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FACE NUMBER INEQUALITIES FOR MATROID COMPLEXES AND COHEN-MACAULAY TYPES OF STANLEY-REISNER RINGS OF DISTRIBUTIVE LATTICES

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We discuss two topics related with combinatorial study of canonical modules of Stanley-Reisner rings, viz., (i) some linear inequalities on the number of faces of a matroid complex and (ii) a formula to compute the Cohen-Macaulay type of the Stanley-Reisner ring of a finite distributive lattice.

Introduction. We study the following two problems in the field of commutative algebra and combinatorics:

(i) What can be said about the number of faces of a matroid complex?

(ii) How can we calculate the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice?

Recently, some topics on Hilbert functions of noetherian graded algebras have been studied by several authors, e.g., **[Sta3]**, **[Sta7]**, **[G-M-R]**, **[R-R]** and **[H5]** from viewpoints of commutative algebra, algebraic geometry and combinatorics. In the first half of the present paper, we are concerned with Hilbert functions of Stanley-Reisner rings of matroid complexes. Via well-known facts **[H-K]**, **[Sta3]** on canonical modules of Cohen-Macaulay graded integral domains, Stanley **[Sta7]** found certain linear inequalities for the Hilbert function of a Cohen-Macaulay graded integral domain. Based on an idea of J. Herzog (cf. Corollary (1.5)), we see that the same linear inequalities as in **[Sta7]** hold for the Hilbert function of the Stanley-Reisner ring of a matroid complex (cf. Theorem (1.8)).

On the other hand, it would be of interest to find a combinatorial formula to compute the Cohen-Macaulay type (i.e., the minimal number of generators of the canonical module) of the Stanley-Reisner ring of a Cohen-Macaulay complex, e.g., **[H7]**. In the latter half of this paper, we find a formula for the computation of the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice. In fact, our main result (cf. Theorem (2.10)) guarantees that the Cohen-Macaulay type of the Stanley-Reisner ring of the order complex of a finite distributive lattice is equal to the number of distinct equivalence classes of a certain equivalence relation (cf. (2.8)) on the set of linear extensions of a finite partially ordered set associated with the distributive lattice.

1. Level rings and matroid complexes.

(1.1) Let k be a field and A a semi-standard k-algebra, that is, A is a commutative graded ring $\bigoplus_{n\geq 0} A_n$ satisfying (i) $A_0 = k$, (ii) A is finitely generated as a k-algebra, and (iii) A is integral over the subalgebra $k[A_1]$ of A generated by A_1. The *Hilbert function* of A is defined to be

$$H(A, n) := \dim_k A_n \quad \text{for } n = 0, 1, \ldots,$$

while the *Hilbert series* of A is given by

$$F(A, \lambda) := \sum_{n=0}^{\infty} H(A, n)\lambda^n.$$

Since A is finitely generated as a $k[A_1]$-algebra and is integral over $k[A_1]$, it follows that A is finitely generated as a $k[A_1]$-module. Hence, well-known properties on Hilbert series, e.g., [**Mat**, pp. 94–95], guarantee that

$$F(A, \lambda) = (h_0 + h_1\lambda + \cdots + h_s\lambda^s)/(1-\lambda)^d$$

for some integers $h_0, h_1, \ldots, h_s$ with $h_s \neq 0$. Here d is the Krull dimension of A. We say that the vector $h(A) := (h_0, h_1, \ldots, h_s)$ is the *h-vector* of A.

(1.2) Suppose that a semi-standard k-algebra $A = \bigoplus_{n\geq 0} A_n$ is Cohen-Macaulay. Let K_A be the canonical module [**H-K**] of A. It is known [**H-K**, Corollary (6.7)] that there exists a graded ideal I of A with $I \cong K_A$ (as graded modules over A, up to shift in grading) if and only if A is generically Gorenstein, i.e., the localization $A_{\mathfrak{q}}$ is Gorenstein for every minimal prime ideal $\mathfrak{q}$ of A. Also, see [**H3**, Lemma (1.7)].

(1.3) PROPOSITION. *Let a Cohen-Macaulay semi-standard k-algebra $A = \bigoplus_{n\geq 0} A_n$ be generically Gorenstein, and let $I = \bigoplus_{n\geq a}(I \cap A_n)$, $I \cap A_a \neq (0)$, be a graded ideal of A with $I \cong K_A$. Suppose that there exists a non-zero divisor $\vartheta \in I \cap A_a$ on A. Then the h-vector $h(A) = (h_0, h_1, \ldots, h_s)$ of A satisfies the linear inequality*

$$(*) \qquad h_0 + h_1 + \cdots + h_i \leq h_s + h_{s-1} + \cdots + h_{s-i}$$

for every $0 \leq i \leq s$.

Proof. Since $\vartheta \in I \cap A_a$ is a non-zero divisor on A, the dimension of $I/\vartheta A$ as an A-module is less than the Krull dimension of A if $\vartheta A \neq I$. Thus the proof of **[Sta7**, Theorem (2.1)] is valid in our situation without modification. □

(1.4) We say that a Cohen-Macaulay semi-standard k-algebra $A = \bigoplus_{n\geq 0} A_n$ is *level* **[Sta2]** if the canonical module $K_A = \bigoplus_{n\geq a}(K_A)_n$ with $(K_A)_a \neq (0)$, $a \in \mathbf{Z}$, of A is generated by $(K_A)_a$ as an A-module. In other words, A is level if and only if the Cohen-Macaulay type of A coincides with the last component of the h-vector of A. Consult, e.g., **[H2**, pp. 343–345].

(1.5) Corollary. *Suppose that a Cohen-Macaulay semi-standard k-algebra $A = \bigoplus_{n\geq 0} A_n$ is both generically Gorenstein and level. Then the h-vector $h(A) = (h_0, h_1, \dots, h_s)$ of A satisfies the linear inequality $(*)$ for every $0 \leq i \leq s$.*

Proof. A routine technique enables us to assume that k is an infinite field. Let $I = \bigoplus_{n\geq a}(I \cap A_n)$, $I \cap A_a \neq (0)$, be a graded ideal of A with $I \cong K_A$. Thanks to Proposition (1.3), what we must show is the existence of a non-zero divisor $\vartheta \in I \cap A_a$ on A. Let $\mathcal{N}_A$ be the set of prime ideals of A which belong to the ideal (0). Since A is Cohen-Macaulay, we know that the Krull dimension of $A/\mathfrak{q}$ equals that of A for each $\mathfrak{q} \in \mathcal{N}_A$. We write $\mathcal{U}$ for the (set-theoretic) union of all prime ideals $\mathfrak{q} \in \mathcal{N}_A$. Recall (e.g., **[Mat**, p. 38]) that the set $\mathcal{U}$ coincides with the set of zero divisors on A. If $I \cap A_a \subset \mathcal{U}$, then $I \cap A_a \subset \mathfrak{q}$ for some $\mathfrak{q} \in \mathcal{N}_A$ since k is infinite (see, e.g., **[Her**, Problem 21, p. 136]). Now, A is level, thus I is generated by $I \cap A_a$ as an A-module. Hence, if $I \cap A_a \subset \mathfrak{q}$ then $I \subset \mathfrak{q}$, thus the Krull dimension of A/I is equal to that of A, which contradicts **[H-K**, Corollary (6.13)]. □

The author is grateful to Professor Jürgen Herzog for suggesting the above proof. We remark that Corollary (1.5) is false if we drop the assumption that A is generically Gorenstein.

(1.6) Let V be a finite set, called the *vertex set*, and Δ a *simplicial complex* on V. Thus Δ is a collection of subsets of V such that (i) $\{x\} \in \Delta$ for every $x \in V$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma$ imply $\tau \in \Delta$. Each element of Δ is called a *face* of Δ. Set $d := \max\{\#(\sigma) ; \sigma \in \Delta\}$. Here $\#(\sigma)$ is the cardinality of σ as a set. Then the *dimension* of Δ is defined to be $\dim \Delta := d - 1$. We say that Δ is *pure* if every maximal face has the same cardinality. We write $f_i = f_i(\Delta)$, $0 \leq i < d$, for the number of faces σ of Δ with $\#(\sigma) = i+1$. Thus $f_0 = \#(V)$. We say that $f(\Delta) := (f_0, f_1, \dots, f_{d-1})$ is the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \dots, h_d)$ of Δ by the formula

$$\sum_{i=0}^{d} f_{i-1}(\lambda - 1)^{d-i} = \sum_{i=0}^{d} h_i \lambda^{d-i}$$

with $f_{-1} = 1$. Consult, e.g., **[Sta4]** and **[Hoc]** for further information.

(1.7) A simplicial complex Δ on the vertex set V is called a *matroid complex* (or G-complex **[Sta2]**) if the following conditions are satisfied:

(i) If $\sigma, \tau \in \Delta$ and $\#(\sigma) < \#(\tau)$, then there exists $x \in \tau$ such that $x \notin \sigma$ and $\sigma \cup \{x\} \in \Delta$.
(ii) $\dim(\Delta - x) = \dim \Delta$ for every $x \in V$. Here $\Delta - x$ is the subcomplex $\{\sigma \in \Delta; x \notin \sigma\}$ of Δ on $V - \{x\}$.

We remark that the above condition (ii) is required only to avoid the inessential case; if $\dim(\Delta - x) < \dim \Delta$ then Δ is a cone over $\Delta - x$ with apex x, thus we should study $\Delta - x$ rather than Δ.

For example, let V be a finite set of non-zero vectors of a vector space over a field and suppose that the dimension of the subspace spanned by V is equal to the dimension of the subspace spanned by $V - \{x\}$ for every $x \in V$. Then the set Δ of linearly independent subsets of V is a matroid complex.

Now, what can be said about the h-vector of an arbitrary matroid complex?

(1.8) THEOREM. *Suppose that $h(\Delta) = (h_0, h_1, \dots, h_d)$ is the h-vector of a matroid complex Δ of dimension $d-1$. Then we have the linear inequality*

$$h_0 + h_1 + \dots + h_i \le h_d + h_{d-1} + \dots + h_{d-i}$$

for every $0 \le i \le d$.

Proof. Let $V = \{X_1, X_2, \dots, X_t\}$ be the vertex set of Δ and $k[\Delta] = k[X_1, X_2, \dots, X_t]/I_\Delta$ the *Stanley-Reisner ring* (**[Sta1]**, **[Rei]**) of Δ over a field k with the standard grading, i.e., each $\deg(X_i) = 1$. Then the Krull dimension of $k[\Delta]$ is d, and the Hilbert series of $k[\Delta]$ is just

$$F(k[\Delta], \lambda) = (h_0 + h_1\lambda + \dots + h_d\lambda^d)/(1-\lambda)^d,$$

see, e.g., [**Sta4**, pp. 62–68]. It is known (and, in fact, not difficult to prove) that a matroid complex is "doubly" Cohen-Macaulay in the sense of [**Bac**]. In other words, $k[\Delta]$ is a level ring with $h_d \neq 0$. See also [**Sta2**]. Moreover, $k[\Delta]$ is generically Gorenstein [**Sta4**, p. 80]. Hence Corollary (1.5) enables us to obtain the required inequality. □

(1.9) *Conjecture.* Work in the same notation as in Theorem (1.8). Then we have the following linear inequalities:

(i) $h_i \leq h_{d-i}$ for every $0 \leq i \leq [d/2]$, and
(ii) $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$.

Consult [**H4**] for further information on the inequalities in the above Conjecture (1.9). We easily see the inequality $h_1 \leq h_2$ when $d \geq 3$. Also, note that, thanks to [**H4**], the above conjecture is weaker than that of [**Sta2**, p. 59].

On the other hand, a log-concavity conjecture on f-vectors of matroid complexes is presented by Mason [**Mas**]. Some partial results on this conjecture are obtained by Dowling [**Dow**] and by Mahoney [**Mah**].

It would, of course, be of great interest to find a combinatorial characterization of the h-vectors of matroid complexes.

The f-vectors (or h-vectors) of various classes of simplicial complexes have been studied by several combinatorialists. We refer the reader to, e.g., [**B-K**] for a survey of the topic.

2. Cohen-Macaulay types of distributive lattices.

(2.1) Given a finite partially ordered set (*poset* for short) P we write $\mathscr{J}(P)$ for the poset which consists of all *poset ideals* (or *order ideals* [**Sta6**, p. 100]) of P, ordered by inclusion. Then $\mathscr{J}(P)$ is a *distributive lattice* [**Sta6**, p. 105]. On the other hand, the fundamental theorem for finite distributive lattices, e.g., [**Sta6**, Theorem (3.4.1)] guarantees that, for every finite distributive lattice L, there exists a unique poset P for which $L \cong \mathscr{J}(P)$.

(2.2) Let $\rho(P;\ell)$ be the number of *chains* [**Sta6**, p. 99]

$$\mathscr{M} : \varnothing = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{\ell+1} = P$$

of *length* $\ell + 1$ (cf. [**Sta6**, p. 99]) in the distributive lattice $\mathscr{J}(P)$ such that

(i) $I_{i+1} - I_i$ is a *clutter* [**Sta6**, p. 100] in P for each $0 \leq i \leq \ell$, and

(ii) for every $1 \le i \le \ell$, there exist $y \in I_{i+1} - I_i$ and $x \in I_i - I_{i-1}$ with $x < y$ in P.

Then $\rho(P;\ell) = 0$ if $\ell < \operatorname{rank}(P)$ and $\rho(P;\operatorname{rank}(P)) \neq 0$. Here $\operatorname{rank}(P)$ is the *rank* **[Sta6**, p. 99] of P.

(2.3) We now study the Stanley-Reisner ring

$$k[\Delta(L)] = k[X_\alpha\,;\,\alpha \in L]/I_{\Delta(L)},$$

with each $\deg(X_\alpha) = 1$, of the *order complex* $\Delta(L)$ (cf. [**Sta6**, p. 120]) of a finite distributive lattice L over a field k. It is well known, e.g., [**B-G-S**] that $k[\Delta(L)]$ is Cohen-Macaulay. We are interested in the Cohen-Macaulay type $\operatorname{type}(k[\Delta(L)])$ of $k[\Delta(L)]$, i.e., the minimal number of generators of the canonical module $K_{k[\Delta(L)]}$ of $k[\Delta(L)]$ as a $k[\Delta(L)]$-module. We refer the reader to, e.g., [**B-G-S**] and [**Sta6**, Chap. 4, §5] for the information on the h-vector of the order complex of a finite distributive lattice. Also, consult [**H1**], [**H3**] and [**H6**] for some topics on commutative algebra related with distributive lattices.

(2.4) Proposition. *The Cohen-Macaulay type* $\operatorname{type}(k[\Delta(L)])$ *of the Stanley-Reisner ring* $k[\Delta(L)]$ *of the order complex* $\Delta(L)$ *of a finite distributive lattice* $L = \mathcal{J}(P)$ *is*

$$(**) \qquad \operatorname{type}(k[\Delta(L)]) = \rho(P;\operatorname{rank}(P)) + \rho(P;\operatorname{rank}(P)+1) + \cdots.$$

Proof. Suppose that $\#(P) = n$, say $P = \{p_1, p_2, \ldots, p_n\}$, and we write $e(I) = (e_1, e_2, \ldots, e_n) \in \mathbf{R}^n$ for the incident vector of a poset ideal I of P, i.e., $e_i = 1$ if $p_i \in I$ and $e_i = 0$ otherwise. Thus in particular $e(\varnothing) = (0, 0, \ldots, 0)$ and $e(P) = (1, 1, \ldots, 1)$. If $\mathcal{M}$ is a chain in L of the form ($\bigstar$) $(\varnothing \subset) I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_\ell$ $(\subset P)$ with each $I_i \in \mathcal{J}(P)$, then we write $[\mathcal{M}]$ for the convex hull of $\{e(I_0), e(I_1), \ldots, e(I_\ell)\}$ in $\mathbf{R}^n$. Thus $[\mathcal{M}]$ is an ℓ-simplex in $\mathbf{R}^n$. Let $\mathcal{C} = \mathcal{C}(L)$ be the set of chains in $L = \mathcal{J}(P)$ and $\mathcal{P} = \mathcal{P}(L)$ the convex hull of $\{e(I)\,;\, I \in \mathcal{J}(P)\}$ in $\mathbf{R}^n$. Hence $\mathcal{P} \subset \mathbf{R}^n$ is a convex polytope of dimension n. We identify $\{[\mathcal{M}]\,;\,\mathcal{M} \in \mathcal{C}\}$ with the order complex $\Delta(L)$ of L. It is known, e.g., [**Sta5**, p. 17] that $\{[\mathcal{M}]\,;\,\mathcal{M} \in \mathcal{C}\}$ is a triangulation of $\mathcal{P}$; hence $\mathcal{P}$ is a geometric realization of $\Delta(L)$.

Now, let $\mathcal{I}$ be the ideal of the Stanley-Reisner ring $k[\Delta(L)] = k[X_\alpha\,;\,\alpha \in L]/I_{\Delta(L)}$ which is generated by those square-free monomials $\prod_{\alpha \in \mathcal{M}} X_\alpha$ with $[\mathcal{M}] \in \Delta(L) - \partial\Delta(L)$. Here $\partial\Delta(L)$ is the boundary of $\Delta(L)$. Then, by virtue of [**Sta4**, Theorem (7.3), p. 81], $\mathcal{I}$ is isomorphic to the canonical module $K_{k[\Delta(L)]}$ of $k[\Delta(L)]$. On the other hand, thanks to [**Sta5**, p. 10], if $\mathscr{M} \in \mathscr{C}$ is of the form $(\bigstar)$, then $[\mathscr{M}] \in \Delta(L) - \partial\Delta(L)$ if and only if the following conditions are satisfied: (i) $I_0 = \varnothing$, (ii) $I_\ell = P$, and (iii) each $I_{i+1} - I_i$ is a clutter. Hence, it follows immediately that the minimal number of generators of $\mathscr{I}$ as a $k[\Delta(L)]$-module is just $(**)$ as required. □

We should remark that the ideal $\mathscr{I}$ in the above proof is generated by $\{\Pi_{\alpha\in\mathscr{M}} X_\alpha ; [\mathscr{M}] \in \Delta(L) - \partial\Delta(L), \#(\mathscr{M}) = \mathrm{rank}(P) + 2\}$ as a $k[\Delta(\mathscr{L})]$-module if and only if $\rho(P; \ell) = 0$ for every $\ell \neq \mathrm{rank}(P)$. In other words,

(2.5) Corollary. *The Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of a finite distributive lattice $L = \mathscr{J}(P)$ is level if and only if $\rho(P; \ell) = 0$ for every $\ell \neq \mathrm{rank}(P)$.*

(2.6) Let $\mathbf{N}$ be the set of non-negative integers and P a finite poset. We say that a map $\sigma: P \to \mathbf{N}$ is *strictly order-preserving* if $x < y$ in P implies $\sigma(x) < \sigma(y)$ in $\mathbf{N}$. We write $\mathscr{B}(P; \ell)$ for the set of strictly order-preserving maps $\sigma: P \to \mathbf{N}$ such that (i) $\sigma(P) = \{0, 1, \dots, \ell\}$ and (ii) $\sigma^{-1}(\{i-1, i\})$ is *not* a clutter in P for every $1 \le i \le \ell$.

(2.7) Lemma. $\rho(P; \ell) = \#(\mathscr{B}(P; \ell))$.

Proof. Given a chain $\mathscr{M} : \varnothing = I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_{\ell+1} = P$ in the distributive lattice $\mathscr{J}(P)$ which satisfies the conditions (i) and (ii) in (2.2), we can define a map $\sigma: P \to \mathbf{N}$ in $\mathscr{B}(P; \ell)$ by $\sigma(x) = i$ if $x \in I_{i+1} - I_i$. On the other hand, if $\sigma \in \mathscr{B}(P; \ell)$, then $\varnothing \subsetneqq \sigma^{-1}(\{0\}) \subsetneqq \sigma^{-1}(\{0, 1\}) \subsetneqq \cdots \subsetneqq \sigma^{-1}(\{0, 1, \dots, \ell - 1\}) \subsetneqq P$ is a chain in $\mathscr{J}(P)$ with the properties (i) and (ii) in (2.2). □

(2.8) We recall that a *linear extension* [**Sta6**, p. 110] of a finite poset P is a strictly order-preserving map $\sigma: P \to \mathbf{N}$ such that $\sigma(P) = \{1, 2, \dots, \#(P)\}$. If σ is a linear extension of P, then there exists a unique sequence $\mathscr{D}(\sigma) = (d_1, d_2, \dots, d_\ell) \in \mathbf{Z}^\ell$, $0 \le \ell = \ell(\sigma) \in \mathbf{Z}$, with $1 \le d_1 < d_2 < \cdots < d_\ell < \#(P)$ such that

(i) $\sigma^{-1}(\{d_i + 1, d_i + 2, \dots, d_{i+1}\})$ is a clutter in P for each $0 \le i \le \ell$, where we set $d_0 = 0$ and $d_{\ell+1} = \#(P)$, and
(ii) for every $1 \le i \le \ell$, there exists $x \in \sigma^{-1}(\{d_{i-1} + 1, \dots, d_i\})$ with $x < \sigma^{-1}(d_i + 1)$ in P.

We say that two linear extensions σ and τ of P are *equivalent* (written as $\sigma \sim \tau$) if $\mathscr{D}(\sigma) = \mathscr{D}(\tau)$ $(= (d_1, d_2, \dots, d_\ell))$ and $\sigma^{-1}(\{1, 2, \dots, d_{i+1}\}) = \tau^{-1}(\{1, 2, \dots, d_{i+1}\})$ for every $0 \le i \le \ell$.

(2.9) Given a linear extension σ of a finite poset P with $\mathscr{D}(\sigma)=(d_1, d_2, \dots, d_\ell)$, we write $I_i(\sigma)$ for the poset ideal $\sigma^{-1}(\{1, 2, \dots, d_i\})$ of P for each $1 \le i \le \ell + 1$, where $d_{\ell+1} = \#(P)$. Also, set $I_0(\sigma) = \varnothing$. Then the chain

$$\mathscr{M}(\sigma) : \varnothing = I_0(\sigma) \subsetneqq I_1(\sigma) \subsetneqq \cdots \subsetneqq I_{\ell+1}(\sigma) = P$$

in the distributive lattice $\mathscr{J}(P)$ possesses the properties (i) and (ii) in (2.2).

On the other hand, for each chain $\mathscr{M}$ in (2.2), there exists a linear extension σ of P with $\mathscr{M} = \mathscr{M}(\sigma)$. Moreover, $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ if and only if σ and τ are equivalent.

We now come to the main result of this section in consequence of Proposition (2.4) with Lemma (2.7) and (2.9).

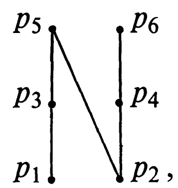
(2.10) THEOREM. *The following quantities on a finite poset P are equal:*

(a) *the Cohen-Macaulay type* $\mathrm{type}(k[\Delta(L)])$ *of the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of the finite distributive lattice $L = \mathscr{J}(P)$,*

(b) *the number of strictly order preserving maps $\sigma: P \to \mathbf{N}$ such that $\sigma^{-1}(\{i-1, i\})$ is not a clutter in P for every $i \in \sigma(P)$ with $i \ge 1$,*

(c) *the number of distinct equivalence classes of the equivalence relation "$\sim$" in (2.8) on the set of linear extensions of the poset P.*

(2.11) EXAMPLE. Let $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be the following finite poset:



and we employ the notation, e.g., 214635 for denoting the linear extension σ of P with $\sigma(p_2) = 1$, $\sigma(p_1) = 2$, $\sigma(p_4) = 3$, $\sigma(p_6) = 4$, $\sigma(p_3) = 5$ and $\sigma(p_5) = 6$. Then the equivalence classes of the equivalence relation "$\sim$" in (2.8) on the set of linear extensions of the poset P are

$$\begin{aligned}&\{123456, 123465, 124356, 124365,\\ &\qquad 213456, 213465, 214356, 214365\},\\ &\{123546, 213546\},\\ &\{124635, 214635\},\\ &\{132546, 132456\},\\ &\{132465\},\\ &\{241635, 241365\},\\ &\{246135\}, \quad \text{and}\\ &\{241356\}.\end{aligned}$$

Hence the Cohen-Macaulay type $\mathrm{type}(k[\Delta(L)])$ of the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of the distributive lattice $L = \mathscr{J}(P)$ is equal to eight. Note that the h-vector of $k[\Delta(L)]$ is $h(k[\Delta(L)]) = (1, 8, 9, 1)$.

It might be of interest to find a "nice" formula to compute the number of distinct equivalence classes of the equivalence relation " $\sim$ " in (2.8) on the set of linear extensions of P when P is, e.g., a *rooted tree* [**Sta6**, p. 294]).

We here turn to the problem of finding a chain condition of P for the Stanley-Reisner ring $k[\Delta(L)]$ to be level.

(2.12) The *altitude* of a finite poset P, written as $\mathrm{alt}(P)$, is defined to be the maximal number $\ell \geq 0$ for which there exists a finite sequence $C_0, C_1, \dots, C_r$ of chains in P such that

(i) every $y \in C_j$ is neither less than nor equal to each $x \in C_i$ if $0 \leq i < j \leq r$, and

(ii) the sum of the cardinalities of C_i's is $\ell + r + 1$.

Obviously, we have $\mathrm{rank}(P) \leq \mathrm{alt}(P)$.

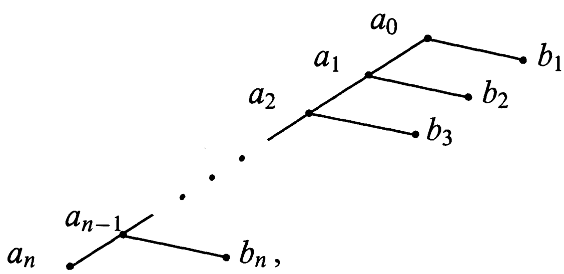
(2.13) Lemma. $\rho(P; \mathrm{alt}(P)) \neq 0$.

Proof. Work in the same notation as in (2.12) with $\ell = \mathrm{alt}(P)$. We write Q for the subposet $C_0 \cup C_1 \cup \cdots \cup C_r$ of P. Then we have $\mathrm{alt}(P) = \mathrm{alt}(Q)$. On the other hand, there exists a unique $\tau \in \mathscr{B}(Q; \mathrm{alt}(Q))$ such that $\tau(\alpha) \leq \tau(\beta)$ if $\alpha \in C_i$ and $\beta \in C_j$ with $0 \leq i < j \leq r$. Let I_i, $0 \leq i \leq \mathrm{alt}(P)$, be the poset ideal of P which consists of those elements $x \in P$ such that $x < \alpha$ for some $\alpha \in Q$ with $\tau(\alpha) \leq i$. In particular $I_0 = \varnothing$. Also, we set $I_{\mathrm{alt}(P)+1} = P$. Then, the chain $\varnothing = I_0 \subsetneqq I_1 \subsetneqq \cdots \subsetneqq I_{\mathrm{alt}(P)+1} = P$ in the distributive lattice $\mathscr{J}(P)$ satisfies the conditions (i) and (ii) in (2.2). Thus $\rho(P; \mathrm{alt}(P)) \neq 0$ as desired. □

Hence, we have $\rho(P; \ell) = 0$ if either $\ell < \mathrm{rank}(P)$ or $\ell > \mathrm{alt}(P)$ and $\rho(P; \mathrm{rank}(P)) \neq 0$, $\rho(P; \mathrm{alt}(P)) \neq 0$. Thus, thanks to Corollary (2.5), we immediately obtain

(2.14) COROLLARY. *The Stanley-Reisner ring* $k[\Delta(L)]$ *of the order complex* $\Delta(L)$ *of a finite distributive lattice* $L = \mathscr{J}(P)$ *is level if and only if* $\mathrm{rank}(P) = \mathrm{alt}(P)$.

(2.15) EXAMPLE. If C_n is the following finite poset



then the Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of the finite distributive lattice $L = \mathscr{J}(C_n)$ is level with the Cohen-Macaulay type $\mathrm{type}(k[\Delta(L)]) = n!$.

(2.16) Recall that the *height* (resp. *depth*) $\mathrm{height}_P(\alpha)$ (resp. $\mathrm{depth}_P(\alpha)$) of an element α of a finite poset P is the maximal number $\ell \geq 0$ for which there exists a chain in P of the form $\alpha_\ell < \alpha_{\ell-1} < \cdots < \alpha_0 = \alpha$ (resp. $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_\ell$). Thus we have $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\alpha) \leq \mathrm{rank}(P)$ for every element $\alpha \in P$. On the other hand, if α and β are incomparable elements of P, then $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\beta) \leq \mathrm{alt}(P)$. We write $P^{(+)}$ for the subposet of P which consists of all elements $\alpha \in P$ with $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\alpha) = \mathrm{rank}(P)$.

(2.17) COROLLARY. *Suppose that the Stanley-Reisner ring* $k[\Delta(L)]$ *of the order complex* $\Delta(L)$ *of a finite distributive lattice* $L = \mathscr{J}(P)$ *is level. If* α *and* β *are incomparable elements of the poset* P, *then we have the inequality* $\mathrm{height}_P(\alpha) + \mathrm{depth}_P(\beta) \leq \mathrm{rank}(P)$. *Thus, in particular, the subposet* $P^{(+)}$ *of* P *is the* ordinal sum **[Sta6**, p. 100] *of clutters.*

(2.18) We say that a finite poset P satisfies the *λ-chain condition* **[Sta6**, p. 219] if $P = P^{(+)}$. It is known, e.g., [**Sta6**, Corollary (4.5.17)] that a poset P satisfies the λ-chain condition if and only if the last non-zero component of the h-vector of the order complex $\Delta(L)$ of the distributive lattice $L = \mathcal{J}(P)$ is equal to one.

(2.19) COROLLARY. *The Stanley-Reisner ring $k[\Delta(L)]$ of the order complex $\Delta(L)$ of a finite distributive lattice $L = \mathcal{J}(P)$ is Gorenstein, i.e.,* $\mathrm{type}(k[\Delta(L)]) = 1$, *if and only if the poset P is the ordinal sum of clutters.*

REFERENCES

[Bac] K. Baclawski, *Cohen-Macaulay connectivity and geometric lattices*, European J. Combin., **3** (1982), 293–305.

[B-G-S] A. Björner, A. Garsia and R. Stanley, *An introduction to Cohen-Macaulay partially ordered sets*, in *Ordered Sets* (I. Rival, ed.), Reidel, Dordrecht/Boston, (1982), 583–615.

[B-K] A. Björner and G. Kalai, *On f-vectors and homology*, in *Combinatorial Mathematics*, Annals of the New York Academy of Sciences, **555** (1989), 63–80.

[Dow] T. A. Dowling, *On the independent set numbers of a finite matroid*, Ann. Discrete Math., **8** (1980), 21–28.

[G-M-R] A. Geramita, P. Maroscia and L. Roberts, *The Hilbert function of a reduced k-algebra*, J. London Math. Soc., (2) **28** (1983), 443–452.

[Her] I. N. Herstein, *Topics in Algebra*, Blaisdell, New York/Toronto/London, 1964.

[H-K] J. Herzog and E. Kunz, *Der Kanonische Modul eines Cohen-Macaulay-Rings*, Lect. Notes in Math., No. 238, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[H1] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in *Commutative Algebra and Combinatorics* (M. Nagata and H. Matsumura, eds.), Advanced Studies in Pure Math., Vol. 11, North-Holland, Amsterdam, (1987), 93–109.

[H2] ——, *Level rings and algebras with straightening laws*, J. Algebra, **117** (1988), 343–362.

[H3] ——, *Canonical ideals of Cohen-Macaulay partially ordered sets*, Nagoya Math. J., **112** (1988), 1–24.

[H4] ——, *What can be said about pure 0-sequences?*, J. Combinatorial Theory Ser. A, **50** (1989), 319–322.

[H5] ——, *Flawless 0-sequences and Hilbert functions of Cohen-Macaulay integral domains*, J. Pure and Appl. Algebra, **60** (1989), 245–251.

[H6] ——, *Hilbert functions of Cohen-Macaulay integral domains and chain conditions of finite partially ordered sets*, J. Pure and Appl. Algebra, **72** (1991), 265–273.

[H7] ——, *Cohen-Macaulay type of the face poset of a plane graph*, to appear.

[Hoc] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, in *Ring Theory* II (B. R. McDonald and R. Morris, eds.), Lect. Notes in Pure and Appl. Math., No. 26, Dekker, New York, (1977), 171–223.

[Mah] C. Mahoney, *On the unimodality of the independent set numbers of a class of matroids*, J. Combinatorial Theory, Ser. B, **39** (1985), 77–85.

[Mas] J. H. Mason, *Matroids: Unimodal conjectures and Motzkin's theorem*, in *Combinatorics* (D. J. A. Welsh and D. R. Woodall, eds.), Inst. Math. Appl., Southend-on-Sea, England, (1972), 207–220.

[Mat] H. Matsumura, *Commutative Ring Theory* (translated by M. Reid), Cambridge Univ. Press, Cambridge, 1986.

[Rei] G. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Advances in Math., **21** (1976), 30–49.

[R-R] L. Roberts and M. Roitman, *On Hilbert functions of reduced and of integral algebras*, J. Pure and Appl. Algebra, **56** (1989), 85–104.

[Sta1] R. Stanley, *The upper bound conjecture and Cohen-Macaulay rings*, Stud. Appl. Math., **54** (1975), 135–142.

[Sta2] ——, *Cohen-Macaulay complexes*, in *Higher Combinatorics* (M. Aigner, ed.), NATO Advanced Study Institute Series, Reidel, Dordrecht/Boston, (1977), 51–62.

[Sta3] ——, *Hilbert functions of graded algebras*, Advances in Math., **28** (1978), 57–83.

[Sta4] ——, *Combinatorics and Commutative Algebra*, Birkhäuser, Boston/Basel/Stuttgart, 1983.

[Sta5] ——, *Two poset polytopes*, Discrete Comput. Geom., **1** (1986), 9–23.

[Sta6] ——, *Enumerative Combinatorics, Volume* I, Wadsworth & Brooks/Cole, Monterey, Calif., 1986.

[Sta7] ——, *On the Hilbert function of a graded Cohen-Macaulay domain*, J. Pure Appl. Algebra, **73** (1991), 307–314.

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