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L^p-BOUNDEDNESS OF THE HILBERT TRANSFORM AND MAXIMAL FUNCTION ALONG FLAT CURVES IN $\mathbb{R}^n$

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We consider the Hilbert transform and maximal function associated to a curve $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$ in $\mathbb{R}^n$. It is well-known that for a plane convex curve $\Gamma(t) = (t, \gamma(t))$ these operators are bounded on L^p, $1 < p < \infty$, if γ' doubles. We give an n-dimensional analogue, $n \geq 2$, of this result.

1. Introduction. Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a curve in $\mathbb{R}^n$, $n \geq 2$, with $\Gamma(0) = 0$. We define the associated Hilbert transform, $\mathcal{H}_\Gamma$ and maximal function $\mathcal{M}_\Gamma$ by

$$\mathcal{H}_\Gamma f(x) = \text{p. v.} \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t}$$

and

$$\mathcal{M}_\Gamma f(x) = \sup_{r>0} \frac{1}{r} \int_0^r |f(x - \Gamma(t))| \, dt,$$

respectively. We use p. v. to indicate that we are taking a principal value integral.

There has been considerable interest in finding conditions on Γ which give $L^2(\mathbb{R}^n)$-boundedness or $L^p(\mathbb{R}^n)$-boundedness, $1 < p < \infty$, of $\mathcal{H}_\Gamma$ and $\mathcal{M}_\Gamma$, when Γ is permitted to be flat (i.e. vanish to infinite order) at the origin; the case of well-curved Γ was dealt with in the 1970's, see for example [**7**].

The aim of this paper is to give an n-dimensional analogue of the following well-known theorem for plane curves.

THEOREM 1.1. [**1**]. *Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^2, \Gamma(t) = (t, \gamma(t))$ be a convex curve such that $\gamma \in C^2(0, \infty)$ is either even or odd and $\gamma(0) = \gamma'(0) = 0$. Suppose that $\exists 1 < \lambda < \infty$ such that $\forall t \in (0, \infty)$*

$$\gamma'(\lambda t) \geq 2\gamma'(t). \tag{1}$$

Then

$$\|\mathcal{H}_\Gamma f\|_p \le C\|f\|_p$$
$$\|\mathcal{M}_\Gamma f\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

Conditions such as (1) are known as doubling conditions; in this case we say that γ' doubles.

In $\mathbb{R}^n$ we shall consider curves $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$ which are of class $C^n(0,\infty)$ and such that $\Gamma(0) = 0$. The convexity hypothesis for plane curves we replace by the "convexity" hypothesis used in the n-dimensional results of [**6**] and [**4**].

So we define determinants $D_j, \quad j = 1, \dots, n$ by

$$D_j = \det \begin{pmatrix} 1 & \gamma_2' & \cdots & \gamma_j' \\ 0 & \gamma_2'' & \cdots & \gamma_j'' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j)} & \cdots & \gamma_j^{(j)} \end{pmatrix}$$

and say that Γ is "convex" if

$$D_j(t) > 0, \qquad j = 2, \dots, n, t \in (0,\infty). \tag{2}$$

We also introduce the determinants N_j, $j = 1, \dots, n$, given by

$$N_j = \det \begin{pmatrix} t & \gamma_2 & \cdots & \gamma_j \\ 1 & \gamma_2' & \cdots & \gamma_j' \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j-1)} & \cdots & \gamma_j^{(j-1)} \end{pmatrix},$$

and as in [**6**] define functions h_j, $j = 1, \dots, n$, by

$$h_j(t) = \frac{N_j(t)}{D_{j-1}(t)}, \tag{3}$$

where we take $D_0 \equiv 1$.

In order to state our theorem we also introduce the differential operators L_k, of [**6**], defined by

$$L_1 f = \frac{df}{dt} \tag{4}$$
$$L_{k+1} f = \frac{h_k}{h_{k+1}'}(L_k f)', \quad k = 1, \dots, n-1.$$

It is also useful to have the following formula, proven via a Sylvester determinant identity in [**6**]:

$$L_k f(t) = \frac{E_k f(t)}{D_k(t)}, \quad k = 1, \dots, n, \tag{5}$$

where

$$E_k f(t) = \det \begin{pmatrix} 1 & \gamma_2'(t) & \cdots & \gamma_{k-1}'(t) & f'(t) \\ 0 & \gamma_2''(t) & \cdots & \gamma_{k-1}''(t) & f''(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \gamma_2^{(k)}(t) & \cdots & \gamma_{k-1}^{(k)}(t) & f^{(k)}(t) \end{pmatrix}.$$

From this we can see, immediately, that

$$L_k \gamma_j = 0, \quad j = 1, \dots, k-1; k = 1, \dots, n \tag{6}$$

$$L_k \gamma_k = 1, \quad k = 1, \dots, n. \tag{7}$$

Our result is the following.

THEOREM 1.2. *Let* $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$, $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$, $n \geq 2$, *be an odd curve in* $\mathbb{R}^n$, *of class* $C^n(0, \infty)$ *such that* $\Gamma(0) = 0$ *and* (2) *is satisfied. Suppose that* $\exists\ A \in \mathrm{GL}(n, \mathbb{R})$ *such that, with* $\widetilde{\Gamma}(t) = (t, \widetilde{\gamma}_2(t), \dots, \widetilde{\gamma}_n(t)) := A\Gamma(t)$, $\widetilde{\Gamma}$ *also satisfies* (2) *and*

$$\lim_{t \to 0} L_j \widetilde{\gamma}_k(t) = 0 \quad j = 1, \dots, n-1, k = j+1, \dots, n. \tag{8}$$

Suppose also that $\exists 1 < \lambda < \infty$ *such that,* $\forall t \in (0, \infty)$,

$$L_k \widetilde{\gamma}_{k+1}(\lambda t) \geq 2 L_k \widetilde{\gamma}_{k+1}(t), \quad k = 1, \dots, n-1. \tag{9}$$

Then

$$\|\mathcal{H}_\Gamma f\|_p \leq C \|f\|_p,$$
$$\|\mathcal{M}_\Gamma f\|_p \leq C \|f\|_p, \quad 1 < p < \infty.$$

REMARKS. (a) Since L^p-boundedness of $\mathcal{H}_\Gamma$ and of $\mathcal{M}_\Gamma$ is a $\mathrm{GL}(n, \mathbb{R})$ invariant property, in the proof we shall assume, without loss of generality, that the initial curve Γ satisfies (8) and (9).

(b) For $n = 2$ our theorem is precisely Theorem 1.1.

(c) It is easily checked that the "convexity" hypothesis, (2), is equivalent to requiring that

$$(L_k\gamma_{k+1})' > 0, \quad k = 1, \dots, n-1.$$

Thus, for the class of "convex" curves our conditions are natural analogues of the γ' doubling condition for plane convex curves (i.e. those for which $(L_1\gamma)' > 0$).

(d) The condition that Γ be odd is convenient but not essential; it may be replaced by other conditions on Γ giving suitable compatibility of the two halves $\Gamma(t)$, $t > 0$ and $\Gamma(t)$, $t < 0$. For example each $\gamma_k, k = 2, \dots, n$ may be either even or odd; this will be clear from the proof.

(e) The role of (8) is to impose a certain ordering of the components of the curve. Further, it follows easily from Lemma 3 of [**6**] (see Lemma 3.1) that each $L_j\gamma_k$ has at most $k-j$ zeros and at most $k-j-1$ changes of monotonicity on $(0,\infty)$; the normalization conditions (8) force the $L_j\gamma_k$ to be positive and increasing, thus much simplifying matters.

We note that if $\lim_{t\to 0} L_j\gamma_k(t)$ exists for all $1 \le j \le k-1 \le n-1$, then we can find an $A \in \mathrm{GL}(n,\mathbb{R})$ such that $\widetilde{\Gamma} = A\Gamma$ satisfies (8). To see this we first define an operator $\mathcal{L}$ by

$$\mathcal{L}\Gamma(t) = \begin{pmatrix} L_1\gamma_1(t) & L_2\gamma_1(t) & \cdots & L_n\gamma_1(t) \\ L_1\gamma_2(t) & L_2\gamma_2(t) & \cdots & L_n\gamma_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ L_1\gamma_n(t) & L_2\gamma_n(t) & \cdots & L_n\gamma_n(t) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ L_1\gamma_2(t) & 1 & 0 & \cdots & 0 & 0 \\ L_1\gamma_3(t) & L_2\gamma_3(t) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_1\gamma_{n-1}(t) & L_2\gamma_{n-1}(t) & L_3\gamma_{n-1}(t) & \cdots & 1 & 0 \\ L_1\gamma_n(t) & L_2\gamma_n(t) & L_3\gamma_n(t) & \cdots & L_{n-1}\gamma_n(t) & 1 \end{pmatrix},$$

using (6) and (7).

It is easily shown that if $A \in T_-$, the subgroup of $\mathrm{GL}(n,\mathbb{R})$ consisting of lower triangular matrices with 1 in the top left-hand corner and positive diagonal entries, then A preserves "convexity", i.e. if

Γ satisfies (2) then so does $A\Gamma$. Moreover, an easy calculation using (5) shows that if $A \in T_-$ and has diagonal entries all equal to 1 then

$$\mathcal{L}(A\Gamma) = A(\mathcal{L}\Gamma).$$

We now let $A = (\lim_{t\to 0} \mathcal{L}\Gamma(t))^{-1}$, where $\lim_{t\to 0} \mathcal{L}\Gamma(t)$ denotes the matrix with entries $\lim_{t\to 0} L_j\gamma_k(t)$. Then $\widetilde{\Gamma} = A\Gamma$ is "convex" and $\lim_{t\to 0} \mathcal{L}\widetilde{\Gamma}(t)$ is the identity matrix, from which we see that $\lim_{t\to 0} L_j\widetilde{\gamma}_k(t) = 0, \quad j = 1, \dots, n-1; \quad k = j+1, \dots, n.$

Curves for which we do not have the existence of $\lim_{t\to 0} L_j\gamma_k(t)$ for all $1 \le j \le k-1 \le n-1$ may still satisfy the hypotheses of our theorem. Consider, for example the "convex" curve in $\mathbb{R}^3$, $\Gamma(t) = (t, t^3, -t^2)$; in this case we have $L_2\gamma_3(t) = -\frac{1}{3t}$. However taking A to be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

we obtain the curve $\widetilde{\Gamma}(t) = (t, t^2, t^3)$, which clearly satisfies the hypotheses (8) and (9).

(f) Theorem 1.1, after a technical adjustment to condition (1), may also be seen to hold for curves which are not $C^2(0,\infty)$ but convex and piecewise-linear. We say that a piecewise-linear γ curve is convex if

$$\frac{\gamma(c)-\gamma(b)}{c-b} \ge \frac{\gamma(b)-\gamma(a)}{b-a}, \quad 0 \le a < b < c.$$

Our method of proof of Theorem 1.2 allows us to extract the following result for piecewise-linear curves in $\mathbb{R}^n$.

Corollary 1.3. *Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$, $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$ be an odd curve such that $\Gamma(0) = 0$ and each γ_k, $k = 2, \dots, n$, is convex and piecewise-linear on $[\lambda^j, \lambda^{j+1}]$, $j \in \mathbb{Z}$, some $\lambda > 1$. Suppose*

$$\frac{\gamma_k(\lambda^{j+1}) - \gamma_k(\lambda^j)}{\gamma_{k-1}(\lambda^{j+1}) - \gamma_{k-1}(\lambda^j)} \ge 2\frac{\gamma_k(\lambda^j) - \gamma_k(\lambda^{j-1})}{\gamma_{k-1}(\lambda^j) - \gamma_{k-1}(\lambda^{j-1})},$$

for $j \in \mathbb{Z}$, $k = 2, \dots, n$.

Then

$$\|\mathcal{H}_\Gamma f\|_p \le C\|f\|_p$$
$$\|\mathcal{M}_\Gamma f\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

2. Sketch of Proof. We define measures μ_k, σ_k on the curve Γ by

$$\int f \, d\mu_k = \frac{1}{\lambda^k(\lambda - 1)} \int_{\lambda^k}^{\lambda^{k+1}} f(\Gamma(t)) \, dt \quad \text{and}$$
$$\int f \, d\sigma_k = \int_{\lambda^k \le |t| \le \lambda^{k+1}} f(\Gamma(t)) \frac{dt}{t},$$

respectively. Then we have the associated Fourier multipliers

$$\hat{\mu}_k(\zeta) = \frac{1}{\lambda^k(\lambda - 1)} \int_{\lambda^k}^{\lambda^{k+1}} e^{i\zeta.\Gamma(t)} \, dt \tag{10}$$

and

$$\hat{\sigma}_k(\zeta) = \int_{\lambda^k \le |t| \le \lambda^{k+1}} e^{i\zeta.\Gamma(t)} \frac{dt}{t}. \tag{11}$$

We adopt the standard approach of decomposing $\mathcal{H}_\Gamma$ as

$$\mathcal{H}_\Gamma f = \sum_k \sigma_k * f$$

and majorizing $\mathcal{M}_\Gamma$ by

$$\mathcal{M}_\Gamma f \le C \sup_k |\mu_k * f|.$$

From [4] the following theorem is easily extracted.

THEOREM 2.1. *Let* $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$, $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$ *be an odd curve in* $\mathbb{R}^n, \Gamma(0) = 0$. *Suppose* $\exists$ *a family of dilation matrices* $\{A_k\} \subseteq \mathrm{GL}(n, \mathbb{R})$ *such that*

(12) (a) $\exists\ \alpha$ *such that* $\left\| A_{k+1}^{-1} A_k \right\| \le \alpha < 1$

(b) A_{k+1}^{-1} supp $\mu_k \subseteq$ *fixed ball*

(c) $|\hat{\mu}_k(\zeta)| \le C|A_k^* \zeta|^{-\varepsilon}$ *for some* $\varepsilon > 0$.

Then

$$\left\| \sup_k |\mu_k * f| \right\|_p \le C\|f\|_p$$
$$\|\mathcal{H}_\Gamma f\|_p \le C\|f\|_p, \qquad 1 < p < \infty.$$

In (8a) we use $\|\cdot\|$ to denote the operator (matrix) norm. We note that the conditions of the theorem do not involve σ_k. This is because, in view of the cancellation property,

$$\int d\sigma_k = 0$$

and the fact that Γ is odd, (12b) and (12c) give also analogous statements for σ_k. Without the assumption that Γ is odd we require also that

$$A_{k+1}^{-1} \operatorname{supp}\ \sigma_k \subseteq \text{ fixed ball}$$

and

$$|\hat{\sigma}_k(\zeta)| \leq C|A_k^* \zeta|^{-\varepsilon} \text{ for some } \varepsilon > 0.$$

Condition (12a) is known as Rivière's condition and enables a Calderón-Zygmund theory with respect to balls $\{A_j B\}$, for B the unit ball in $\mathbb{R}^n$, and thence an "annular" Littlewood-Paley decomposition to be developed.

Conditions (12b) and (12c) give decay estimates for $\hat{\mu}_k$ (and $\hat{\sigma}_k$) which may be combined with the Littlewood-Paley theory, along with a bootstrapping argument, to give the result. In [**4**] the authors find conditions on Γ under which (12c) holds, (12a) and (12b) being easily satisfied with an appropriate choice of the dilation matrices.

Our approach is to consider, for each k, the points $\zeta \in \mathbb{R}^n$ where (12c) may fail and to develop a conical Littlewood-Paley decomposition to deal with these "bad" ζ, in the spirit of [**1**] or [**5**].

In Section 3 we shall give some essential properties of "convex" curves and define our choice of dilation matrices $\{A_k\}$. In Section 4 we consider the set of $\zeta \in \mathbb{R}^n$ where the required decay estimates for $\hat{\mu}_k$, $\hat{\sigma}_k$ may fail and show that these ζ are contained in a cone C_k. Next we give conditions on Γ, of which there are $\frac{1}{2}n(n-1)$, under which these C_k form a Littlewood-Paley decomposition and show how they may be reduced to the $n-1$ conditions, (9), in the statement of our theorem. Finally in Section 5 we indicate how to combine the conical Littlewood-Paley theory of Section 4 with the "annular" Littlewood-Paley theory of Theorem 2.1 to complete the proof.

3. "Convexity" and dilation matrices. Most of the consequences of "convexity" that we shall need are dealt with in [**6**].

First, from Lemma 2 of [**6**] we know that for a "convex" curve we have, for $k = 2, \dots, n$, $t \in (0, \infty)$

$$h_k(t) > 0 \qquad \text{and} \qquad h'_k(t) > 0. \tag{13}$$

The tool we have for estimating oscillatory integrals such as $\hat{\mu}_k$ is Van der Corput's lemma; in order to be able to use this we need to know that $\zeta.\Gamma'$ has a bounded number of changes of monotonicity on each $[\lambda^k, \lambda^{k+1})$. This is given in Lemma 3 of [**6**].

LEMMA 3.1. ([**6**, Lemma 3]). *Let $\Gamma \in C^n(0, \infty)$ be a "convex" curve in $\mathbb{R}^n$, $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$ such that $\Gamma(0) = 0$. Then for $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{R}^n$, $L_n(\zeta.\Gamma) = \zeta_n$ and for $j = 1, 2, \dots, n, L_j(\zeta.\Gamma)$ has at most $n - j$ zeros in $(0, \infty)$, provided $\zeta_n \neq 0$.*

The proof of this in [**6**] establishes the identity (5) mentioned previously, the result then following easily. We shall also need the following:

LEMMA 3.2. *Let $\Gamma \in C^n(0, \infty)$, $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_n(t))$, $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a "convex" curve in $\mathbb{R}^n$, satisfying also (8), i.e.*

$$\lim_{t \to 0} L_k \gamma_{j+1}(t) = 0, \quad j = k, \dots, n-1, \ k = 1, \dots, n-1.$$

Then for $t \in (0, \infty)$

$$(L_k \gamma_j)'(t) > 0 \qquad \text{and} \qquad (L_k \gamma_j)(t) > 0, \tag{14}$$

$$k = 1, \dots, n-1, \ j = k+1, \dots, n.$$

In particular $\gamma''_j > 0$, $j = 2, \dots, n$.

Proof. We recall that, for $k = 1, \dots, n-1$,

$$L_{k+1} f = \frac{h_k}{h'_{k+1}} (L_k f)'.$$

So by (7) we have, for $k = 1, \dots, n-1$, $t \in (0, \infty)$

$$(L_k \gamma_{k+1})'(t) = \frac{h'_{k+1}(t)}{h_k(t)} > 0,$$

using (13). Then (8) gives us also

$$L_k\gamma_{k+1}(t) > 0, \quad k = 1, \dots, n-1, \ t \in (0, \infty).$$

We now fix $j \in \{k+1, \dots, n\}$ and suppose that for some $k \in \{1, \dots, j\}$, $t \in (0, \infty)$,

$$(L_k\gamma_j)'(t) > 0 \quad \text{and} \quad L_k\gamma_j(t) > 0.$$

Then, for $t \in (0, \infty)$,

$$(L_{k-1}\gamma_j)'(t) = \frac{h_k'(t)}{h_{k-1}(t)} L_k\gamma_j(t) > 0,$$

using again (13). We also have $L_{k-1}\gamma_j(t) > 0$, $t \in (0, \infty)$, using (8). The result now follows by induction. □

COROLLARY 3.3. *Let Γ be as in the lemma. Suppose also that $\Gamma(0) = 0, \gamma_k'(0) = 0, k = 2, \dots, n$. Then for $k = 2, \dots, n$*

(a) *γ_k' is increasing and non-negative on $(0, \infty)$*

(b) *γ_k is increasing and non-negative on $(0, \infty)$*

(c) *$\gamma_k(\lambda^{j+1}) \geq \lambda\gamma_k(\lambda^j), \quad \forall j \in \mathbb{Z}$.*

Proof. Immediate from Lemma 3.2. □

LEMMA 3.4. *Let Γ be as in Lemma 3.2. Then, for $t \in (0, \infty)$,*

$$\left(\frac{L_k\gamma_{j+1}}{L_k\gamma_j}\right)'(t) > 0, \quad \forall j = k, \dots, n-1, \ k = 1, \dots, n-1.$$

Proof. We proceed by induction. Let $k \in \{1, \dots, n-1\}$ be fixed. Then

$$\left(\frac{L_k\gamma_{k+1}}{L_k\gamma_k}\right)' = (L_k\gamma_{k+1})' = \frac{h_{k+1}'}{h_k} > 0.$$

Now we suppose that

$$\left(\frac{L_m\gamma_{k+1}}{L_m\gamma_k}\right)' > 0, \quad \text{for some } m \in \{2, \dots, k\}.$$

Then

$$\left(\frac{L_m\gamma_{k+1}}{L_m\gamma_k}\right)' = \left(\frac{(L_{m-1}\gamma_{k+1})'}{(L_{m-1}\gamma_k)'}\right)' > 0. \tag{15}$$

So by the Second Mean Value Theorem, if $\varepsilon \in (0,t)$,

$$\frac{L_{m-1}\gamma_{k+1}(t) - L_{m-1}\gamma_{k+1}(\varepsilon)}{L_{m-1}\gamma_k(t) - L_{m-1}\gamma_k(\varepsilon)} = \frac{(L_{m-1}\gamma_{k+1})'(\eta)}{(L_{m-1}\gamma_k)'(\eta)},$$

for some $\eta \in (0,t)$. Then, by (15) and (8),

$$\frac{L_{m-1}\gamma_{k+1}(t)}{L_{m-1}\gamma_k(t)} < \frac{(L_{m-1}\gamma_{k+1})'(t)}{(L_{m-1}\gamma_k)'(t)}. \tag{16}$$

Hence, using (16) and (14),

$$\left(\frac{L_{m-1}\gamma_{k+1}}{L_{m-1}\gamma_k}\right)' = \frac{(L_{m-1}\gamma_k)'}{(L_{m-1}\gamma_k)} \left\{ \frac{(L_{m-1}\gamma_{k+1})'}{(L_{m-1}\gamma_k)'} - \frac{L_{m-1}\gamma_{k+1}}{L_{m-1}\gamma_k} \right\} > 0.$$

Thus, by induction, for each fixed $k \in \{1, \dots, n-1\}$ we have

$$\left(\frac{L_m\gamma_{k+1}}{L_m\gamma_k}\right)' > 0, \quad \forall m = 1, \dots, k.$$

□

We now turn to defining our dilation matrices $\{A_k\}$. The choice of these is motivated by the fact that we are looking for a theory which admits piecewise-linear curves; we want, therefore, the A_k to have entries involving at most 1 derivative of γ_k, $k = 2, \dots, n$.

We define the diagonal matrix A by

$$A(t) = \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & \gamma_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma_n(t) \end{pmatrix}$$

and put $A_j = A(\lambda^j)$, $j \in \mathbb{Z}$.

That these matrices satisfy (12a) and (12b) is trivial, using Corollary 3.3.

4. A conical Littlewood-Paley decomposition. We wish to consider the $\zeta \in \mathbb{R}^n$ where we cannot expect (12c) to hold. By Lemma 3.1c) we know that $\zeta.\Gamma'$ has at most $(n-2)$ changes of monotonicity in $(0,\infty)$, thence must have a bounded number of changes of monotonicity in each interval $[\lambda^k, \lambda^{k+1})$.

So, by Van der Corput's lemma, if

$$|\zeta.\Gamma'(t)| \geq \frac{C}{\lambda^k}|A_k^*\zeta| \quad \forall t \in [\lambda^k, \lambda^{k+1}), \tag{17}$$

then

$$|\hat{\mu}_k(\zeta)| \leq C|A_k^*\zeta|^{-1}.$$

We consider, therefore, the set of ζ where (17) may fail, i.e.

$$\bigcup_{t\in[\lambda^k,\lambda^{k+1})} C_k^t,$$

where

$$C_k^t := \left\{\zeta \in \mathbb{R}^n : |\zeta.\Gamma'(t)| < \frac{\varepsilon}{\lambda^k}|A_k^*\zeta|\right\}.$$

Here $\varepsilon > 0$ may be as small as we like.

PROPOSITION 4.1. (a) *Let Γ be a "convex" $C^n(0,\infty)$ curve in $\mathbb{R}^n$. Then $\exists$ cones C_k such that*

$$\bigcup_{t\in[\lambda^k,\lambda^{k+1})} C_k^t \subseteq C_k := \bigcup_{m=1}^{n-1} (C_{km} \cup \widetilde{C}_{km})$$

where

$$C_{km} = \left\{\zeta : \sum_{j=m}^{n} \zeta_j L_m\gamma_j(\lambda^k) < \varepsilon \sum_{j=m}^{n} |\zeta_j| L_m\gamma_j(\lambda^k) \text{ and } \right.$$
$$\left. \sum_{j=m}^{n} \zeta_j L_m\gamma_j(\lambda^{k+1}) > -\varepsilon \sum_{j=m}^{n} |\zeta_j| L_m\gamma_j(\lambda^{k+1}) \right\}$$

and $\zeta \in C_{km} \iff -\zeta \in \widetilde{C}_{km}$.

(b) *Let Γ be piecewise-linear on $[\lambda^k, \lambda^{k+1}], k \in \mathbb{Z}$ and γ_j convex, $j = 2, \ldots, n$. Then*

$$\bigcup_{t\in[\lambda^k,\lambda^{k+1})} C_k^t \subseteq C_{k1} \cup \widetilde{C}_{k1},$$

where $C_{k1}, \widetilde{C}_{k1}$ are as defined in (a).

Proof. Let $\zeta \in \bigcup_{t\in[\lambda^k,\lambda^{k+1})} C_k^t$. We suppose first that $\zeta.\Gamma'$ is monotone-increasing on $[\lambda^k, \lambda^{k+1})$. Then $\forall\, t \in [\lambda^k, \lambda^{k+1})$,

$$L_1(\zeta.\Gamma)(\lambda^k) = \zeta.\Gamma'(\lambda^k) \leq \zeta.\Gamma'(t) \leq \zeta.\Gamma'(\lambda^{k+1}) = L_1(\zeta.\Gamma)(\lambda^{k+1}).$$

Hence

$$L_1(\zeta.\Gamma)(\lambda^k) < \frac{\varepsilon}{\lambda^k}|A_k^*\zeta| \tag{18}$$

and

$$L_1(\zeta.\Gamma)(\lambda^{k+1}) > -\frac{\varepsilon}{\lambda^k}|A_k^*\zeta|. \tag{19}$$

By Corollary 3.3 and the definition of the A_k we have

$$\frac{1}{\lambda^k}|A_k^*\zeta| \leq \sum_{j=1}^{n} \gamma_j'(\lambda^k)|\zeta_j| \leq \sum_{j=1}^{n} \gamma_j'(\lambda^{k+1})|\zeta_j|,$$

which, together with (18) and (19), gives

$$\sum_{j=1}^{n} \zeta_j L_1 \gamma_j(\lambda^k) = L_1(\zeta.\Gamma)(\lambda^k)$$
$$< \varepsilon \sum_{j=1}^{n} |\zeta_j| \gamma_j'(\lambda^k) = \varepsilon \sum_{j=1}^{n} |\zeta_j| L_1 \gamma_j(\lambda^k)$$

and

$$\sum_{j=1}^{n} \zeta_j L_1 \gamma_j(\lambda^{k+1}) = L_1(\zeta.\Gamma)(\lambda^{k+1})$$
$$> -\varepsilon \sum_{j=1}^{n} |\zeta_j| \gamma_j'(\lambda^{k+1}) = -\varepsilon \sum_{j=1}^{n} |\zeta_j| L_1 \gamma_j(\lambda^{k+1}).$$

Thus, if $\zeta \cdot \Gamma'$ is monotone-increasing on $[\lambda^k, \lambda^{k+1})$, then $\zeta \in C_{k1}$. Similarly, if $\zeta \cdot \Gamma'$ is monotone-decreasing on $[\lambda^k, \lambda^{k+1})$, then $\zeta \in \widetilde{C}_{k1}$.

We note here that if Γ is piecewise-linear on $[\lambda^k, \lambda^{k+1}]$, then $\zeta \cdot \Gamma'$ is constant on $(\lambda^k, \lambda^{k+1})$; by a suitable definition of $\zeta \cdot \Gamma'(\lambda^k)$ we may take $\zeta \cdot \Gamma'$ to be constant on $[\lambda^k, \lambda^{k+1})$ and thus (b) is proven.

We now suppose that $\zeta \cdot \Gamma'(t)$ is not monotone on $[\lambda^k, \lambda^{k+1})$. Then $\exists\, t_0 \in [\lambda^k, \lambda^{k+1})$ such that $\zeta \cdot \Gamma''(t_0) = 0$. Then

$$L_2(\zeta \cdot \Gamma)(t_0) = \frac{h_1}{h_2'} \zeta \cdot \Gamma''(t_0) = 0.$$

If then $L_2(\zeta \cdot \Gamma)$ is monotone-increasing on $[\lambda^k, \lambda^{k+1})$,

$$L_2(\zeta \cdot \Gamma)(\lambda^k) \leq 0 = L_2(\zeta \cdot \Gamma)(t_0) \leq L_2(\zeta \cdot \Gamma)(\lambda^{k+1})$$

and so $\zeta \in C_{k2}$; similarly $\zeta \in \widetilde{C}_{k2}$ if $L_2(\zeta \cdot \Gamma)$ is monotone-decreasing on $[\lambda^k, \lambda^{k+1})$. If $L_2(\zeta \cdot \Gamma)$ is not monotone on $[\lambda^k, \lambda^{k+1})$, then $\exists\ t_1 \in [\lambda^k, \lambda^{k+1})$ such that $(L_2(\zeta \cdot \Gamma))'(t_1) = 0$, from which we obtain $L_3(\zeta \cdot \Gamma)(t_1) = 0$ and so if $L_3(\zeta \cdot \Gamma)$ is monotone on $[\lambda^k, \lambda^{k+1})$, we obtain $\zeta \in C_{k3} \cup \widetilde{C}_{k3}$. We repeat this process iteratively. By Lemma 3.1 $L_n(\zeta \cdot \Gamma)(t) = \zeta_n$ so it follows that $L_{n-1}(\zeta \cdot \Gamma)$ must be monotone on $[\lambda^k, \lambda^{k+1})$ and hence the final possibility is that $\zeta \in C_{k(n-1)} \cup \widetilde{C}_{k(n-1)}$. □

We now wish to find conditions on Γ under which these cones give a Littlewood-Paley decomposition for $L^p(\mathbb{R}^n)$. The next result, in the same spirit as the lacunary Littlewood-Paley decomposition of [**5**], leads to the choice of these conditions. First we give our definition of lacunarity.

Definition 4.2. Let $\{\mathcal{E}_k(n, \varepsilon)\}$ be a family of cones in $\mathbb{R}^n$ given by

$$\mathcal{E}_k(n,\varepsilon) = \left\{ \zeta \in \mathbb{R}^n : \sum_{j=1}^n \alpha_k^j \zeta_j < \varepsilon \sum_{j=1}^n \alpha_k^j |\zeta_j| \quad \text{and} \right.$$
$$\left. \sum_{j=1}^n \alpha_{k+1}^j \zeta_j > -\varepsilon \sum_{j=1}^n \alpha_{k+1}^j |\zeta_j| \right\},$$

where α_k^j are positive reals, $j = 1, \ldots, n$, $k \in \mathbb{Z}$ and $\varepsilon > 0$ is small. If

$$\frac{\alpha_{k+1}^j}{\alpha_k^j} \geq 2\frac{\alpha_{k+1}^{j-1}}{\alpha_k^{j-1}}, \quad \forall\, k \in \mathbb{Z},\ j = 2, \ldots, n, \tag{20}$$

then the $\mathcal{E}_k(n, \varepsilon)$ are said to be lacunary.

We define "smoothed-out" characteristic functions $\Psi_k^{n,\varepsilon}$ of the cones $\mathcal{E}_k(n, \varepsilon)$ as follows.

Let $\Psi^{n,\varepsilon}$ be a C^∞ function away from 0, homogeneous of degree zero such that

$$\Psi^{n,\varepsilon}(\zeta) = \begin{cases} 1 & \sum_{j=1}^n \zeta_j < \varepsilon \sum_{j=1}^n |\zeta_j| \\ 0 & \sum_{j=1}^n \zeta_j > -2\varepsilon \sum_{j=1}^n |\zeta_j|, \end{cases}$$

and put

$$\Psi_k^{n,\varepsilon}(\zeta) = \Psi^{n,\varepsilon}(\alpha_k^1\zeta_1, \ldots, \alpha_k^n\zeta_n)\Psi^{n,\varepsilon}(-\alpha_{k+1}^1\zeta_1, \ldots, -\alpha_{k+1}^n\zeta_n).$$

Associated to these $\Psi_k^{n,\varepsilon}$ are operators T_k given by

$$\widehat{(T_k f)}(\zeta) = \Psi_k^{n,\varepsilon}(\zeta)\hat{f}(\zeta), \quad k \in \mathbb{Z}.$$

THEOREM 4.3. *If $\{\mathcal{E}(n,\varepsilon)\}$ is a lacunary family of cones in $\mathbb{R}^n$ then*

$$\left\| \left(\sum_k |T_k f|^2 \right)^{1/2} \right\|_p \leq C\|f\|_p, \quad 1 < p < \infty.$$

Proof. It suffices to show that $\sum_k \pm\Psi_k^{n,\varepsilon}$ is a multiplier for $L^p(\mathbb{R}^n)$, $1 < p < \infty$, independently of the choice of $\pm$; the result then follows by a standard Rademacher function argument. We use the formulation of the Marcinkiewicz multiplier theorem given in [**2**]. So, we let ϕ^n be a $C_0^\infty(\mathbb{R}^n)$ function such that $0 \leq \phi^n \leq 1$ and

$$\phi^n(\zeta) = \begin{cases} 1 & 1 \leq |\zeta_j| \leq 2, \ j = 1, \dots, n \\ 0 & \text{off } \frac{1}{2} \leq |\zeta_j| \leq 4, \ j = 1, \dots, n, \end{cases}$$

and define $L_\alpha^2(\mathbb{R}^n)$ to be the n-parameter Sobolev space given by

$$L_\alpha^2(\mathbb{R}^n) = \left\{ g : \|g\|_{L_\alpha^2}^2 = \int |\hat{g}(\zeta)|^2 \prod_{i=1}^n (1+\zeta_i^2)^\alpha d\zeta < \infty \right\}.$$

Then, by Theorem A of [**2**], it suffices to show that

$$\sup_{i_1,\dots,i_n} \left\| \sum_k \pm\Psi_k^{n,\varepsilon}(2^{i_1}\zeta_1, \dots, 2^{i_n}\zeta_n)\phi^n(\zeta) \right\|_{L_\alpha^2(\mathbb{R}^n)} < \infty, \tag{21}$$

for some $\alpha > \frac{1}{2}$.

We show (21) for $\alpha = 1$ and for convenience take $\varepsilon = \frac{1}{2^{2n}}$. Our proof is by induction on n; the argument for $n = 2$ is contained in the inductive step and therefore we omit it.

Suppose, therefore, that

$$\sup_{i_1,\dots,i_{n-1}} \left\| \sum_k \pm\Psi_k^{n-1,\widetilde{\varepsilon}}(2^{i_1}\zeta_1, \dots, 2^{i_{n-1}}\zeta_{n-1})\phi^{n-1}(\zeta) \right\|_{L_1^2(\mathbb{R}^{n-1})} < \infty,$$

with $\widetilde{\varepsilon} = \frac{1}{2^{2n-2}}$, under the hypothesis that

$$\frac{\alpha_{k+1}^j}{\alpha_k^j} \geq 2\frac{\alpha_{k+1}^{j-1}}{\alpha_k^{j-1}}, \quad \forall\, k \in \mathbb{Z},\ j = 2, \dots, n-1 \tag{22}$$

and consider

$$\sup_{i_1,\dots,i_n}\left\|\sum_k \pm\Psi_k^{n,\varepsilon}(2^{i_1}\zeta_1,\dots,2^{i_n}\zeta_n)\phi^n(\zeta)\right\|_{L_1^2(\mathbb{R}^n)},$$

assuming that (22) now holds also for $j=n$.

We suppose that, for some k, $\Psi_k^{n,\varepsilon}(2^{i_1}\zeta_1,\dots,2^{i_n}\zeta_n)\phi^n(\zeta)\neq 0$, i.e.

$$\sum_{j=1}^n \alpha_k^j 2^{i_j}\zeta_j < 2\varepsilon\sum_{j=1}^n \alpha_k^j 2^{i_j}|\zeta_j|$$

$$\sum_{j=1}^n \alpha_{k+1}^j 2^{i_j}\zeta_j > -2\varepsilon\sum_{j=1}^n \alpha_{k+1}^j 2^{i_j}|\zeta_j|$$

and

$$\frac{1}{2}\le|\zeta_j|\le 4,\quad j=1,\dots,n.$$

Case 1. Suppose that for some $j_0\in\{1,\dots,n\}$

$$\alpha_k^{j_0}2^{i_{j_0}}|\zeta_{j_0}|\le\frac{1}{2^{2n}}\sum_{j=1}^n\alpha_k^j2^{i_j}|\zeta_j|$$

and

$$\alpha_{k+1}^{j_0}2^{i_{j_0}}|\zeta_{j_0}|\le\frac{1}{2^{2n}}\sum_{j=1}^n\alpha_{k+1}^j2^{i_j}|\zeta_j|.$$

In this instance we find that

$$\begin{aligned}&\Psi_k^{n,\varepsilon}(2^{i_1}\zeta_1,\dots,2^{i_n}\zeta_n)\neq 0\\ &\qquad\Longrightarrow \Psi_k^{n-1,\widetilde{\varepsilon}}(2^{i_1}\zeta_1,\dots,2^{i_{j_0-1}}\zeta_{j_0-1},2^{i_{j_0+1}}\zeta_{j_0+1},\dots,2^{i_n}\zeta_n)\neq 0.\end{aligned}$$

Taking $2^{i_{j_0}}=1$, which we may by homogeneity of $\Psi^{n,\varepsilon}$, the problem is reduced to the $(n-1)$-dimensional case and we are done, by the inductive hypothesis.

Case 2. Suppose that for all $j\in\{1,\dots,n\}$ either

$$\alpha_k^j2^{i_j}|\zeta_j|\ge\frac{1}{2^{2n}}\sum_{j=1}^n\alpha_k^j2^{i_j}|\zeta_j| \tag{23}$$

or

$$\alpha_{k+1}^j2^{i_j}|\zeta_j|\ge\frac{1}{2^{2n}}\sum_{j=1}^n\alpha_{k+1}^j2^{i_j}|\zeta_j|. \tag{24}$$

Let us suppose that

$$\alpha_k^n 2^{i_n}|\zeta_n| \geq \frac{1}{2^{2n}} \sum_{j=1}^{n} \alpha_k^j 2^{i_j}|\zeta_j|.$$

Then by the lacunarity conditions (20) we have

$$\alpha_{k+m}^n 2^{i_n}|\zeta_n| \geq 2 \sum_{j=1}^{n-1} \alpha_{k+m}^j 2^{i_j}|\zeta_j| \quad \forall\, m \geq N, \text{ say.}$$

Then if $\zeta_n > 0$ we find

$$\begin{aligned}\sum_{j=1}^{n} \alpha_{k+m}^j 2^{i_j}\zeta_j &\geq \alpha_{k+m}^n 2^{i_n}|\zeta_n| - \sum_{j=1}^{n-1} \alpha_{k+m}^j 2^{i_j}|\zeta_j| \\ &\geq \frac{1}{3} \sum_{j=1}^{n} \alpha_{k+m}^j 2^{i_j}|\zeta_j|,\end{aligned}$$

whilst if $\zeta_n < 0$ we have

$$\sum_{j=1}^{n} \alpha_{k+m}^j 2^{i_j}\zeta_j \leq -\frac{1}{3} \sum_{j=1}^{n} \alpha_{k+m}^j 2^{i_j}|\zeta_j|.$$

Thus

$$\Psi_{k+m}^{n,\varepsilon}(2^{i_1}\zeta_1, \ldots, 2^{i_n}\zeta_n) = 0 \quad \forall\, m \geq N. \tag{25}$$

If we assume that (24) holds for $j = n$ then the same argument follows. Further, $\forall\ \zeta$ with $\Psi_k^{n,\varepsilon}(2^{i_1}\zeta_1, \ldots, 2^{i_n}\zeta_n) \neq 0$, for each $j_0 \in \{1, \ldots, n\}$, we have either

$$\alpha_k^{j_0} 2^{i_{j_0}}|\zeta_{j_0}| \leq \frac{1+2\varepsilon}{1-2\varepsilon} \sum_{j \neq j_0} \alpha_k^j 2^{i_j}|\zeta_j|$$

or

$$\alpha_{k+1}^{j_0} 2^{i_{j_0}}|\zeta_{j_0}| \leq \frac{1+2\varepsilon}{1-2\varepsilon} \sum_{j \neq j_0} \alpha_{k+1}^j 2^{i_j}|\zeta_j|.$$

This, together with (23), (24), lacunarity and $\phi^n(\zeta) \neq 0$ gives that $2^{i_j} \sim 1 \forall j = 1, \ldots, n$. So using (25) we obtain

$$\sup_{i_1,\ldots,i_n} \left| \sum_k \pm \Psi_k^{n,\varepsilon}(2^{i_1}\zeta_1, \ldots, 2^{i_n}\zeta_n)\phi^n(\zeta) \right| < \infty.$$

It is trivial to check that differentiating with respect to any ζ_j causes no problem. This concludes the proof. □

Let us now see how may apply Theorem 4.3 to our cones C_k. It is clear that if we have a Littlewood-Paley theory for each $\{C_{km}\}$, $\{\widetilde{C}_{km}\}, m = 1, \ldots, n-1$, where we consider $C_{km}, \widetilde{C}_{km}$ as cones in $\mathbb{R}^{n-m+1}$, then this will suffice to give a Littlewood-Paley theory for the C_k. We define now

$$\Phi_{km}(\zeta) = \Psi_k^{n,\varepsilon}(0, \ldots, 0, \zeta_m, L_m\gamma_{m+1}(\lambda^k)\zeta_{m+1}, \ldots, L_m\gamma_n(\lambda^k)\zeta_n)$$
$$\times \Psi_k^{n,\varepsilon}(0, \ldots, 0, -\zeta_m, -L_m\gamma_{m+1}(\lambda^{k+1})\zeta_{m+1}, \ldots, -L_m\gamma_n(\lambda^{k+1})\zeta_n)$$

and put

$$\Phi_k(\zeta) = \sum_{m=1}^{n-1} \Phi_{km}(\zeta);$$

we associate to Φ_k the operator S_k given by

$$\widehat{(S_k f)}(\zeta) = \Phi_k(\zeta)\hat{f}(\zeta). \tag{26}$$

PROPOSITION 4.4. *If*

$$\frac{L_m\gamma_{j+1}(\lambda^{k+1})}{L_m\gamma_j(\lambda^{k+1})} \geq 2\frac{L_m\gamma_{j+1}(\lambda^k)}{L_m\gamma_j(\lambda^k)}, \tag{27}$$

$\forall\, k \in \mathbb{Z}$, $j = m, \ldots, n-1$; $m = 1, \ldots, n-1$, *then*

$$\left\| \left(\sum_k |S_k f|^2 \right)^{1/2} \right\|_p \leq C\|f\|_p, \quad 1 < p < \infty.$$

Proof. If, for fixed m,

$$\frac{L_m\gamma_{j+1}(\lambda^{k+1})}{L_m\gamma_j(\lambda^{k+1})} \geq 2\frac{L_m\gamma_{j+1}(\lambda^k)}{L_m\gamma_j(\lambda^k)} \quad \forall\, k \in \mathbb{Z},\ j = m, \ldots, n$$

then the family of cones $\{C_{km}\}$, and hence also $\{\widetilde{C}_{km}\}$, may be considered as lacunary in $\mathbb{R}^{n-m+1}$,i.e. $\sum_k \pm\Phi_{km}(\zeta)$ is a multiplier in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Thus if (27) is satisfied we have that $\sum_k \sum_{m=1}^{n-1} \pm\Phi_{km}(\zeta)$ is a multiplier for $L^p(\mathbb{R}^n)$. This gives the result. □

Thus, assuming (27), the cones C_k give a Littlewood-Paley decomposition of $\mathbb{R}^n$. Let us now see how the $\frac{1}{2}n(n-1)$ conditions of (27) relate to the conditions in the statement of our theorem, i.e. (9).

LEMMA 4.5. *Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a "convex" curve such that $\Gamma \in C^n(0,\infty)$, $\Gamma(0) = 0$ and*

(28)
$$\lim_{t\to 0} L_m\gamma_{k+1}(t) = 0 \text{ for } m = 1,\dots,n-1,\ k = m,\dots,n-1.$$

Suppose $\exists\, 1 < \lambda < \infty$ such that

$$L_k\gamma_{k+1}(\lambda t) \geq 2L_k\gamma_{k+1}(t) \quad k = 1,\dots,n-1. \tag{29}$$

Then $\exists\, 1 < \mu < \infty$ such that

$$\frac{L_m\gamma_{k+1}(\mu t)}{L_m\gamma_k(\mu t)} \geq 2\frac{L_m\gamma_{k+1}(t)}{L_m\gamma_k(t)}, \tag{30}$$

$m = 1,\dots,n-1,\ k = m,\dots,n-1.$

Proof. Fix k. Clearly, by hypothesis (29), (30) holds for $m = k$, with $\mu = \lambda$. We now suppose that (30) holds for $m = j$ and show that it is then also true for $m = j-1$. Now

$$\begin{aligned} L_{j-1}\gamma_k(\lambda t) &= \frac{L_{j-1}\gamma_k(\lambda t)}{L_{j-1}\gamma_{k-1}(\lambda t)} \cdot \frac{L_{j-1}\gamma_{k-1}(\lambda t)}{L_{j-1}\gamma_{k-2}(\lambda t)} \cdots L_{j-1}\gamma_j(\lambda t) \\ &\geq 2L_{j-1}\gamma_k(t), \end{aligned}$$

by Lemma 3.4 and (29). Then

$$\begin{aligned} \frac{L_{j-1}\gamma_{k+1}(\mu^3 t)}{L_{j-1}\gamma_k(\mu^3 t)} &\geq \frac{1}{2} \cdot \frac{L_{j-1}\gamma_{k+1}(\mu^3 t) - L_{j-1}\gamma_{k+1}(\mu^2 t)}{L_{j-1}\gamma_k(\mu^3 t) - L_{j-1}\gamma_k(\mu^2 t)} \\ &\geq \frac{1}{2} \cdot \frac{(L_{j-1}\gamma_{k+1})'(\mu^2 t)}{(L_{j-1}\gamma_k)'(\mu^2 t)}, \end{aligned}$$

by the Second Mean Value Theorem and Lemma 3.4. Thus

$$\begin{aligned}\frac{L_{j-1}\gamma_{k+1}(\mu^3 t)}{L_{j-1}\gamma_k(\mu^3 t)} &\geq \frac{1}{2}\cdot\frac{L_j\gamma_{k+1}(\mu^2 t)}{L_j\gamma_k(\mu^2 t)}\\ &\geq 2\frac{L_j\gamma_{k+1}(t)}{L_j\gamma_k(t)}, \text{ by inductive hypothesis}\\ &= 2\frac{(L_{j-1}\gamma_{k+1})'(t)}{(L_{j-1}\gamma_k)'(t)}\\ &\geq 2\frac{L_{j-1}\gamma_{k+1}(t)}{L_{j-1}\gamma_k(t)},\end{aligned}$$

by Second Mean Value Theorem, Lemma 3.4 and hypothesis (28). □

Lemma 4.5 and Proposition 4.4 together give us

PROPOSITION 4.6. *Let* $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}^n$ *be a "convex" curve such that* $\Gamma \in C^n(0,\infty), \Gamma(0) = 0$ *and* $\lim_{t\to 0} L_j\gamma_k(t) = 0, \quad \forall\ j = 1,\dots,n-1;\ k = j+1,\dots,n$. *Suppose that* $\exists\ 1 < \lambda < \infty$ *such that*

$$L_k\gamma_{k+1}(\lambda t) \geq 2L_k\gamma_{k+1}(t), \quad k = 1,\dots,n-1.$$

Then

$$\left\|\left(\sum_k |S_k f|^2\right)^{1/2}\right\|_p \leq C\|f\|_p.$$

In view of Proposition 4.1 (b), which defines the cones for a piecewise-linear curve, we also have a corresponding result for piecewise-linear curves if we replace the hypotheses of Proposition 4.6 with those of Corollary 1.3.

5. Proof of Theorem 1.2. We now have a family of dilation matrices $\{A_k\}$ satisfying

(31) $$\exists\ \alpha \text{ such that } \|A_{k+1}^{-1}A_k\| \leq \alpha < 1$$

(32) $$A_{k+1}^{-1}\operatorname{supp}\mu_k \subseteq \text{ fixed ball}$$

and a family of cones $\{C_k\}$ with associated operators S_k given by (26) satisfying the Littlewood-Paley inequality

$$\left\| \left(\sum_k |S_k f|^2 \right)^{1/2} \right\|_p \leq C\|f\|_p \tag{33}$$

and such that

$$\zeta \notin C_k \Longrightarrow |\hat{\mu}_k(\zeta)| \leq C|A_k^* \zeta|^{-1}. \tag{34}$$

We let $f = S_k f + (I - S_k)f$, $k \in \mathbb{Z}$, and consider first $\sup_k |\mu_k * f|$. We use the standard technique of combining a bootstrapping argument with the Littlewood-Paley theory to obtain an L^p-result, starting with just the L^2-result. Now,

$$\begin{aligned} \left\| \sup_k |\mu_k * f| \right\|_2 &\leq \left\| \sup_k |\mu_k * S_k f| \right\|_2 + \left\| \sup_k |\mu_k * (I - S_k) f| \right\|_2 \\ &= A + B. \end{aligned}$$

By (33), Plancherel's theorem and the fact that the μ_k have unit mass we immediately have

$$A \leq C\|f\|_2.$$

For B we use comparison of μ_k with a measure ν_k given by

$$\nu_k(x) = \rho\left(A_{k+1}^{-1} x\right) \det A_{k+1}^{-1},$$

where $\rho \in C_0^\infty$, $0 \leq \rho \leq 1$ and $\int \rho = 1$. It is easily verified that $\sup_k |\nu_k * f|$ is majorized by the Hardy-Littlewood maximal operator associated to balls $A_k B$, where B is the unit ball in $\mathbb{R}^n$, and thus, by [**3**], Proposition 2.2,

$$\left\| \sup_k |\nu_k * f| \right\|_p \leq C\|f\|_p, \quad 1 < p < \infty. \tag{35}$$

Then

$$\begin{aligned} B \leq &\left\| \sup_k |(\mu_k - \nu_k) * (I - S_k) f| \right\|_2 + \left\| \sup_k |\nu_k * f| \right\|_2 \\ &+ \left\| \sup_k |\nu_k * S_k f| \right\|_2. \end{aligned}$$

Now, by the same argument as used for A,

$$\left\| \sup_k |\nu_k * S_k f| \right\|_2 \le C\|f\|_2,$$

so it remains to show that

$$\left\| \sup_k |(\mu_k - \nu_k) * (I - S_k) f| \right\|_2 \le C\|f\|_2.$$

Taking into account (34) the proof of this is essentially contained in the proof of Proposition 5.1, [**3**]. To pass from the L^2-result to the L^p-result we have the following analogue of Proposition 5.1, [**3**].

PROPOSITION 5.1. *Suppose*

$$\left\| \sup_k |\mu_k * f| \right\|_{\tilde{p}} \le C\|f\|_{\tilde{p}}, \qquad \textit{for some } 1 < \tilde{p} \le 2.$$

Then

$$\left\| \sup_k |\mu_k * f| \right\|_p \le C\|f\|_p \quad \forall\, p > \frac{2\tilde{p}}{\tilde{p}+1}.$$

Proof. First we note that, under the hypothesis of the proposition,

$$\left\| \left(\sum_k |\mu_k * f|^2 \right)^{1/2} \right\|_p \le C \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_p, \tag{36}$$

$\forall\, \frac{2\tilde{p}}{\tilde{p}+1} < p < \frac{2\tilde{p}}{\tilde{p}-1}$, exactly as in [**3**]. Then

$$\begin{aligned}\left\| \sup_k |\mu_k * f| \right\|_p \le{}& \left\| \left(\sum_k |\mu_k * S_k f|^2 \right)^{1/2} \right\|_p + \left\| \left(\sum_k |\nu_k * S_k f|^2 \right)^{1/2} \right\|_p \\ &+ \left\| \sup_k |\nu_k * f| \right\|_p + \left\| \sup_k |(\mu_k - \nu_k) * (I - S_k) f| \right\|_p \\ ={}& A + B + D + E.\end{aligned}$$

Now (36) together with (33) gives suitable bounds for A and B, $\forall \frac{2\tilde{p}}{\tilde{p}+1} < p < \frac{2\tilde{p}}{\tilde{p}-1}$, whilst $D \le C\|f\|_p$, $\forall 1 < p < \infty$, by (35). It remains, therefore, to bound E. Again the proof that $E \leq C\|f\|_p$, $\forall \frac{2\tilde{p}}{\tilde{p}+1} < p$ is essentially contained in Proposition 5.1, [**3**]. □

Proposition 5.1 completes the proof of L^p-boundedness of $\sup_k |\mu_k * f|$ and thence of $\mathcal{M}_\Gamma$. Noting that from (33) we may also obtain

$$\left\|\sum_k S_k f_k\right\|_p \leq C \left\|\left(\sum_k |S_k f_k|^2\right)^{1/2}\right\|_p, \quad 1 < p < \infty,$$

we may now deduce the result for $\mathcal{H}_\Gamma$ from that for $\mathcal{M}_\Gamma$, following the argument in [**3**].

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