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ON A PLANCHEREL FORMULA FOR CERTAIN DISCRETE, FINITELY GENERATED, TORSION-FREE NILPOTENT GROUPS

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We prove a Plancherel formula for elementarily exponentiable, discrete, finitely generated, torsion-free nilpotent groups.

1. Introduction. Let Γ be a discrete, finitely generated, torsion-free, nilpotent Lie group. If $\Gamma^{(k)}$ denotes its descending central series, we will call Γ n-step nilpotent if $\Gamma^{(n)} \neq \{1\}$ but $\Gamma^{(n+1)} = \{1\}$. Malcev has shown that any Γ of this type may be embedded as a discrete cocompact subgroup of a simply connected, connected nilpotent Lie group (see [**1**], Chapter 1); thus we may utilize all that is known about uniform subgroups of these groups, which is summarized beautifully in ([**3**], Secs. 5.1 and 5.2).

This work extends the pioneering work of R. Howe on the representation theory of groups of this type, and uses much of the machinery he developed (see [**5**]). The techniques used to prove the Plancherel formula for Γ are essentially those used in [**4**] to prove a similar Plancherel formula for discrete groups which are the rational points of a nilpotent Lie group. The corresponding result for Γ follows easily once we observe that, for a certain type of character $\{\tilde{\lambda}\}$ in the Pontryagin dual of the center of Γ, the Γ-orbit of any extension λ of $\tilde{\lambda}$ to a character on the Lie algebra $\mathcal{L}$ of Γ is dense in the set $\lambda + z(\mathcal{L})^{\perp} \subseteq \widehat{\mathcal{L}}$ (Proposition 2.5).

Let $\mathcal{L}_{\mathbb{R}}$ be a real finite-dimensional r-step nilpotent Lie algebra, and let $\mathcal{L} \subseteq \mathcal{L}_{\mathbb{R}}$ be a discrete additive subgroup of $\mathcal{L}_{\mathbb{R}}$. A calculation with the Campbell-Baker-Hausdorff formula shows that if $\mathcal{L}$ is

1. an additive discrete subgroup of $\mathcal{L}_{\mathbb{R}}$, not necessarily of cofinite volume, and
2. $\mathcal{L}$ satisfies $[\mathcal{L}, \mathcal{L}] \subseteq r!\mathcal{L}$,

then $\Gamma = \exp \mathcal{L}$ forms a discrete subgroup of the connected, simply connected nilpotent Lie group $N = \exp \mathcal{L}_{\mathbb{R}}$. If $\mathcal{L}$ satisfies condition 1, we will refer to $\mathcal{L}$ as a *lattice*. If $\mathcal{L}$ satisfies both conditions we will say that $\mathcal{L}$, and $\Gamma = \exp \mathcal{L}$, are *elementarily exponentiable*, or *e.e.* for short. If $\mathcal{L}$ is *e.e.*, and i is an *e.e.* lattice contained in $\mathcal{L}$ which is closed under the bracket operation, we will call i an *ideal* of $\mathcal{L}$ if i is $\mathrm{Ad}^\star(\mathcal{L})$−invariant. Note that for i to be *e.e.*, we must have $[i, i] \subseteq r!i$, where r is the length of $\mathcal{L}_{\mathbb{R}}$. We assume throughout this paper that the Γ under consideration are *e.e.*.

We define the rising central series of ideals of $\mathcal{L}$ as follows; set $\mathcal{L}_{(0)} = (0)$, and define $\pi_n \colon \mathcal{L} \longrightarrow \mathcal{L} \backslash \mathcal{L}_{(n-1)}$ to be the natural quotient map for each $n \geq 1$. Then $\mathcal{L}_{(n)}$ is defined to be the preimage under π_n of the center of $\mathcal{L} \backslash \mathcal{L}_{(n-1)}$. Thus $\mathcal{L}_{(n)}$ is an increasing sequence of ideals of $\mathcal{L}$, satisfying $\mathcal{L}_{(r)} = \mathcal{L}$, since $\mathcal{L}$ is r-step nilpotent.

We show in what follows that the $L_{(n)}$ are *e.e.*.

LEMMA 1.1. *$\mathcal{L}_{(n)} = \mathcal{L} \cap (\mathcal{L}_{\mathbb{R}})_{(n)}$, for $n \in 0, ..., r$.*

Proof. For $n = 0$, the result is trivial.

Suppose now that $\mathcal{L}_{(k)} = \mathcal{L} \cap (\mathcal{L}_{\mathbb{R}})_{(k)}$. By definition, $\mathcal{L}_{(k+1)} = \pi_k^{-1}(z(\mathcal{L} \backslash \mathcal{L}_{(k)}))$. We show:

(a). $\mathcal{L}_{(k+1)} \subseteq \mathcal{L} \cap (\mathcal{L}_{\mathbb{R}})_{(k+1)}$.

For if $X \in \mathcal{L}_{(k+1)}$, $[X, Y] = Z \in \mathcal{L}_{(k)}$ for all $Y \in \mathcal{L}$. Let $\overline{\mathcal{L}}$ be the image of $\mathcal{L}$ in $\mathcal{L}_{\mathbb{R}} \backslash \mathcal{L}_{\mathbb{R}(k)}$, and let $\overline{X}$ be the image of X. It follows from Theorems 5.1.4 and 5.2.3 in [**3**] that $\overline{\mathcal{L}}$ is uniform in $\mathcal{L}_{\mathbb{R}} \backslash \mathcal{L}_{\mathbb{R}(k)}$. For $X \in \mathcal{L}_{(k+1)}$, $\overline{X}$ commutes with all $\overline{Y} \in \overline{L}$. We apply Theorem 5.1.5 from [**3**] to see that $\overline{X} \in z(\mathcal{L}_{\mathbb{R}} \backslash \mathcal{L}_{\mathbb{R}(k)})$. It follows that $X \in \mathcal{L}_{\mathbb{R}(k+1)}$.

(b). $\mathcal{L} \cap \mathcal{L}_{\mathbb{R}k+1} \subseteq \mathcal{L}_{(k+1)}$.

If $X \in \mathcal{L} \cap \mathcal{L}_{\mathbb{R}(k+1)}$, then $[X, Y] = Z \in \mathcal{L} \cap \mathcal{L}_{\mathbb{R}(k)} = \mathcal{L}_{(k)}$. This completes the proof of the lemma. □

It now follows easily that $\mathcal{L}_{(k)}$ is *e.e.* for all k; suppose $Z = [X, Y] \in [\mathcal{L}_{(k)}, \mathcal{L}_{(k)}]$. Then since $\mathcal{L}$ is *e.e*, $Z = r!X$ for some $X \in \mathcal{L}$. By definition, $[\mathcal{L}_{(k)}, \mathcal{L}_{(k)}] \subseteq [\mathcal{L}_{(k)}, \mathcal{L}] \subseteq \mathcal{L}_{(k-1)} \subseteq \mathcal{L}_{(k)} = \mathcal{L}_{\mathbb{R}(k)} \cap \mathcal{L}$; so $Z \in \mathcal{L}_{\mathbb{R}(k)}$, hence $X \in \mathcal{L}_{\mathbb{R}(k)}$. Thus $X \in \mathcal{L}_{(k)}$, and so $Z \in r!\mathcal{L}_{(k)}$. Therefore, $\mathcal{L}_{(k)}$ is *e.e.*.

We will refer to a subgroup $\Gamma' \subseteq \Gamma$ (or a subalgebra $\mathcal{L}' \subseteq \mathcal{L}$) as *saturated* if $x^n \in \Gamma'$ implies $x \in \Gamma'$ ($kX \in \mathcal{L}'$ implies $X \in \mathcal{L}'$).

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2. Generic coadjoint orbits. Let $\mathcal{L}$ be an *e.e.*lattice in a real nilpotent Lie algebra $\mathcal{L}_{\mathbb{R}}$, so that $\Gamma = \exp(\mathcal{L}_{\mathbb{R}})$ is uniform in $N = \exp(\mathcal{L}_{\mathbb{R}})$. We assume throughout this section that a strong Malcev basis $\{X_1, X_2, ..., X_n\}$ for $\mathcal{L}_{\mathbb{R}}$ through its ascending central series has been chosen so that it satisfies:

1. The $\mathbb{Z}$-span of $\{X_1, ..., X_n\}$ is the lattice $\mathcal{L}$ in $\mathcal{L}_{\mathbb{R}}$:

2. The real span of $\{X_1, ..., X_i\}$ forms an ideal of $\mathcal{L}_{\mathbb{R}}$, and in particular the real span of $\{X_1, ..., X_k\}$ is the center of $\mathcal{L}_{\mathbb{R}}$ (we will also regard $\mathcal{L}_{\mathbb{R}}$ as having the inner product defined by setting $< X_i, X_j >= \delta_{i,j}$);

3. $\Gamma = \exp(\mathbb{Z}X_1)\exp(\mathbb{Z}X_2)...\exp(\mathbb{Z}X_n)$. Thus we may coordinatize Γ as follows: $\gamma \leftrightarrow (x_1, ..., x_n) \in \mathbb{Z}^n$ whenever $\gamma = \exp x_1X_1 \cdot ... \cdot \exp x_nX_n$.

That a basis for $\mathcal{L}_{\mathbb{R}}$ may be chosen satisfying all these conditions is shown in Section 5.1 of [**3**], and as part of the proof of Proposition 5.4.11 in [**3**].

We think of $\mathcal{L}$ as the Lie algebra of Γ; as an abelian group it is isomorphic to $\mathbb{Z}^n$ via the map

$$\begin{aligned} \Psi\colon \mathbb{Z}^n &\longrightarrow \mathcal{L} \\ (a_1, ..., a_n) &\longmapsto a_1X_1 + ... + a_nX_n. \end{aligned}$$

Thus the natural Pontryagin dual of $\mathcal{L}$ is $T^n \cong (\mathbb{R}\backslash\mathbb{Z})^n$ via the map

$$\begin{aligned} \Phi\colon T^n &\longrightarrow \widehat{\mathcal{L}} \\ (\bar{\lambda}_1, ..., \bar{\lambda}_n) &\longmapsto \lambda \end{aligned}$$

where $\lambda(a_1, ..., a_n) = \exp(2\pi i(\lambda_1 a_1 + ... + \lambda_n a_n))$, for any choice $\{\lambda_i\}$ of representatives for the elements $\bar{\lambda}_i$ of $\mathbb{R}\backslash\mathbb{Z}$.

Let $z(\mathcal{L})$ denote the center of the Lie algebra $\mathcal{L}$; if $z(\mathcal{L}_{\mathbb{R}})$ is the center of the real Lie algebra $\mathcal{L}_{\mathbb{R}}$, then $z(\mathcal{L}) = \mathcal{L} \bigcap z(\mathcal{L}_{\mathbb{R}})$ (Lemma 5.1.5 in [**3**]). Hence $z(\mathcal{L})$ is a saturated subalgebra of $\mathcal{L}$.

For a fixed $\lambda \in \widehat{\mathcal{L}}$, let $\mathbf{i}_\lambda$ be the largest ideal of $\mathcal{L}$ contained in the subalgebra $\mathbf{r}_\lambda = \{X \in \mathcal{L}\colon \lambda[X, Y] = 1 \text{ for all } Y \in \mathcal{L}\}$. Then $R_\lambda = \exp(\mathbf{r}_\lambda)$ is the isotropy subgroup of λ under the $\mathrm{Ad}^\star$-action of Γ; in particular it is shown in Lemma 2 of [**5**] that $\mathbf{r}_\lambda$ is an *e.e.*lattice in $\mathcal{L}$. Furthermore, $\mathbf{i}_\lambda = \bigcap_{\gamma\in\Gamma}\mathrm{Ad}(\gamma)\mathbf{r}_\lambda$ is an *e.e.*ideal of $\mathcal{L}$ (the intersection of *e.e.* subalgebras is *e.e.*, and the $\mathrm{Ad}(\gamma)\mathbf{r}_\lambda$ are each *e.e.*, being the images of an *e.e.* subalgebra under a Lie algebra automorphism). It follows that $\exp(\mathbf{i}_\lambda) = I_\lambda \subseteq R_\lambda$ is a normal subgroup of $\mathcal{L}$ which always contains $\exp(z(\mathcal{L})) = z(\Gamma)$. Let $r\colon \widehat{\mathcal{L}} \longrightarrow \widehat{z(\mathcal{L})}$ send an element $\lambda \in \widehat{\mathcal{L}}$ to its restriction to $z(\mathcal{L})$. In what follows, we will show that for all $\tilde{\lambda} \in \widehat{z(\mathcal{L})}$ except those in a set of Haar measure zero, the elements of $r^{-1}(\tilde{\lambda})$ satisfy $\mathbf{i}_\lambda = z(\mathcal{L})$.

LEMMA 2.1. *Suppose $\lambda = \Phi(\bar{\lambda}_1, ..., \bar{\lambda}_n)$ where $\bar{\lambda}_i \in \mathbb{R}\backslash\mathbb{Z}$. Then λ has trivial kernel in $\mathcal{L}$ if and only if for some choice of representatives $\lambda_i \in \mathbb{R}$ of $\bar{\lambda}_i \in \mathbb{R}\backslash\mathbb{Z}$, the set $\{\lambda_1, ..., \lambda_n, 1\}$ is linearly independent over $\mathbb{Q}$.*

Proof. Suppose λ has trivial kernel, and suppose that for some elements $q_1, ..., q_n, q \in \mathbb{Q}$, we have

$$\lambda_1 q_1 + ... + \lambda_n q_n + q = 0$$

for some choice of representatives $\{\lambda_i\}$ for $\{\bar{\lambda}_i\}$. After multiplication of this equation by some integer, we have $\lambda_1 a_1 + ... + \lambda_n a_n + a = 0$ for some integers $\{a_i\}$. Then $\lambda_1 a_1 + ... + \lambda_n a_n = -a$, so $(a_1, ..., a_n) \in$ kernel (λ). Thus $(a_1, ..., a_n) = 0$. It follows immediately that $q_i = 0$ for each $i = 1, ..., n$, and $q = 0$.

Conversely, suppose that λ has a nontrivial kernel, and that $(a_1, ..., a_n) \neq 0$ is an element of the kernel of λ. Let $\{\lambda_i\}$ be any choice of representatives for the elements $\{\bar{\lambda}_i\}$ which determine λ; then $a_1\lambda_1 + ... + a_n\lambda_n = k$ for some element $k \in \mathbb{Z}$. The set $\{\lambda_1, ..., \lambda_n, 1\}$ is thus a linearly dependent set over $\mathbb{Q}$. □

LEMMA 2.2. *Let i be an e.e.ideal of the lattice $\mathcal{L}$. Let $\mathcal{L}_{(n)}$ be the rising central series of $\mathcal{L}$. Suppose $i \supseteq z(\mathcal{L})$, $i \neq z(\mathcal{L})$. Then $i \cap (\mathcal{L}_{(2)} - z(\mathcal{L}))$ is nonempty.*

Proof. Since everything in sight is *e.e.*, and since $I = \exp(i)$ is normal, we prove the result on the group level. Let $\overline{I}$ denote the image of I in $\overline{G} = G\backslash z(\Gamma)$. Then since $\overline{I}$ is a nontrivial normal subgroup of $\overline{G}$, its intersection with the center of $\overline{G}$ is nontrivial. It follows that $I \cap (\Gamma_{(2)} - z(\Gamma))$ is nonempty, and so the result on the algebra level follows. □

LEMMA 2.3. *For* $\lambda \in \widehat{\mathcal{L}}$, $\mathbf{i}_\lambda = z(\mathcal{L})$ *if and only if there exists no element* $X \in \mathcal{L}_{(2)}$ *such that* $\mathrm{ad}(X)$ *maps* $\mathcal{L}$ *into the kernel of* $\tilde{\lambda}$, *where* $\tilde{\lambda} = \lambda|_{z(\mathcal{L})}$.

Proof. Suppose $X \in \mathcal{L}_{(2)} - z(\mathcal{L})$, and $\mathrm{ad}(X)$ maps $\mathcal{L}$ into the kernel of $\tilde{\lambda}$. Take K to be the additive subgroup in $\mathcal{L}$ generated by X and the elements of $z(\mathcal{L})$. Then K is an ideal of $\mathcal{L}$, since $[K, \mathcal{L}] \subseteq z(\mathcal{L})$, and clearly $K \subseteq \mathbf{r}_\lambda$. Therefore $\mathbf{i}_\lambda \neq z(\mathcal{L})$.

Conversely, suppose $\mathbf{i}_\lambda \neq z(\mathcal{L})$. Then $\mathbf{i}_\lambda$ properly contains $z(\mathcal{L})$. By the result of Lemma 2.2, we may choose $X \in \mathcal{L}_{(2)} - z(\mathcal{L})$ such that $X \in \mathbf{i}_\lambda$; then $\mathrm{ad}(X)$ maps $\mathcal{L}$ into the kernel of $\tilde{\lambda}$. This completes the proof of Lemma 2.3. □

Throughout the rest of this paper, λ will denote an element of $\widehat{\mathcal{L}}$, and $\tilde{\lambda}$ will denote its restriction to $z(\mathcal{L})$.

COROLLARY. *If* $\mathbf{i}_\lambda = z(\mathcal{L})$, *then for all* $\phi \in r^{-1}(\tilde{\lambda})$, $i_\phi = z(\mathcal{L})$. *Therefore, if* $\ker \tilde{\lambda} \subseteq z(\mathcal{L})$ *is trivial,* $\mathbf{i}_\lambda = z(\mathcal{L})$.

PROPOSITION 2.4. *For a set* $\{\tilde{\lambda}\} \subseteq \widehat{z(\mathcal{L})}$ *of full Haar measure in* $\widehat{z(\mathcal{L})}$, $\ker \tilde{\lambda}$ *is trivial.*

Proof. Suppose $\tilde{\lambda}$ corresponds to the element $(\bar{\lambda}_1, \ldots, \bar{\lambda}_k) \in T^k$, and that for some set of representatives in $\mathbb{R}$ of the $\bar{\lambda}_i$, the set $\{\lambda_1, \ldots, \lambda_k, 1\}$ is linearly dependent over $\mathbb{Q}$. Then for some set of integers $a_1, \ldots, a_k, a \in \mathbb{Z}$, not all zero, we have that $a_1\lambda_1 + \ldots + a_k\lambda_k = a$. If we choose some other set of representatives for the $\bar{\lambda}_i$, the previous expression changes only by an integral constant.

It follows that the preimage in $\mathbb{R}^k$ of the set of elements $\{\bar{\lambda}_i\}$ satisfying this linear dependence condition consists of the union of hyperplanes of the form

$$a_1x_1 + a_2x_2 + \ldots + a_kx_k = a,$$

where a_i, a vary over the elements of $\mathbb{Z}$. These are of measure zero in $\mathbb{R}^k$ individually, hence their (countable) union is of measure zero. The corresponding set in T^n is therefore of measure zero, and thus the elements $(\bar{\lambda}_1, \ldots, \bar{\lambda}_k) \in T^n$ which satisfy the linear independence condition of Lemma 2.1 are of full measure in T^n. This completes the proof of Proposition 2.4. □

We finish this section by proving

PROPOSITION 2.5. *If ker* $\tilde{\lambda}$ *is trivial, then the closure of* $\mathrm{Ad}^\star(\Gamma)\lambda$ *is* $\lambda + z(\mathcal{L})^\perp$.

Suppose that $\gamma \in \Gamma$, and λ has coordinates $(\lambda_1, ..., \lambda_n) \in T^n$ via the map Φ. Then $\mathrm{Ad}^\star(\gamma) = (\lambda_1, ..., \lambda_k, p_{k+1}(\gamma : \lambda), ..., p_n(\gamma : \lambda)$, where each $p_i(\gamma : \lambda)$ is a polynomial in the coordinates of γ (with respect to the map Ψ), with coefficients from the $\mathbb{Q}$–span of $\{\lambda_1, ..., \lambda_n\}$.

In $\widehat{\mathcal{L}}$, $z(\mathcal{L})^\perp$ consists of all elements of the form $(0, ..., 0, \lambda_{k+1}, ..., \lambda_n)$, for $\lambda_i \in \mathbb{R}\backslash\mathbb{Z}$.

We wish to show that if $\lambda_1, ..., \lambda_k$ satisfy the equivalent conditions of Lemma 2.1 (where $z(\mathcal{L})$ plays the role of $\mathcal{L}$), then the set $\{\mathrm{Ad}^\star(\gamma)\lambda - \lambda \colon \gamma \in \Gamma\}$ is dense in $z(\mathcal{L})^\perp$. In what follows, λ of this type will be called "generic", as will coadjoint orbits of the form $\mathcal{O}_\lambda = \lambda + z(\mathcal{L})^\perp$ where λ is generic.

We may regard the polynomials $p_i(\gamma \colon \lambda)$ as polynomials in $\{x_1, ..., x_n\}$, $x_i \in \mathbb{Z}$, by identifying γ with $(x_1, ..., x_n)$ via Ψ.

LEMMA 2.6. (H. Weyl, [**2**]). *Suppose that* $\{p_i\}_{i=1}^n$ *is a set of polynomials in one integer variable, with coefficients in* $\mathbb{R}\backslash\mathbb{Z}$. *If for each set of integers* $a_1, ..., a_n$, *not all zero, the polynomial* $a_1p_1 + ... + a_np_n$ *has an irrational coefficient, then the set of points* $\{(p_1(k), ..., p_n(k)) \colon k \in \mathbb{Z}\}$ *is dense in* $(\mathbb{R}\backslash\mathbb{Z})^n$.

Now let $T = (t_1, ..., t_s) \in N^s$, and let $X^T = x_1^{t_1} ... x_s^{t_s}$ be a multinomial in s integer variables.

LEMMA 2.7. *Suppose* $\{X^{T_i}\}_{i=1}^r$ *is a set of distinct multinomials in s integer variables. Then there is* $\phi \colon \mathbb{Z} \longrightarrow \mathbb{Z}^s$ *so that* $X^{T_i} \circ \phi$ *are monomials and are distinct.*

Proof. We put a lexicographic order on N^s as follows: let $i \in 1, ..., s$ be the greatest integer with $m_i \neq m_i'$; then $(m_1, ..., m_s) > (m_1', ..., m_n')$ if $m_i > m_i'$. A simple induction argument shows that for the finite set $\{T_i\} \subseteq N^s$ there is $N \in N^s$, $N = (N_1, ..., N_s)$, so that if $T_i > T_j$ in the ordering on N^s, then $N \cdot T_i > N \cdot T_j$ ($N \cdot T$ denotes the usual dot product).

Now define $\phi(x) = (x^{N_1}, ..., x^{N_s})$; then we have $X^{T_i} \circ \phi = x^{T_i \cdot N}$, and so the monomials $X^{T_i} \circ \phi$ remain distinct. □

LEMMA 2.8. *Suppose $\{p_i\}_{i=1}^n$ is a set of polynomials in s integer variables, with coefficients in $\mathbb{R}\backslash\mathbb{Z}$. If for all integers $a_1, ..., a_n$, not all zero, $p = a_1p_1 + ... + a_np_n$ has an irrational coefficient, then the image of $\mathbb{Z}^s$ under $(p_1, ..., p_n)$ is dense in $(\mathbb{R}\backslash\mathbb{Z})^n$.*

Proof. Let $\{X^{T_i}\}$ be the set of monomials which appear in the polynomials p_i, and let ϕ be as in Lemma 2.7. We note that the monomials in $a_1p_1 + ... + a_np_n$ are a subset of the $\{X^{T_i}\}$, so that they remain distinct if composed with ϕ; and so for all $a_1, ..., a_n$, not all zero, $a_1p_1 \circ \phi + ... + a_np_n \circ \phi$ has an irrational coefficient. We invoke Lemma 2.6, and the result follows.

Thus we will have proven Proposition 2.5 if we can show that whenever λ is generic, the polynomial

$$P(x\colon \lambda) = a_{k+1}p_{k+1}(x\colon \lambda) + ... + a_np_n(x\colon \lambda)$$

(where $a_{k+1}, ..., a_n$ are integers, not all zero) has an irrational coefficient.

Assume λ is generic. We begin by writing $\mathrm{Ad}^\star(\gamma) = \mathrm{Ad}^\star(x_1, ..., x_n)$ as a matrix with respect to the coordinates $(\lambda_1, .., \lambda_n)$ of λ given by the map Φ. The condition $[\mathcal{L}, \mathcal{L}] \subseteq r!\mathcal{L}$ implies that the entries of the matrix $\mathrm{Ad}^\star(x_1, ..., x_n)$ are elements of $\mathbb{Z}[x_1, ..., x_n]$.

Therefore $\mathrm{Ad}^\star(\gamma)(\lambda_1, ..., \lambda_k, ..., \lambda_n) = (\lambda_1, ..., \lambda_k, p_{k+1}(\gamma\colon \lambda), ..., p_n(\gamma\colon \lambda))$. $\mathrm{Ad}^\star(\gamma)$ is given by the matrix

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots 0 \\
p_{k+1,1} & p_{k+1,2} & p_{k+1,3} & \cdots & p_{k+1,k} & 1 & 0 & \cdots 0 \\
p_{k+2,1} & p_{k+2,2} & p_{k+2,3} & \cdots & p_{k+2,k} & p_{k+2,k+1} & 1 & \cdots 0 \\
p_{k+3,1} & p_{k+3,2} & p_{k+3,3} & \cdots & p_{k+3,k} & p_{k+3,k+1} & p_{k+3,k+2} & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
p_{n,1} & p_{n,2} & p_{n,3} & \cdots & p_{n,k} & p_{n,k=1} & p_{n,k+2} & \cdots 1
\end{pmatrix}$$

with each $p_{i,j} \in \mathbb{Z}[x_1, ..., x_n]$.

We break the problem down as follows.

1. $p_i(\gamma, \lambda) = p_{i,1}(\gamma)\lambda_1 + p_{i,2}(\gamma)\lambda_2 + ... + p_{i,k}(\gamma)\lambda_k + ... + p_{i,i-1}(\gamma)\lambda_{i-1} + \lambda_i$, for i=k+1,...,n.

2. Let $\{a_{k+1}, ..., a_n\}$ be any set of integers, not all zero. Define

$$\begin{aligned} p(\gamma, \lambda) &= a_{k+1}p_{k+1}(\gamma, \lambda) + ... + a_n p_n(\gamma, \lambda) \\ &= \sum_{i=k+1}^{n} a_i \left\{ \sum_{t=1}^{i-1} \lambda_t p_{i,t}(\gamma) + \lambda_i \right\} \\ &= \sum_{t=1}^{k} \lambda_t \left\{ \sum_{i=k+1}^{n} a_i p_{i,t} \right\} + \sum_{t=k+1}^{n} \lambda_t \left\{ \sum_{i=t}^{n} a_i p_{i,t} \right\}. \end{aligned}$$

We note that $p_{i,i} = 1$ for all i. Let $A = (0, ..., 0, a_{k+1}, ..., a_n) \in \mathcal{L}$. If we write $\mathrm{Ad}(\gamma^{-1})A = (\Phi_1, ..., \Phi_n)$, then we have $p(\gamma, \lambda) = \sum_{i=1}^{n} \lambda_i \Phi_i$. We write $p(\gamma, \lambda) = p_c(\gamma, \lambda) + p_n(\gamma, \lambda)$ where $p_c = \sum_{i=1}^{k} \lambda_i \Phi_i$ and $p_n = \sum_{i=k+1}^{n} \lambda_i \Phi_i$.

We first show that if $\{\lambda_1, ..., \lambda_k\}$ are as in lemma 2.1, then the polynomial p_c has a nontrivial irrational coefficient. Since the Φ_i, $i = 1, ..., k$, consist of polynomials with integral coefficients and $\{\lambda_1, ..., \lambda_k\}$ are linearly independent over $\mathbb{Q}$, all the coefficients of the polynomial p_c are irrational. Therefore, we need only show that p_c is not a constant polynomial, or equivalently that some Φ_i, $i = 1, ..., k$, is not constant.

These are the central components of the vector giving $\mathrm{Ad}(\gamma^{-1})A$, where not all of the entries a_i are zero. Therefore the orbit described is that of a non-central element of $\mathcal{L}$, and so it will suffice to show that for any non-central element T of $\mathcal{L}$, $\mathrm{Ad}(\Gamma)(T)$ has nondegenerate orthogonal projection onto the center of $\mathcal{L}_{\mathbb{R}}$.

LEMMA 2.9. *Let $\{X_1, ..., X_n\}$ be as before, and let P_z be projection onto the real* span *of $\{X_1, ..., X_k\}$ with $\mathbb{R}$ –* span *$\{X_{k+1}, ..., X_n\}$ as kernel. Then if T is a non-central element of $\mathcal{L}_{\mathbb{R}}$, $P_z(\mathrm{Ad}(\Gamma)T - T)$ is not identically zero.*

Proof. It suffices to show that $P_z(\mathrm{Ad}(N)T - T)$ is not identically zero, where $N = \exp(\mathcal{L}_{\mathbb{R}})$. Let $\{t_1, ..., t_s\}$ be a subset of $\mathbb{R}$ and $\{Y_1, ..., Y_s\}$ be a subset of $\mathcal{L}_{\mathbb{R}}$.

If we use the formula $\mathrm{Ad}(\exp Y)T = \exp\ (\mathrm{ad}(Y)T)$ to write $\mathrm{Ad}(\exp t_1 Y_1 \cdot \exp t_2 Y_2 \cdot ... \cdot \exp t_s Y_s)T - T$ as a polynomial expression in $\{t_1, ..., t_s\}$ with coefficients in $\mathcal{L}_{\mathbb{R}}$, we see that the coefficient of the monomial $t_1 t_2 ... t_s$ is a rational multiple of $[Y_1, [Y_2, [\ldots [Y_s, T]] \ldots]]$. Now suppose that $P_z(\mathrm{Ad}(N)T - T) = 0$; then we must have $[Y_1, [Y_2, [...[Y_s, T]...]]] = 0$. However, since T is not central, there exists a sequence of elements $\{Y_1, ..., Y_s\} \subseteq \mathcal{L}_{\mathbb{R}}$ such that $W = [Y_1, [Y_2, [...[Y_s, T]...]]] \in z(\mathcal{L})$, $W \neq 0$. Then $P_z(W) = W$ is nonzero, giving a contradiction, and completing the proof of Lemma 2.9.

LEMMA 2.10. *The polynomial* $p(\gamma, \lambda)$ *has an irrational coefficient (of a nontrivial monomial) whenever* λ *is generic.*

Proof. Suppose that $p(\gamma, \lambda)$ has entirely rational coefficients. Then since $p_c(\gamma, \lambda)$ is nontrivial, $p_n(\gamma, \lambda) = p_r(\gamma, \lambda) - p_c(\gamma, \lambda)$, where $p_r(\gamma, \lambda)$ has rational coefficients. Since we have $p_n(\gamma, \lambda) = \sum_{i=k+1}^{n} \lambda_i \Phi_i$, where the Φ_i have integer coefficients, we must have some subset $\{\lambda_{\sigma(t)}\}_{t=1}^{l}$ of coefficients which are from the $\mathbb{Q}$-span of $\{\lambda_1, ..., \lambda_k\}$, and the rest must be rational. Thus we have

$$\sum_{i=1}^{k} \lambda_i \Phi_i + \sum_{t=1}^{l} \lambda_{\sigma(t)} \Phi_{\sigma(t)} = K,$$

where K is some real constant. However, the $\lambda_{\sigma(t)}$, $t = 1, ..., l$, satisfy $\lambda_{\sigma(t)} = \sum_{i=1}^{k} q_{t,i} \lambda_i$, where the $q_{t,i}$ are rational. Therefore we may write the equation above as

$$\sum_{i=1}^{k} \lambda_i \Phi_i + \sum_{t=1}^{l} \left\{ \sum_{s=1}^{k} q_{s,t} \lambda_s \right\} \Phi_{\sigma(t)} = K.$$

Since the λ_i are linearly independent over $\mathbb{Q}$, we must have, for each i, that

$$\Phi_i + \sum_{t=1}^{l} q_{t,i} \Phi_{\sigma(t)} = K_i,$$

for some real constant K_i. Let $v_i = X_i + \sum_{t=1}^{l} q_{i,t} X_{\sigma(t)} \in \mathcal{L}_{\mathbb{R}}$, for $i = 1, ..., k$. The above implies that the function $\gamma \longmapsto < v_i, \mathrm{Ad}(\gamma) A >$ is a constant function, and so that the projection of $\mathrm{Ad}(N)A$ onto the subspace $W = \mathbb{R}$-span $\{v_i\}$ is degenerate. We will show that this is impossible, giving a contradiction.

LEMMA 2.11. *Let* $\mathcal{O}$ *be a non-trivial* Ad-*orbit of* N. *Then* $P_W(\mathcal{O})$ *is nondegenerate, i.e.,* $P_W(\mathrm{Ad}(\Gamma)A - A)$ *is nonzero.*

Proof. Let $\{Y_1, ..., Y_s\}$ be as in the proof of Lemma 2.9., such that $[Y_1, [...[Y_{s-1}, [Y_s, A]]...]]$ is a nonzero element of $z(\mathcal{L})$. Then if we express $\mathrm{Ad}(\exp t_1 Y_1 \cdot \ldots \cdot \exp t_s Y_s)A$ as a polynomial in $t_1, \ldots, t_s$ taking on values in $\mathcal{L}_{\mathbb{R}}$, we see that the monomial $t_1 \ldots t_s$ appears as a coefficient only of central elements $\{X_1, \ldots, X_k\}$. Let $i \leq k$ be such that $[Y_1, [\ldots[Y_s, A]\ldots]]$ has a nontrivial X_i-component. Then $< v_i, \mathrm{Ad}(\exp t_1 Y_1 \ldots \exp t_s Y_s)A >$ is a polynomial with a nonzero coefficient of $t_1 \ldots t_s$, so it is nontrivial, and so $P_W(\mathcal{O})$ is nondegenerate.

This completes the proof of Lemma 2.10; taken together with Lemma 2.8, this completes the proof of Proposition 2.5. □

3. Traceable factor representations associated with generic coadjoint orbits. Suppose now that λ is a generic element of $\widehat{\mathcal{L}}$, with $\mathrm{Ad}^*(\Gamma)$- orbit closure $\mathcal{O}_\lambda$. We define τ_λ to be the representation of Γ induced from the restriction of λ to $z(\mathcal{L})$, regarded as a character on $z(\Gamma)$ (this is possible because $z(\mathcal{L}) = z(\mathcal{L}_{\mathbb{R}}) \cap \mathcal{L}$ by Theorem 5.1.5 in [**3**], and because $\exp z(\mathcal{L}) = z(\Gamma)$); τ_λ is defined on the Hilbert space

$$H_\lambda = \left\{ f : \Gamma \longrightarrow \mathbb{C} \middle| \int_{\Gamma \backslash z(\Gamma)} |f|^2 dx < \infty f(z\gamma) = \lambda(z) f(\gamma), \right.$$
$$\left. z \in z(\Gamma), \gamma \in \Gamma \right\},$$

with inner product $< f, g >= \int_{\Gamma \backslash z(\Gamma)} f \cdot \overline{g}\ dx$. Since elements in $\mathcal{O}_\lambda$ agree on the center of $\mathcal{L}$, τ_λ depends only upon the coadjoint orbit closure $\mathcal{O}_\lambda$. Recall that τ_λ is a factor representation if $CR(\tau_\lambda) = \tau_\lambda(\Gamma)' \cap \tau_\lambda(\Gamma)'' = \mathbb{C}I$ (in general, A' denotes the commutator of the set A). In what follows, we will show that if $\mathcal{O}_\lambda$ is a generic coadjoint orbit closure in $\widehat{\mathcal{L}}$, then τ_λ is a factor representation.

Lemma 3.1. (Lemma 1, [**4**]). *Let $U \in \tau_\lambda(\Gamma)'$, and let H_λ be the Mackey space as defined above for the representation τ_λ. Then U is entirely determined by its value on the function $\delta_1 \in H_\lambda$ defined by*

$$\delta_1(\gamma) = \begin{cases} 0 & \gamma \notin z(\Gamma) \\ \lambda(\gamma) & \gamma \in z(\Gamma) \end{cases}.$$

Lemma 3.2. (Lemma 2, [**4**]). *If $U \in CR(\tau_\lambda)$, then U is convolution by an element of H_λ which is constant on conjugacy classes.*

Furthermore, if $f \in H_\lambda$ is constant on conjugacy classes in Γ, and if convolution by f is a bounded operator on H_λ, then convolution by f is in $CR(\tau_\lambda)$.

By Lemma 3.2, to show that τ_λ is a factor it suffices to show that the only element of H_λ which is constant on conjugacy classes of Γ is δ_1; for convolution by δ_1 is the identity map on H_λ, and so all elements of $CR(\tau_\lambda)$ are multiples of the identity.

THEOREM 3.3. *If λ is a generic element of $\widehat{\mathcal{L}}$, then τ_λ is a factor representation.*

Proof. Let $\Gamma^{(i)}$ denote the i-th element of the rising central series of Γ, and let $\gamma \in \Gamma$, $\gamma \notin \Gamma^{(2)}$. We will show that a function constant on the conjugacy class $C(\gamma)$ cannot be in H_λ unless it is zero there.

Let I_γ denote the isotropy subgroup in Γ of the element $\gamma z(\Gamma)$, where Γ acts on $\Gamma\backslash z(\Gamma)$ by conjugation. If $\gamma \notin \Gamma^{(2)}$, then I_γ is a proper subgroup of Γ. The cosets of $z(\Gamma)$ intersected by the conjugacy class $C(\gamma)$ are in bijective correspondence with $\Gamma\backslash I_\gamma$; therefore, if I_γ were also a saturated subgroup, the number of cosets intersected by the conjugacy class of γ would be infinite, and so a function in H_λ which is constant on conjugacy classes would have to be zero on $C(\gamma)$.

Therefore we prove

LEMMA 3.4. *If $\gamma \in \Gamma$, and $\gamma \notin \Gamma^{(2)}$, then I_γ is a saturated subgroup of Γ.*

Proof. Let $\gamma = \exp T$, $x = \exp X$, for $T, X \in \mathcal{L}$. If $x^n \in I_\gamma$ for some n, then we have $x^n \gamma x^{-n} \gamma^{-1} \in z(\Gamma)$. By the Campbell-Baker-Hausdorff formula,

$$\begin{aligned} x^n \gamma x^{-n} \gamma^{-1} &= \exp nX \exp T \exp -nX \exp -T \\ &= \exp \left\{ n[X,T] + \frac{1}{2}(n^2[X,[X,T]] - n[T,[X,T]]) + \dots \right\} \\ &= \exp P(n) \in z(\Gamma), \end{aligned}$$

where successive terms involve brackets of increasing order. Clearly the polynomial $P(n)$ is in $z(\mathcal{L})$, and since I_γ is a subgroup of Γ, $P(kn) \in z(\mathcal{L})$ as well for each $k \in \mathbb{Z}$. Therefore each individual term of $P(nk)$, viewed as a polynomial in k, is in $z(\mathcal{L})$. Using the Campbell-Baker-Hausdorff formula again, we rewrite the above as

$$\begin{aligned}&\exp nkX(\exp T \exp -nkX \exp -T)\\ &= \exp nkX \exp -(\mathrm{Ad}(\exp T)nkX)\\ &= \exp\left(nkX - \mathrm{Ad}(\exp T)nkX - \frac{1}{2}[nkX, \mathrm{Ad}(\exp T)nkX] - ...\right),\end{aligned}$$

where successive terms involve powers of k which are higher than 2. It follows that $n\{X - \mathrm{Ad}(\exp T)X\} \in z(\mathcal{L})$, and therefore, since $z(\mathcal{L})$ is saturated, $X - \mathrm{Ad}(\exp T)X$ is in $z(\mathcal{L})$. We have

$$\mathrm{Ad}(\exp T)X - X = [T, X] + \frac{1}{2}[T, [T, X]] + \frac{1}{6}[T, [T, [T, X]]] + ...$$

and we wish to see that $[T, X] \in z(\mathcal{L})$. We expand $[T, X]$ in terms of a strong malcev basis $\{X_i\}$ through the lower central series of $\mathcal{L}$; then we may write $[T, X] = a_1X_1 + ... + a_sX_s$, where $a_s \neq 0$. Since $[T, [T, X]]$ and subsequent terms belong to ideals which are further down in the lower central series, X_s will be absent from basis expansions of these terms and so we must have $X_s \in z(\mathcal{L})$, and therefore $[T, X] \in z(\mathcal{L})$. Therefore $x\gamma x^{-1}\gamma^{-1} \in z(\Gamma)$, and so I_γ is saturated.

Now we suppose that $x \in \Gamma^{(2)}$, $x \notin z(\Gamma)$. The conjugacy class of x is thus a nontrivial subset of the coset $xz(\Gamma)$. Since λ is a generic character, the kernel of $\tilde{\lambda}$ in $z(\Gamma)$ is trivial; therefore a function in H_λ, with left $z(\Gamma)$-covariance, could not possibly be constant on $C(x)$ unless it were zero on $C(x)$.

Therefore, we see that a function in H_λ which is constant on conjugacy classes is supported only upon $z(\Gamma)$, and hence must be a multiple of δ_1. This completes the proof of Theorem 3.3. □

What follows is proved in Section 3 in [**4**], and applies here with $I_\lambda = z(\Gamma)$.

THEOREM 3.5. (Theorem 3, [**4**]). *If λ is generic, τ_λ is a traceable factor representation, with trace (for $f \in \mathcal{L}^1(\Gamma)$)*

$$\Theta_\lambda(f) = \sum_{u \in z(\Gamma)} \lambda(u)f(u) = < f, \delta_1 >,$$

where

$$\delta_1(u) = \begin{cases} \lambda(u) & \text{if } u \in z(\Gamma) \\ 0 & \text{if } u \notin z(\Gamma). \end{cases}$$

Furthermore, we have the orbital trace formula

$$\Theta_\lambda(f) = \int_{\bar{\mathcal{O}}_\lambda} F^\wedge(\chi) d\chi,$$

where $d\chi$ is the lift of Haar measure on $z(\mathcal{L})^\perp$ to the closure $\lambda + z(\mathcal{L})^\perp$ of $\mathcal{O}_\lambda$, and $F = f \circ \exp \in L^1(\mathcal{L})$. $F^\wedge(\chi)$ denotes the usual Fourier transform of F.

These are the same traces R. Howe found as elements of dual cones of primitive ideals in the primitive ideal space of Γ (see Proposition 3 of [**5**]).

Now let $F \in C_c(\mathcal{L})$, so that $f = F \circ \log \in C_c(\Gamma)$. If $\{X_1, \ldots, X_n\}$ is our chosen basis, we can define an inclusion $i\colon z(\hat{\mathcal{L}}) \longrightarrow \hat{\mathcal{L}}$ as follows: $\tilde{\lambda} \longmapsto \lambda$ if

$$\lambda(a_1 X_1 + \ldots + a_n X_n) = \tilde{\lambda}(a_1 X_1 + \ldots + a_k X_k),$$

for all $\vec{a} \in \mathbb{Z}^n$. Then by Fourier inversion on the abelian group $\mathcal{L}$,

$$f(e) = F(0) = \int_{\widehat{\mathcal{L}}} F^\wedge(\xi) d\xi = \int_{\widehat{z(\mathcal{L})}} \left\{ \int_{z(\mathcal{L})^\perp} F^\wedge(\lambda + \chi) d\chi \right\} d\lambda,$$

where Haar measures are normalized so that their supports have measure 1.

We let $d_\lambda(\chi)$ be the lift of normalized Haar measure on $z(\mathcal{L})^\perp$ to $\lambda + z(\mathcal{L})^\perp$; if λ is generic, then this is the measure on the closure of $\mathcal{O}_\lambda$ which appears in the orbital trace formula for τ_λ. Since *a.a.* $\tilde{\lambda} \in \widehat{z(\mathcal{L})}$ are generic, the above becomes

$$\int_{\widehat{z(\mathcal{L})}} \left\{ \int_{z(\mathcal{L})^\perp} F^\wedge(\lambda + \chi) d\chi \right\} d\lambda = \int_{\widehat{z(\mathcal{L})}} \left\{ \int_{\bar{\mathcal{O}}_\lambda} F^\wedge(\chi) d_\lambda(\chi) \right\} d\lambda = \int_{\widehat{z(\mathcal{L})}} \Theta_\lambda(f) d\lambda.$$

We have proven

THEOREM 3.6. (Plancherel Formula). *Suppose $f \in C_c(\Gamma)$, and that for generic $\lambda \in \widehat{\mathcal{L}}$, Θ_λ is the trace associated with the factor* *representation* τ_λ *induced from* $\tilde{\lambda}$ *on* $z(\mathcal{L})$. *Then* $\tilde{\lambda} \longmapsto \Theta_\lambda(f)$ *is defined for a.a.* λ, *is integrable on* $\widehat{z(\mathcal{L})}$, *and we have*

$$f(e) = \int_{\widehat{z(\mathcal{L})}} \Theta_\lambda(f) d\mu(\lambda),$$

where μ *is Haar meaasure on* $\widehat{z(\mathcal{L})}$, *normalized so that* $\mu(\widehat{z(\mathcal{L})}) = 1$.

References

[1] L. Auslander, L. Green and F. Hahn, *Flows on Homogeneous Spaces*, Annals of Math. Studies, Vol. 53, Princeton University Press, Princeton, N.J., 1963.

[2] J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge University Press, 1957.

[3] L. Corwin and F. Greenleaf, *Representations of Nilpotent Lie Groups and their Applications*, Part I, *Basic Theory and Examples*, Cambridge University Press, Cambridge, 1990.

[4] L. Corwin and C. Pfeffer, *On factor representations of discrete rational nilpotent groups and the Plancherel formula*, Pacific. J. Math., **162** (1994), 261-175.

[5] R. Howe, *On representations of discrete, finitely generated, torsion-free, nilpotent groups*, Pacific J. Math., **73**, No. 2 (1977), 281-306.

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