_j)\qquad (j=1,2,\dots,n).$$

Multiplying both sides of (27) by $\phi'_\mu(\eta_j)$ and summing over j, we obtain, using (26), a new system of equations:

$$(28)\qquad 2\cdot\pi_\mu=\sum_{\nu=1}^{s}A_{\mu\nu}\lambda_\nu\qquad (\mu=1,2,\dots,s).$$

The matrix of the system (28) is approximately diagonal in view of (26). We have the decomposition

$$(A_{\mu\nu})=\begin{pmatrix}A_{11}&&&0\\&A_{22}&&\\&&\ddots&\\0&&&A_{ss}\end{pmatrix}\cdot\begin{pmatrix}1&&&*\\&1&&\\&&\ddots&\\ *&&&1\end{pmatrix}=D\cdot(I+B),$$

(I = identity matrix), where the entries of B are

$$\le C_r\cdot n^{-\gamma}/(n-C_r\cdot n^{-\gamma})$$

in absolute value. Let us calculate the inverse matrix $(A_{\mu\nu})^{-1}$:

$$(A_{\mu\nu})^{-1}=(I-B+B^2-+\cdots)\cdot D^{-1}=(I+B_1)\cdot D^{-1},$$

where the entries of B_1 are $\le C_r\cdot n^{-\gamma}/(n-(s+1)C_r\cdot n^{-\gamma})$ in absolute value. Hence the entries of the inverse $(A_{\mu\nu})^{-1}$ are in absolute value

$$\le(n^{1+\gamma}-s\cdot C_r)\cdot(n^{1+\gamma}-(s+1)C_r)^{-1}\cdot(n-C_r\cdot n^{-\gamma})^{-1}$$

in the main diagonal, and by

$$\le C_r\cdot n^{\gamma}\cdot(n^{1+\gamma}-(s+1)C_r)^{-1}\cdot(n^{1+\gamma}-C_r)^{-1}$$

elsewhere. Inserting these estimates into (28) and (27), we obtain the following inequalities:

$$(29)\qquad |\lambda_\nu|\le 2\sigma_r\cdot n^{\gamma}\cdot(n^{1+\gamma}-(s+1)C_r)^{-1}\qquad(\nu=1,\dots,s)$$

and

$$|h_j|\le sK\sigma_r\cdot n^{\gamma}\cdot(n^{1+\gamma}-(s+1)C_r)^{-1}\qquad(j=1,\dots,n).$$

If we replace $\eta'_j = \eta_j - h_j$ in (24), the new error terms ρ'_ν are bounded in absolute value by

$$\begin{aligned}(30)\qquad \sigma_{r+1} &\leq \sum_{j=1}^{n} \frac{1}{2} h_j^2 \cdot \max_{\nu} \max_{x \in I_j} |\phi''_\nu(x)| \\ &\leq \frac{n}{2} K^3 s^2 \sigma_r^2 (n - (s+1) C_r n^{-\gamma})^{-2}.\end{aligned}$$

By (29) and (22) the new constant C_{r+1} in (23) can be chosen as small as

$$(31)\qquad C_{r+1} \leq C_r + \sigma_r \cdot s K^2 n^{1+\gamma} (n - (s+1) C_r n^{-\gamma})^{-1}.$$

Keeping in mind that $\sigma_0 \leq C_0 n^{-\gamma}$ by (21), it is not difficult to prove by induction from (30) and (31) that if we choose the number n of interpolation points large enough, the following inequalities are true:

$$\sigma_r \leq (C_0 n^{-\gamma})^{2^r}, \quad C_{r+1} - C_r \leq c_2 \cdot 2^{-r-1} \cdot C_0 \quad \text{and} \quad C_r \leq 2 c_2 \cdot C_0 .$$

Moreover, it follows from the second half of (29) that the total displacement of the initial points ξ_j does not exceed

$$c_3 \cdot n^{-1} \cdot \sum_{r=0}^{\infty} \sigma_r \leq c_4 \cdot n^{-1-\gamma}.$$

Hence, in view of (19), all the limit points t_j of the sequences $\xi_j, \dots,$ $\eta_j, \eta'_j, \dots$ are contained in the interval $(-1, 1)$. This finishes the proof of the theorem.

In order to derive the Main Lemma from the preceding theorem we have to prove that the bound K_1 in condition (15) can be chosen such as to be independent of certain parameters connected with the choice of the domain D.

LEMMA. *Let $w(x)$ be a weight function on $[-1, 1]$, satisfying the conditions $\int_{-1}^{1} w(x)\,dx = 1$ and $w(x) \geq L_2(1 - |x|)^\beta$, where β, L_2 are positive constants. Let $\Psi = \{T_0(x), \dots, T_{2r}(x)\}$ be the system of functions on $[-1, 1]$ defined by*

$$T_{2j}(x) = \left(\frac{1 - \cos \varepsilon x}{\varepsilon^2/2}\right)^j \qquad (j = 0, \dots, r)$$

and

$$T_{2j+1}(x) = T_{2j}(x) \cdot \frac{\sin \varepsilon x}{\varepsilon} \qquad (j = 0, \dots, r-1).$$

Here ε denotes a positive real parameter. If the functions T_k are orthonormalized successively with respect to $w(x)$ by the Gram-Schmidt process, and if $\varepsilon < \varepsilon_0(L_2, r, \beta)$ holds, then the functions of the new system are bounded on $[-1, 1]$ by a constant which depends on r, L_2 and β, but not on ε.

Proof. Let $\Psi_1 = \{g_0 = T_0, g_1, \dots, g_{2r}\}$ be the orthonormal system resulting from Ψ. Each g_j has a unique representation of the form

$$g_j(x) = b_{j0}T_0(x) + \cdots + b_{jj}T_j(x). \tag{32}$$

Assume that for some s, $0 \le s < 2r$, the following inequality holds:

$$|b_{jk}| \le K(s, L_2, \beta) = K_s \qquad (j = 0, 1, \dots, s;\ k = 0, 1, \dots, j). \tag{33}$$

Note that (33) is true for $s = 0$ with $K_0 = 1$. We proceed by induction on s. We orthogonalize the function $T_{s+1}(x)$ with respect to $g_0, \dots, g_s$ by setting

$$f_{s+1}(x) = T_{s+1}(x) - \sum_{j=0}^{s}\langle g_j, T_{s+1}\rangle g_j(x) = \sum_{j=0}^{s+1} a_j T_j(x) \tag{34}$$

and

$$g_{s+1}(x) = f_{s+1}(x)/\|f_{s+1}\|_2.$$

Here as usual we define $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,dx$ and $\|f\|_2^2 = \langle f, f\rangle$.

Note that $|T_j(x)| \le 1$ on $[-1, 1]$; hence $|\langle g_j, T_{s+1}\rangle| \le 1$. From (32) and (33) it follows that

$$|a_j| \le (s+1)\cdot K_s \tag{35}$$

for $j = 0, \dots, s+1$. All we have to prove is that $\|f_{s+1}\|_2$ is bounded from below. From the inequality

$$|T_j(x) - x^j| \le c_1(s)\cdot\varepsilon^2,$$

valid from $x \in [-1, 1]$, $j = 0, 1, \dots, s+1$, and $\varepsilon < 1$ it follows that $f_{s+1}(x)$ admits an approximation by a monic polynomial, i.e.

$$f_{s+1}(x) = x^{s+1} + d_s x^s + \cdots + d_0 + R(x) = p_s(x) + R(x),$$

where $|R(x)| < \varepsilon^2 \cdot c_2(s, L_2, \beta)$. Using expansion of $p_s(x)$ into

Legendre polynomials, we easily obtain:

$$\max_{[-1,1]} |f_{s+1}(x)| \geq c_3(s) - c_2 \cdot \varepsilon^2,$$

where $c_3(s) > 0$.

Furthermore, using $|f'_{s+1}(x)| \leq \sum |a_j| \cdot |T'_j(x)| < c_4(s, L_2, \beta)$, we find that $|f_{s+1}(x)| > \frac{1}{2}c_3(s)$ holds on an interval of length $\geq \delta(s, L_2, \beta) > 0$, provided that ε is small enough. From the assumption $w(x) \geq L_2(1-|x|)^\beta$ we obtain the estimate $\|f_{s+1}\|_2 \geq c_5(s, L_2, \beta) > 0$, which proves the assertion in view of the relations (34) and (35).

COROLLARY. *As the derivatives* $T'_j(x)$ *and* $T''_j(x)$ $(j = 0, \dots, 2r)$ *are bounded on* $[-1, 1]$, *uniformly in* $\varepsilon > \varepsilon_0$, *it follows from* (32) *that the assertion of the lemma is also true for the derivatives* g'_j *and* g''_j.

The proof of the Main Lemma is now completed as follows.

Let $D = \{\theta_{1\mu} \leq \theta_\mu \leq \theta_{2\mu}, \phi_1 \leq \phi \leq \phi_2\} \subset S$ be the given domain. First we note that it is sufficient to prove the Main Lemma for domains D for which the differences $\theta_{2\mu} - \theta_{1\mu}$ $(\mu = 1, \dots, d-2)$ and $\phi_2 - \phi_1$ are sufficiently small. In order to obtain the assertion for domains of arbitrary size, we only have to stick together a bounded number of suitable "small" D's. Secondly we note that it is sufficient to prove the existence of the bound $n_0(r)$ for each coordinate separately. Without restriction, we choose the coordinate θ_1, the proof for the other coordinates being essentially the same. We are hence given the interval of integration $\theta_{11} \leq \theta \leq \theta_{21}$, the weight function $\sin^{d-2}\theta_1$, and the system of functions $\Omega_r(\theta_1) = \{1, \cos\theta_1, \dots, \cos r\theta_1, \sin\theta_1, \dots, \sin r\theta_1\}$. By a suitable linear transformation, replacing the variable θ_1 by x, we obtain the interval $-1 \leq x \leq 1$, the weight function

$$w(x) = \sin^{d-2}\varepsilon(x - x_0) \Big/ \int_{-1}^{1} \sin d^{d-2}\varepsilon(x - x_0)\, dx,$$

where $\varepsilon = \frac{1}{2}(\theta_{21} - \theta_{11})$ and $x_0 = (\theta_{21} + \theta_{11})/(\theta_{21} - \theta_{11})$, and the system $\Omega'_r = \{1, \cos\varepsilon x, \dots, \sin r\varepsilon x\}$. We replace the system Ω'_r by the equivalent system $\Omega''_r = \{1, G_0, G_1, \dots, G_{2r}\}$, where $G_0, \dots, G_{2r}$ are arbitrary primitives of the functions $g_0, g_2, \dots, g_{2r}$ defined in the proof of the lemma. By the lemma, the assumptions of the theorem are now satisfied with $\beta = d - 2$, L_1 and L_2 depending on d only, and $K_1 \leq c(L_2, \beta, r) = c(r)$, as soon as $\varepsilon < \varepsilon_0(L_2, \beta, r) = \varepsilon_0(r)$. This finishes the proof of the Main Lemma.

References

[1] P. D. Seymour and T. Zaslavsky, *Averaging sets*, Adv. in Math., **52** (1984), 213–240.

[2] K. B. Stolarsky, *Sums of distances between points of a sphere* II, Proc. Amer. Math. Soc., **41** (1973), 575–582.

[3] G. Wagner, *On the product of distances to a point set on a sphere* J. Australian Math. Soc., (Series A) **47** (1989), 466–482.

[4] ——, *On means of distances on the surface of a sphere* (*Lower bounds*), Pacific J. Math., **144** (1990), 389–398.

Received February 27, 1991.

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