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ON THE RANGE OF AN UNBOUNDED PARTLY ATOMIC VECTOR-VALUED MEASURE

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A well-known theorem of Liapunov states that the range of a bounded, non-atomic, finite-dimensional, vector-valued measure is closed and convex. In this paper we study the range of an unbounded finite-dimensional vector-valued measure that is at least partly atomic. In the one-dimensional case we show that if the range is dense in an interval $[0, a]$ for some $a > 0$ then it contains $[0, a]$. In the general case of arbitrary dimension d we shall use the following notation. If $e_1, \ldots, e_d$ are linearly independent vectors in $\mathbb{R}^d$ let C° denote the interior of the convex cone $C = \{a_1e_1 + \cdots + a_de_d : a_1, \ldots, a_d \geq 0\}$. Then, if $x = a_1e_1 + \cdots + a_de_d$ and $y = b_1e_1 + \cdots + b_de_d$ are in C, $x < y$ and $x \leq y$ shall mean that $a_k < b_k$ and that $a_k \leq b_k$, respectively, for $k = 1, \ldots, d$. Finally, if $a \in C^\circ$ define $(0, a) = \{x \in C^\circ : 0 < x < a\}$ and $(0, a] = \{x \in C^\circ : 0 < x \leq a\}$. Now let μ be a measure such that any bounded subset of its range is in C, and such that the set of all $\mu(E)$ in C such that E contains no atom is bounded. We show that if R_μ is dense in $(0, a]$ for some $a \in C^\circ$ then it contains $(0, a]$.

A well-known theorem of Liapunov [1] states that the range of a bounded, non-atomic, finite-dimensional vector measure is closed and convex. In the case of an unbounded measure the range remains convex but need not be closed. Additional properties of the range in this case have been studied by C. Olech [2]. When a measure is at least partly atomic, one cannot hope to extend these results even in the one-dimensional case. There is, however, an extension when the range of the measure is dense, and this result appears to be non-trivial for an unbounded measure. Our proof will include the easy case when the measure is bounded.

Furthermore, a typical argument in showing that a linear operator has an inverse is to show first that it has a dense range and then conclude, via a closed range theorem, that its range is closed. It is natural to ask to what extent this type of argument holds for not necessarily linear but simply additive mappings. A measure is one of the fundamental structures in mathematics. It is not linear but it is additive on disjoint subsets. The results in this paper give conditions under which we can conclude that if the range of a finite-dimensional vector measure is dense then it is closed.

We consider first the one-dimensional case. Let $(S, \mathscr{M}, \mu)$ be a positive measure, that is, a space S, a σ-algebra $\mathscr{M}$ of subsets of S and a countably additive function $\mu\colon \mathscr{M} \to [0, \infty]$, and let R_μ denote its range. Notice that we allow R_μ to contain ∞.

THEOREM 1. *Let μ be a positive measure. If R_μ is dense in $[0, a]$ for some $a > 0$ then it contains $[0, a]$.*

Proof. It is enough to prove it for a σ-finite measure. Otherwise, consider a countable collection of elements of $\mathscr{M}$ whose measures are dense in $[0, a]$. These elements generate a σ-finite measure, and if its range contains $[0, a]$ so does R_μ.

Now, let R_A and R_N denote the ranges of the atomic and non-atomic components of μ, respectively, and for any $x \in (0, a]$ define

$$\mathscr{A}_x = \{A \in \mathscr{M} : A \text{ is an atom and } \mu(A) \in (0, x)\},$$

and let $x_0 \in (0, a]$. We shall assume that $x_0 \notin R_\mu$ and arrive at a contradiction. Consider first the case in which $\mathscr{A}_{x_0}$ is *not μ-summable,* which we define to mean that the measures of its elements do not have a finite sum. Then there is a positive integer p_1 and disjoint atoms $A_1^1, \dots, A_{p_1+1}^1 \in \mathscr{A}_{x_0}$ with $\mu(A_{p_1+1}^1) \le \cdots \le \mu(A_1^1)$ such that

$$\tfrac{1}{2}x_0 \le \sum_{i=1}^{p_1} \mu(A_i^1) < x_0 < \sum_{i=1}^{p_1+1} \mu(A_i^1)$$

and

$$0 < x_1 \overset{\text{def}}{=} x_0 - \sum_{i=1}^{p_1} \mu(A_i^1) \le \tfrac{1}{2}x_0.$$

Notice that $x_1 < \mu(A_i^1)$ for $i = 1, \dots, p_1$. If $\mathscr{A}_{x_1}$ is not μ-summable, we can restart the process with x_1 in place of x_0, and this can be done again if possible. At the nth stage, $n \ge 1$, we have a positive integer p_n, and disjoint atoms $A_1^n, \dots, A_{p_n+1}^n \in \mathscr{A}_{x_{n-1}}$, which are automatically disjoint from those chosen at any previous stage, such that $\mu(A_{p_n+1}^n) \le \cdots \le \mu(A_1^n)$,

$$\tfrac{1}{2}x_{n-1} \le \sum_{i=1}^{p_n} \mu(A_i^n) < x_{n-1} < \sum_{i=1}^{p_n+1} \mu(A_i^n),$$

$$0 < x_n \overset{\text{def}}{=} x_{n-1} - \sum_{i=1}^{p_n} \mu(A_i^n) = x_0 - \sum_{j=1}^{n} \sum_{i=1}^{p_j} \mu(A_i^j) \le \tfrac{1}{2}x_{n-1},$$

and $x_n < \mu(A_i^j)$ for $i = 1, \ldots, p_n$ and $j = 1, \ldots, n$. If this process can be repeated indefinitely then the x_n form an infinite sequence that, by construction, converges to zero, and then

$$x_0 = \sum_{j=1}^{\infty} \sum_{i=1}^{p_j} \mu(A_i^j) = \mu\left(\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{p_j} A_i^j\right) \in R_A \subset R_\mu,$$

contradicting the original assumption.

Now, if $\mathscr{A}_{x_n}$ is μ-summable for some $n \geq 0$, then the restriction of R_A to $[0, x_n]$ is a closed set and, since the restriction of R_N to $[0, x_n]$ is also closed, it follows that the restriction of R_μ to $[0, x_n]$, which is the sum of the two previous restrictions restricted to $[0, x_n]$, is a closed set. Since it is also dense, there are sets A and E in $\mathscr{M}$ such that A is either empty or a union of atoms disjoint from the A_i^j, E contains no atom, and $x_n = \mu(A) + \mu(E)$. Therefore,

$$\begin{aligned} x_0 &= \mu(A) + \mu(E) + \sum_{j=1}^{n} \sum_{i=1}^{p_j} \mu(A_i^j) \\ &= \mu\left[A \cup E \cup \left(\bigcup_{j=1}^{n} \bigcup_{i=1}^{p_j} A_i^j\right)\right] \in R_\mu, \end{aligned}$$

contradicting, again, the original assumption. □

It should be remarked that this theorem does not hold if the measure is not positive. For if μ is purely atomic and the measures of its atoms are -1 and all the rational numbers in $[1, \infty)$ then its range is dense in $[0, 1]$ but does not contain $[0, 1]$.

To deal with the general case of arbitrary dimension d we develop some additional notation and terminology. If $e_1, \ldots, e_d$ are linearly independent vectors in $\mathbb{R}^d$ consider the convex cone

$$C = \{x = a_1 e_1 + \cdots + a_d e_d : a_1, \ldots, a_d \geq 0\},$$

let C° denote the interior of C, and if $x = a_1 e_1 + \cdots + a_d e_d$ and $y = b_1 e_1 + \cdots + b_d e_d$ are in C, $x < y$ and $x \leq y$ shall mean that $a_k < b_k$ and that $a_k \leq b_k$, respectively, for $k = 1, \ldots, d$. If $a, b \in C$ and $a < b$ define

$$(a, b) = \{x \in C^\circ : a < x < b\}$$

and

$$(a, b] = \{x \in C^\circ : a < x \leq b\}.$$

Now, let $(S, \mathscr{M}, \mu)$ be a vector-valued measure in C, that is, a space S, a σ-algebra $\mathscr{M}$ of subsets of S and a countably additive function μ defined on $\mathscr{M}$ and taking values in C or infinity. Infinity is defined as any linear combination of the e_k in which at least one coefficient is $+\infty$. Let R_μ denote the range of μ.

THEOREM 2. *Let μ be a vector-valued measure in C such that the set of all $\mu(E) \in C$ such that E contains no atom is bounded. If R_μ is dense in $(0, a]$ for some $a \in C^\circ$ then it contains $(0, a]$.*

Proof. As in the one-dimensional case, it is enough to prove it for a σ-finite measure, and again we denote by R_A and R_N the ranges of its atomic and non-atomic components. For simplicity we carry out the proof in the two-dimensional case, and then indicate how it is to be modified in the general case.

If $x = a_1e_1 + a_2e_2$ is in C, let $0x$ denote the segment $\{tx: 0 < t \leq 1\}$ and, for $k = 1, 2$, let x^k and C_{kx} denote the vector a_ke_k and the closed triangle with vertices $0, x$ and x^k, respectively. Finally, define

$$\mathscr{A}_{kx} = \{A \in \mathscr{M}: A \text{ is an atom and } \mu(A) \in C_{kx} \cap (0, x)\}$$

and define $y \in C$ to be a *μ-cluster point* for $\mathscr{A}_{kx}$ if for every open disc $D \subset \mathbb{R}^2$ centered at y there are infinitely many atoms in $\mathscr{A}_{kx}$ whose measures are in $D \cap C$.

Now, let $x_0 \in (0, a]$ be arbitrary and assume first that for any $x \in 0x_0$ the sets $\mathscr{A}_{1x}$ and $\mathscr{A}_{2x}$ are not μ-summable. Then there is a positive integer p_1 and disjoint atoms $A_1^1, \dots, A_{p_1}^1$ such that

$$\tfrac{1}{2}x_0 \leq \sum_{i=1}^{p_1} \mu(A_i^1) < x_0$$

and

$$0 < x_1 \overset{\text{def}}{=} x_0 = \sum_{i=1}^{p_1} \mu(A_i^1) \leq \tfrac{1}{2}x_0 .$$

If for any $x \in 0x_1$ the sets $\mathscr{A}_{1x}$ and $\mathscr{A}_{2x}$ are not μ-summable, we can restart the process with x_1 in place of x_0, and this can be done again if possible. At the nth stage, $n \geq 1$, we have a positive integer p_n and disjoint atoms $A_1^n, \dots, A_{p_n}^n$, which are disjoint from those chosen at any previous stage, such that

$$\tfrac{1}{2}x_{n-1} \leq \sum_{i=1}^{p_n} \mu(A_i^n) < x_{n-1}$$

and

$$0 < x_n \overset{\text{def}}{=} x_{n-1} - \sum_{i=1}^{p_n} \mu(A_i^n) = x_0 - \sum_{j=1}^{n} \sum_{i=1}^{p_j} \mu(A_i^j) \le \tfrac{1}{2} x_{n-1}.$$

If this process can be repeated indefinitely then the x_n form an infinite sequence that, by construction, converges to the origin, and then

$$x_0 = \sum_{j=1}^{\infty} \sum_{i=1}^{p_j} \mu(A_i^j) = \mu\left(\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{p_j} A_i^j \right) \in R_A \subset R_\mu.$$

If, on the other hand, there is an $n \ge 0$, a point $x \in 0x_n$ and a value of k, which we assume to be $k = 1$, such that $\mathscr{A}_{1x}$ is μ-summable, let x_c denote either x_n if $\mathscr{A}_{1x_n}$ is μ-summable or, otherwise, the closest point to the origin in $0x_n$ with the property that the closed segment from x_c to x_c^1 contains a μ-cluster point for $\mathscr{A}_{1x_n}$. In the second case there is a positive integer q_0 and disjoint atoms $B_1^0, \ldots, B_{q_0}^0 \in \mathscr{A}_{1x_n}$, which are disjoint from those chosen at any previous stage, such that if

$$y_0 \overset{\text{def}}{=} x_n - \sum_{i=1}^{q_0} \mu(B_i^0)$$

then $y_0^1 \in 0x_c^1$ and, in addition, $y_0^1 < x_c^1$ if $\mathscr{A}_{1x_c}$ is not μ-summable. In the first case, when $\mathscr{A}_{1x_n}$ is μ-summable, define $y_0 = x_c = x_n$ and take B_i^0 to be the empty set for each i so that the equation above remains valid. Then choose $x \in 0x_c$ such that $\mathscr{A}_{1x}$ is μ-summable and $y_0^1 \in 0x^1$, and notice that the set Σ_1 of all sums of measures of elements of $\mathscr{A}_{1x}$ is closed and disjoint from $0x^1$.

Now, if $\mathscr{A}_{2y_0}$ is not μ-summable, then there is a positive integer q_1 and disjoint atoms $B_1^1, \ldots, B_{q_1}^1 \in \mathscr{A}_{2y_0}$, which are disjoint from those chosen at any previous stage, such that

$$0 < y_1 \overset{\text{def}}{=} y_0 - \sum_{i=1}^{q_1} \mu(B_i^1)$$

and $y_1^2 \le y_0^2/2$. This process can now be repeated as many times as possible. At the mth stage, $m \ge 1$, we have a positive integer q_m and disjoint atoms $B_1^m, \ldots, B_{q_m}^m \in \mathscr{A}_{2y_{m-1}}$, which are disjoint from those chosen at any previous stage, such that

$$0 < y_m \overset{\text{def}}{=} y_{m-1} - \sum_{i=1}^{q_m} \mu(B_i^m)$$

and $y_m^2 \leq y_{m-1}^2/2$. If this process can be repeated indefinitely then the y_m form a sequence that, by construction, converges to some point $s_1 \in \{0\} \cup 0x^1$. Next we claim that the restriction of R_μ to $0x^1$ is dense in $0x^1$. In fact, every point $s \in 0x^1$ is the limit of a sequence $\{s_k\}$ in $(0, a]$ where each s_k is the sum of four elements: $s_{k1} \in \{0\} \cup \Sigma_1$, $s_{k2} \in C_{2x}$, $s_{k3} \in R_N$ and s_{k4}, which is either 0 or is in the restriction of R_A to $0x^1$. But $s_{k1} \to 0$ and $s_{k2} \to 0$ as $k \to \infty$, and, by considering a subsequence if necessary, s_{k3} converges to some $s_N \in R_N$, since this last set is closed by Liapunov's theorem. Notice that $s_N \in \{0\} \cup 0x^1$, and then $s_N + s_{k4}$ is arbitrarily close to s for k large enough. This proves the claim and then, by Theorem 1, the restriction of R_μ to $0x^1$ contains $0x^1$. It follows that $s_1 = \mu(E_1)$ where $E_1 \in \mathscr{M}$ is disjoint from any of the atoms chosen above. Therefore,

$$\begin{aligned} x_0 &= x_n + \sum_{j=1}^{n} \sum_{i=1}^{p_j} \mu(A_i^j) \\ &= \mu(E_1) + \sum_{m=0}^{\infty} \sum_{i=1}^{q_m} \mu(B_i^m) + \sum_{j=1}^{n} \sum_{i=1}^{p_j} \mu(A_i^j) \\ &= \mu \left[E_1 \cup \left(\bigcup_{m=0}^{\infty} \bigcup_{i=1}^{q_m} B_i^m \right) \cup \left(\bigcup_{j=1}^{n} \bigcup_{i=1}^{p_j} A_i^j \right) \right] \in R_\mu . \end{aligned}$$

If the process described above cannot be repeated indefinitely, then there is an $m \geq 0$ such that, $\mathscr{A}_{2y_m}$ is μ-summable. Then the set Σ_2 of all sums of elements of $\mathscr{A}_{2y_m}$ is closed, and an argument like the one above applied to y_m^1 and y_m^2 shows that $y_m = \mu(E_1) + \mu(E_2)$, where E_1 and E_2 are disjoint from each other and from any previously chosen atoms. In this case

$$\begin{aligned} x_0 &= x_n + \sum_{j=1}^{n} \sum_{i=1}^{p_j} \mu(A_i^j) \\ &= \mu(E_1) + \mu(E_2) + \sum_{j=0}^{m} \sum_{i=1}^{q_m} \mu(B_i^m) + \sum_{j=1}^{n} \sum_{i=1}^{p_j} \mu(A_i^j) \\ &= \mu \left[E_1 \cup E_2 \cup \left(\bigcup_{j=0}^{m} \bigcup_{i=1}^{q_m} B_i^m \right) \cup \left(\bigcup_{j=1}^{n} \bigcup_{i=1}^{p_j} A_i^j \right) \right] \in R_\mu . \end{aligned}$$

This completes the proof in the two-dimensional case.

The same proof is valid in the general case with the following changes. Let H_{kx} denote the hyperplane of codimension one consisting of all points of the form $x^k + \sum_{j \neq k} a_j e_j$, $a_j \in \mathbb{R}$, and let C_{kx} denote now the closed convex hull of $\{0\} \cup (H_{kx} \cap (0, x])$. Then the segment from x_c to x_c^1 above is replaced with the set of all points in H_{1x_c} that are in the closure of C_{1x_c}. With these changes the previous proof is valid until the definition of y_0. The y_m are also defined as above, if $\mathscr{A}_{ky_{m-1}}$ is not μ-summable for each $k \geq 2$, but the B_i^m are chosen from $\bigcup_{k=2}^{d} \mathscr{A}_{ky_{m-1}}$ and so that $y_m^k \leq y_{m-1}^k/2$ for $k \geq 2$. If the y_m form an infinite sequence that converges to a point $s_1 \in \{0\} \cup 0x^1$, the previous proof is still valid. Otherwise, one or more of the $\mathscr{A}_{ky_m}$ become μ-summable along the way. It should be clear how the components of y_m that do not become zero are in R_μ and how this can be done using sets disjoint from all others. □

It is not possible to assert that R_μ contains $[0, a]$, for if μ is purely atomic and the measures of its atoms are the points with positive rational coordinates in $(0, 1)$ then R_μ does not contain any point on the coordinate axes except the origin.

References

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