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DIAGONALIZING HILBERT CUSP FORMS

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We develop an operator $C_{\mathfrak{q}}(\Psi_Q)$ on the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ of Hilbert cuspforms as an alternative to the Hecke operator $T_{\mathfrak{q}}$ for primes $\mathfrak{q}$ dividing $\mathcal{N}$. For $\mathbf{f} \in \mathcal{S}_k(\mathcal{N}, \Psi)$ a newform, we have $\mathbf{f} \mid C_{\mathfrak{q}}(\Psi_Q) = \mathbf{f} \mid T_{\mathfrak{q}}$. We are able to decompose the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ into a direct sum of common eigenspaces of $\{T_{\mathfrak{p}},\ C_{\mathfrak{q}}(\Psi_Q) : \mathfrak{p} \nmid \mathcal{N},\ \mathfrak{q} \mid \mathcal{N}\}$, each of dimension one. Each common eigenspace is spanned by an element with the property that its eigenvalue with respect to $T_{\mathfrak{p}}$ (resp. $C_{\mathfrak{q}}(\Psi_Q)$) is its $\mathfrak{p}^{\text{th}}$ (resp $\mathfrak{q}^{\text{th}}$) Fourier coefficient. We finish by deriving bounds for the eigenvalues of $C_{\mathfrak{q}}(\Psi_Q)$.

Introduction. Let $\mathcal{S}_k(\mathcal{N}, \Psi)$ denote the space of Hilbert cusp forms of Hecke character Ψ. Shemanske and Walling [**7**] characterized the newform theory for $\mathcal{S}_k(\mathcal{N}, \Psi)$ which is analogous to that derived in [**1**] for the elliptic modular case. They decompose the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ into a direct sum of common eigenspaces for the Hecke operators $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$. The non-zero elements of the one-dimensional common eigenspaces are called newforms, and a newform can be normalized such that its $\mathfrak{p}^{\text{th}}$ Fourier coefficient is equal to its eigenvalue for $T_{\mathfrak{p}}$. They also show that each common eigenspace of $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$ has a basis of the form $\{\mathbf{g} \mid B_{\mathfrak{L}} : \mathbf{g} \in \mathcal{S}_k(\mathcal{M}, \Psi)$ a newform , $\mathcal{M} \mid \mathcal{N}, \mathfrak{L} \mid \mathcal{N}\mathcal{M}^{-1}\}$. While the Hecke operators $\{T_{\mathfrak{q}} : \mathfrak{q} \mid \mathcal{N}\}$ act invariantly on these eigenspaces, there generally does not exist a basis for these eigenspaces which consists of eigenforms for $\{T_{\mathfrak{q}} : \mathfrak{q} \mid \mathcal{N}\}$.

In this work, we resolve this particular difficulty by replacing $T_{\mathfrak{q}}$, $\mathfrak{q} \mid \mathcal{N}$ by the operator $C_{\mathfrak{q}}(\Psi_Q)$. It is defined using the Hecke operator $T_{\mathfrak{q}}$ and the Hilbert analog of the Atkin-Lehner W_Q operator of [**7**], and hence depends upon a choice of Hecke character Ψ_Q. We are able to diagonalize the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ with respect to the family $\{T_{\mathfrak{p}},\ C_{\mathfrak{q}}(\Psi_Q) : \mathfrak{p} \nmid \mathcal{N},\ \mathfrak{q} \mid \mathcal{N}\}$. Further, we are able to establish that each common eigenspace is one-dimensional and is spanned by a form whose $\mathfrak{p}$th (resp $\mathfrak{q}^{\text{th}}$) Fourier coefficient is its eigenvalue with respect to $T_\mathfrak{p}$ (resp. $C_\mathfrak{q}(\Psi_Q)$) (Theorem 2.7). In addition, for a newform $\mathbf{f} \in \mathcal{S}_k(\mathcal{N}, \Psi)$, we show that $\mathbf{f} \mid C_\mathfrak{q}(\Psi_Q) = \mathbf{f} \mid T_\mathfrak{q}$ regardless of the choice made for Ψ_Q, and hence the newform theory of [**7**] is left intact when we replace $T_\mathfrak{q}$ by $C_\mathfrak{q}(\Psi_Q)$. Our results generalize those in the elliptic modular case, where the C_q operator was first introduced by Pizer in [**5**] for trivial character, and by Li in [**4**] for non-trivial character.

We finish by investigating the eigenvalues of $C_\mathfrak{q}(\Psi_Q)$. In the case of elliptic cusp forms, one has the sharp Deligne bound of $2q^{(k-1)/2}$ for the magnitude of the q^{th} Fourier coefficient of a newform. In the Hilbert case, the best corresponding bound is Shahidi's bound of $2N(\mathfrak{q})^{(k-1)/2+1/5}$ given in [**6**]. If one tries to adapt the methods of [**4**] to the Hilbert case, this weaker bound gives rise to complications when dealing with ideals of low norm. Because of these difficulties, we implement a significantly different method of proof to arrive at bounds for the magnitude of the eigenvalues of $C_\mathfrak{q}(\Psi_Q)$ (Theorem 3.2). Essentially, the bound is $2N(\mathfrak{q})^{k/2}$, except for the case where $\mathfrak{q} \| \mathcal{N}$ or $N(\mathfrak{q}) < 11$.

1. Notation. For the most part, we follow the notation of [**9**] and [**10**]. Let K be a totally real number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}$ and different $\mathfrak{d}$. Let $\mathcal{H}$ denote the complex upper half-plane, and $GL_2^+(K)$ be the group of 2×2 matrices with entries in K and totally positive determinant. We define an action of $GL_2^+(K)$ on $\mathcal{H}^n$ by

$$A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \ldots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}} \right)$$

where $a^{(i)}$ denotes the i-th conjugate of a over $\mathbb{Q}$. Also, for $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, we denote the product $\prod_i (c^{(i)} z_i + d^{(i)})^{k_i}$ by $(cz + d)^k$ and $\prod_i (a^{(i)} d^{(i)} - b^{(i)} c^{(i)})^{k_i}$ by $(\det A)^k$.

Define for $N \in \mathbb{Z}_+$ the set $\Gamma_N = \{A \in SL_2(\mathcal{O}) : A - I_2 \in NM_2(\mathcal{O})\}$, and denote by $M_k(\Gamma_N)$ the complex vector space of all holomorphic functions f on $\mathcal{H}^n$ such that
$f(A \cdot z) = (\det\ A)^{-k/2}(cz + d)^k f(z)$ for $A \in \Gamma_N$ and which are holomorphic at all of the cusps of Γ_N. Let $M_k = \cup_{N=1}^{\infty} M_k(\Gamma_N)$.

For an integral ideal $\mathcal{N}$ and a fractional ideal $\mathcal{I}$, set

$$\Gamma_0(\mathcal{N},\ \mathcal{I})$$
$$= \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O} & \mathcal{I}^{-1}\mathfrak{d}^{-1} \\ \mathcal{N}\mathcal{I}\mathfrak{d} & \mathcal{O} \end{pmatrix} : \det A \in \mathcal{O}^\times,\ \det A \gg 0 \right\}.$$

By a numerical character modulo $\mathcal{N}$, we mean a character $\psi : (\mathcal{O}/\mathcal{N})^\times \to \mathbb{C}^\times$. As in [**9**] and [**10**], we define for a numerical character ψ modulo $\mathcal{N}$ and a character θ on the totally positive units, the space $M_k(\Gamma_0(\mathcal{N},\ \mathcal{I}),\ \psi,\ \theta)$ which consists of all functions $f \in M_k$ such that

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z\right) = \psi(a)\theta(ad-bc)(ad-bc)^{-k/2}(cz+d)^k f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathcal{N}, \mathcal{I})$. As in [**10**, (9.20)], we shall assume that $\psi(\epsilon)\theta(\epsilon^2) = \operatorname{sgn}(\epsilon)^k$ for all $\epsilon \in \mathcal{O}^\times$. This imposes no real restriction since without this assumption, the space of modular forms is zero. We note the existence of an $m \in \mathbb{R}^n$ such that $\theta(a) = a^{im}$ for all totally positive units a. While this m is not unique, we will fix an m which satisfies the previous equality for the remainder of the article.

Fix a complete set of strict ideal class representatives $\mathcal{I}_1, \ldots, \mathcal{I}_h$, and denote $\Gamma_0(\mathcal{N},\ \mathcal{I}_\lambda)$ by Γ_λ. Then we put

$$\mathfrak{M}_k(\mathcal{N}, \psi, \theta) = \prod_{i=1}^{h} M_k(\Gamma_\lambda,\ \psi,\ \theta).$$

We are interested in the h-tuples $(f_1, \ldots, f_h) \in \mathfrak{M}_k(\mathcal{N}, \psi, \theta)$.

In order to make the notation easier to handle, we follow Shimura and describe the above h-tuples as functions on an idele group. To do this, we must define an assortment of objects. Let $K_A^\times$ denote the set of ideles of K and let G_A be the adelization of $GL_2(K)$, which can be identified with $GL_2(K_A)$. Note that $GL_2(K)$ can be embedded in G_A as the set of diagonal elements, and when viewed this way, they will be denoted G_K. Also, let $G_\infty = GL_2(\mathbb{R})^n$ and $G_{\infty+} = GL_2^+(\mathbb{R})^n$. In the following, we will use $\tilde{a},\ \tilde{b},\ \tilde{c}, \ldots$ to denote elements of $K_A^\times$ and $w,\ x,\ y,\ z$ to denote elements of G_A. If $\mathcal{N}$ is an integral ideal of $\mathcal{O}$ and $\mathfrak{p}$ a prime ideal, define the subsets $Y_\mathfrak{p}(\mathcal{N})$

and $W_{\mathfrak{p}}(\mathcal{N})$ of $GL_2(K_{\mathfrak{p}})$ as follows:

$$Y_{\mathfrak{p}}(\mathcal{N}) = \left\{ x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O}_{\mathfrak{p}} & \mathfrak{d}^{-1}\mathcal{O}_{\mathfrak{p}} \\ \mathcal{N}\mathfrak{d}\mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \end{pmatrix} : \right. \\ \left. \det x \in K_{\mathfrak{p}}^{\times}, (a\mathcal{O}_{\mathfrak{p}},\ \mathcal{N}\mathcal{O}_{\mathfrak{p}}) = 1 \right\},$$

$$W_{\mathfrak{p}}(\mathcal{N}) = \{ x \in Y_{\mathfrak{p}}(\mathcal{N}) : \mathrm{ord}_{\mathfrak{p}}(\det x) = 0 \} .$$

We then use these to define

$$Y(\mathcal{N}) = G_A \cap \left(G_{\infty+} \times \prod_{\mathfrak{p}} Y_{\mathfrak{p}}(\mathcal{N}) \right);$$

$$W(\mathcal{N}) = G_{\infty+} \times \prod_{\mathfrak{p}} W_{\mathfrak{p}}(\mathcal{N}).$$

If $\tilde{a} \in K_A^{\times}$, then $\tilde{a}\mathcal{O}$ denotes the fractional ideal of $\mathcal{O}$ which is canonically identified with $\tilde{a}$, and, similarly, for any ideal $\mathcal{I}$ of $\mathcal{O}$, we set $\tilde{a}\mathcal{I} = (\tilde{a}\mathcal{O})\mathcal{I}$. Also, let $(\tilde{a})_{\mathcal{N}}$ (resp. $(\tilde{a})_0$, $(\tilde{a})_{\infty}$) denote the $\mathcal{N}$-th part (resp. finite part, infinite part) of $\tilde{a}$. Fix h elements $\tilde{t}_1, \ldots, \tilde{t}_h$ in $K_A^{\times}$ such that $\tilde{t}_{\lambda}\mathcal{O} = I_{\lambda}$, $(\tilde{t}_{\lambda})_{\infty} = 1$ and for each $\tilde{t}_{\lambda}$, define $x_{\lambda} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \tilde{t}_{\lambda} \end{smallmatrix}\right)$. Also, fix $\tilde{t}_{\mathfrak{d}}$ so that $\tilde{t}_{\mathfrak{d}}\mathcal{O} = \mathfrak{d}$, $(\tilde{t}_{\mathfrak{d}})_{\infty} = 1$. By Strong Approximation, one can see that

$$G_A = \bigcup_{\lambda=1}^{h} G_K x_{\lambda} W(\mathcal{N}) = \bigcup_{\lambda=1}^{h} G_K x_{\lambda}^{-\iota} W(\mathcal{N})$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{\iota} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

Given a numerical character ψ modulo $\mathcal{N}$, define a homomorphism $\psi_Y : Y(\mathcal{N}) \to \mathbb{C}^{\times}$ by $\psi_Y\left(\left(\begin{smallmatrix} \tilde{a} & * \\ * & * \end{smallmatrix}\right)\right) = \psi(\tilde{a}_{\mathcal{N}} \mod \mathcal{N})$. Following Shimura [**10**, (9.20)], if $(f_1, \ldots, f_h) \in \mathfrak{M}_k(\mathcal{N}, \psi, \theta)$, we define the $\mathbb{C}$–valued function $\mathbf{f}$ on G_A by

$$\mathbf{f}(\alpha x_{\lambda}^{-\iota} w) = \psi_Y(w^{\iota}) \det\, (w_{\infty})^{im} f_{\lambda} \| w_{\infty}(\mathbf{i})$$

where $\alpha \in G_K$, $w \in W(\mathcal{N})$, $\mathbf{i} = (i,\ i, \ldots,\ i)$, and

$$f_{\lambda} \| \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = (ad - bc)^{k/2}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

Given $\mathbf{f}$, one can recover $f_1, \ldots, f_h$, and thus we say $\mathbf{f} = (f_1, \ldots, f_h)$.

As in [**9**] and [**10**], we identify $\mathfrak{M}_k(\mathcal{N},\psi,\theta)$ with the functions $\mathbf{f}: G_A \to \mathbb{C}$ such that

i) $\mathbf{f}(\alpha x w) = \psi_Y(w^\iota)\mathbf{f}(x)$ for all $\alpha \in G_K$, $x \in G_A$, $w \in W(\mathcal{N})$, with $w_\infty = 1$ and

ii) For every λ, there exists an $f_\lambda \in M_k$ such that $\mathbf{f}(x_\lambda^{-\iota} w_\infty) = \det(w_\infty)^{im} f_\lambda \| w_\infty(\mathbf{i})$ for all $w_\infty \in G_{\infty+}$.

We denote the space of such functions by $\mathfrak{M}_k(\mathcal{N},\psi,m)$ where $m \in \mathbb{R}^n$ is the fixed element with $\theta(a) = a^{im}$ for all totally positive units a. We denote the corresponding subspace of cusp forms by $\mathfrak{S}_k(\mathcal{N},\psi,m)$.

By a Hecke character, we shall mean a multiplicative character Ψ on $K_A^\times$ such that $\Psi(\tilde{a}) = 1$ for all $\tilde{a} \in K^\times$. We will denote numerical characters by lower case Greek letters, and Hecke characters by upper case Greek letters. Let $\psi_\infty : K_A^\times \to \mathbb{C}^\times$ be given by $\psi_\infty(\tilde{a}) = \operatorname{sgn}(\tilde{a}_\infty)^k |a_\infty|^{2im}$, with $m \in \mathbb{R}^n$ as above. We then say that a Hecke character Ψ extends $\psi\psi_\infty$ if $\Psi(\tilde{a}) = \psi(\tilde{a}_\mathcal{N} \mod \mathcal{N})\psi_\infty(\tilde{a})$ for all $\tilde{a} \in K_\infty^\times \times \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^\times$. If the previous equality holds for $\psi_\infty(\tilde{a}) = \operatorname{sgn}(\tilde{a}_\infty)^k$, then we say Ψ extends ψ. Let Ψ be a character of $K_A^\times$, and denote by $\mathcal{M}_k(\mathcal{N},\ \Psi)$ the subspace of $\mathfrak{M}_k(\mathcal{N},\psi,m)$ consisting of $\mathbf{f}$ such that $\mathbf{f}(\tilde{s}x) = \Psi(\tilde{s})\mathbf{f}(x)$ for all $\tilde{s} \in K_A^\times$. Since $\mathbf{f}(\tilde{s}x) = \mathbf{f}(x)$ for all $\tilde{s} \in K^\times$, we have $\mathcal{M}_k(\mathcal{N},\ \Psi) = \{0\}$ unless Ψ is a Hecke character, and, in addition, by [**9**, (9.22)], we know such a Ψ must extend $\psi\psi_\infty$. It is shown in [**11**] that there exists only h such Hecke characters, and that $\mathfrak{M}_k(\mathcal{N},\psi,m) = \oplus_\Psi \mathcal{M}_k(\mathcal{N},\ \Psi)$. Let $\mathcal{S}_k(\mathcal{N},\ \Psi)$ denote the subspace of cusp forms in $\mathcal{M}_k(\mathcal{N},\ \Psi)$.

It is easy to show that if $\mathcal{N}$ is the K-modulus of ψ, then the conductor of a Hecke character Ψ which extends $\psi\psi_\infty$ divides $\mathcal{N}\mathfrak{P}_\infty$. This allows us to define an ideal class character Ψ^* modulo $\mathcal{N}\mathfrak{P}_\infty$ by

$$\Psi^*(\mathfrak{q}) = \begin{cases} 0 & \text{if } (\mathfrak{q},\ \mathcal{N}) \neq 1 \\ \Psi(\tilde{\pi}_\mathfrak{q}) & \text{if } (\mathfrak{q},\ \mathcal{N}) = 1, \end{cases}$$

where $(\tilde{\pi}_\mathfrak{q})_\infty = 1$ and $\tilde{\pi}_\mathfrak{q}\mathcal{O} = \mathfrak{q}$. Observe that for any $\tilde{a} \in K_A^\times$ such that $(\tilde{a}\mathcal{O},\ \mathcal{N}) = 1$, we have $\Psi(\tilde{a}) = \Psi^*(\tilde{a}\mathcal{O})\psi(\tilde{a}_\mathcal{N})\psi_\infty(\tilde{a})$. In addition, note that both Ψ and Ψ^* have modulus 1.

Let $\mathbf{f} = (f_1, \ldots, f_h) \in \mathcal{M}_k(\mathcal{N},\ \Psi)$, with $f_\lambda \in M_k(\Gamma_\lambda,\ \psi,\theta)$. Then

f_λ has the Fourier expansion

$$f_\lambda(z) = a_\lambda(0) + \sum_{0 \ll \xi \in \mathcal{I}_\lambda} a_\lambda(\xi) \exp(2\pi i Tr(\xi z)).$$

As in Shimura, we set

$$C(\mathfrak{m}, \mathbf{f}) = \begin{cases} N(\mathfrak{m})^{k_0/2} a_\lambda(\xi) \xi^{-k/2 - im} & \text{if } \mathfrak{m} = \xi I_\lambda^{-1} \subseteq \mathcal{O} \\ 0 & \text{if } \mathfrak{m} \not\subseteq \mathcal{O} \end{cases}$$

where $k_0 = \max\{k_1, \dots, k_n\}$. We call $C(\mathfrak{m}, \mathbf{f})$ the Fourier coefficient of $\mathbf{f}$ at $\mathfrak{m}$, and we use these Fourier coefficients to associate a Dirichlet series to $\mathbf{f}$, namely

$$D(w, \mathbf{f}) = \sum_{\mathfrak{m} \subseteq \mathcal{O}} C(\mathfrak{m}, \mathbf{f}) N(\mathfrak{m})^{-w}.$$

Note that, while the Fourier coefficients of $\mathbf{f}$ determine $\mathbf{f}$, the Dirichlet series does not.

Finally, we define some basic operators on elements of $\mathfrak{M}_k(\mathcal{N}, \psi, m)$. For more details, one is directed to [**9**]. First, we define the slash operator for $\mathbf{f} \in \mathfrak{M}_k(\mathcal{N}, \psi, m)$ and $z \in G_A$ by $\mathbf{f} \mid z(x) = \mathbf{f}(xz^\iota)$. For $\mathfrak{n}$ an ideal of $\mathcal{O}$, we follow [**9**] and define $\mathbf{f} \mid B_\mathfrak{n} = N(\mathfrak{n})^{-k_0/2} \mathbf{f} \mid \left(\begin{smallmatrix} 1 & 0 \\ 0 & \tilde{\mathfrak{n}}^{-1} \end{smallmatrix}\right)$, where $\tilde{\mathfrak{n}} \in K_A^\times$ is such that $\tilde{\mathfrak{n}}\mathcal{O} = \mathfrak{n}$ and $\tilde{\mathfrak{n}}_\infty = 1$. One can then show that $B_\mathfrak{n}$ maps $\mathcal{M}_k(\mathcal{N}, \ \Psi)$ to $\mathcal{M}_k(\mathcal{N}\mathfrak{n}, \ \Psi)$, and $C(\mathfrak{m}, \mathbf{f} \mid B_\mathfrak{n}) = C(\mathfrak{m}\mathfrak{n}^{-1}, \mathbf{f})$. Thus, $\mathbf{f} \mid B_\mathfrak{n} \mid B_\mathfrak{m} = \mathbf{f} \mid B_{\mathfrak{n}\mathfrak{m}}$. Finally, for $\mathfrak{m}$ an integral ideal of $\mathcal{O}$, we have from [**9**] the Hecke operator $T_\mathfrak{m}^\mathcal{N}$ of level $\mathcal{N}$. It is shown that $T_\mathfrak{m} = T_\mathfrak{m}^\mathcal{N}$ maps $\mathcal{M}_k(\mathcal{N}, \ \Psi)$ to $\mathcal{M}_k(\mathcal{N}, \ \Psi)$, regardless of whether $(\mathfrak{m}, \ \mathcal{N}) = 1$ and [**9**, 2.20] gives $C(\mathfrak{m}, T_\mathfrak{n}) = \sum_{\mathfrak{m}+\mathfrak{n} \subseteq \mathfrak{a}} \Psi^*(\mathfrak{a}) N(\mathfrak{a})^{k_0 - 1} C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathbf{f})$. We note that both $B_\mathfrak{n}$ and $T_\mathfrak{n}$ take cusp forms to cusp forms.

2. The $C_\mathfrak{q}(\Psi_Q)$ operator. In this section, we introduce the operator $C_\mathfrak{q}(\Psi_Q)$ and develop its properties. For the most part, these properties mimic those of $T_\mathfrak{q}$, $\mathfrak{q} \mid \mathcal{N}$, with the additional property that $C_\mathfrak{q}(\Psi_Q)$ is normal with respect to the Petersson inner product. We then establish a multiplicity one condition on $\mathcal{S}_k(\mathcal{N}, \Psi)$ with respect to the operators $\{T_\mathfrak{p}, \ C_\mathfrak{q}(\Psi_Q) : \mathfrak{p} \nmid \mathcal{N}, \ \mathfrak{q} \mid \mathcal{N}\}$ (Theorem 2.7).

Fix a space $\mathcal{M}_k(\mathcal{N}, \Psi) \subseteq \mathfrak{M}_k(\mathcal{N}, \psi, m)$, where Ψ is a Hecke character which extends $\psi\psi_\infty$. To define the $C_\mathfrak{q}(\Psi_Q)$ operator, we will need the Hilbert analog of the Atkin-Lehner W_Q operator, as defined in [7]. For the convenience of the reader, we state its definition as follows. For a prime divisor $\mathfrak{q}$ of $\mathcal{N}$, let $\mathcal{Q} = \mathfrak{q}^{\mathrm{ord}_\mathfrak{q}(\mathcal{N})}$, and choose a Hecke character Ψ_Q which extends ψ_Q (here, we write $\psi = \psi_Q \psi_{\mathcal{N}\mathcal{Q}^{-1}}$, where ψ_Q (resp. $\psi_{\mathcal{N}\mathcal{Q}^{-1}}$) is a character modulo $\mathcal{Q}$ (resp. modulo $\mathcal{N}\mathcal{Q}^{-1}$)). In the following, if $\psi_Q \equiv 1$, we will always choose $\Psi_Q \equiv 1$ to extend it. Choose a matrix $y = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right) \in G_A$ so that $y_\infty = 1$, $\det y\mathcal{O} = \mathcal{Q}$ and $\tilde{a}\mathcal{O}$, $\tilde{d}\mathcal{O} \subseteq \mathcal{Q}$, $\tilde{b}\mathcal{O} \subseteq \mathfrak{d}^{-1}$, and $\tilde{c}\mathcal{O} \subseteq \mathcal{N}\mathfrak{d}$. Then the W_Q operator for $\mathbf{f} \in \mathcal{S}_k(\mathcal{N}, \Psi)$ is defined by

$$
\begin{aligned}
&\mathbf{f} \mid W_Q(\Psi_Q)(x) \\
&\qquad = \overline{\Psi_Q}(\det x)\overline{\psi_Q}(\tilde{b}\tilde{t}_\mathfrak{d} \mod \mathcal{Q})\overline{\psi_\mathcal{M}}(\tilde{a} \mod \mathcal{M})\mathbf{f} \mid y(x).
\end{aligned}
$$

This operator is independent of the choice of $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, and sends $\mathcal{S}_k(\mathcal{N}, \Psi)$ to $\mathcal{S}_k(\mathcal{N}, \Psi\overline{\Psi}_Q^2)$. Define $C_\mathfrak{q}(\Psi_Q)$ as follows

$$
C_\mathfrak{q}(\Psi_Q) = \begin{cases} T_\mathfrak{q} & \text{if } \psi \text{ is not a character mod } \mathcal{N}\mathfrak{q}^{-1} \\[2ex] T_\mathfrak{q} + W_Q(1)T_\mathfrak{q}W_Q^{-1}(1) + N(\mathfrak{q})^{(k_0/2)-1}W_Q(1) & \\ \qquad \text{if } \psi \text{ is a character mod } \mathcal{N}\mathfrak{q}^{-1} \text{ and } \mathfrak{q} \| \mathcal{N} & \\[2ex] T_\mathfrak{q} + W_Q(\Psi_Q)T_\mathfrak{q}W_Q^{-1}(\Psi_Q) & \\ \qquad \text{if } \psi \text{ is a character mod } \mathcal{N}\mathfrak{q}^{-1} \text{ and } \mathfrak{q}^2 \mid \mathcal{N}. & \end{cases}
$$

Here, $T_\mathfrak{q}$ is as in [9], and $W_Q^{-1}(\Psi_Q) = \psi_Q(-1)\overline{\Psi}\Psi_Q(\tilde{\pi}_\mathfrak{q})W_Q(\overline{\Psi}_Q)$, by [7, Proposition 2.2]. It is easy to check that the above is an endomorphism of the space $\mathcal{S}_k(\mathcal{N}, \Psi)$.

In what follows, we let $\mathfrak{p}$ denote a prime which does not divide $\mathcal{N}$, and let $\mathfrak{q}$ denote a prime which does divide $\mathcal{N}$. Also, suppose we have fixed a Hecke character Ψ_Q which extends ψ_Q for each $\mathfrak{q} \mid \mathcal{N}$. We now establish properties of $C_\mathfrak{q}(\Psi_Q)$.

PROPOSITION 2.1. *$C_\mathfrak{q}(\Psi_Q)$ commutes with $T_\mathfrak{p}$, $\mathfrak{p} \nmid \mathcal{N}$, and $C_{\mathfrak{q}'}(\Psi_{Q'})$,*

$\mathfrak{q}' \mid \mathcal{N}$.

Proof. This is because

$$\begin{aligned} T_{\mathfrak{p}}T_{\mathfrak{q}} &= T_{\mathfrak{q}}T_{\mathfrak{p}}, \\ T_{\mathfrak{p}}W_Q(\Psi_Q) &= \Psi_Q^*(\mathfrak{p})W_Q(\Psi_Q)T_{\mathfrak{p}}, \\ T_{\mathfrak{q}}T_{\mathfrak{q}'} &= T_{\mathfrak{q}'}T_{\mathfrak{q}}, \\ \text{and} \quad T_{\mathfrak{q}'}W_Q(\Psi_Q) &= \Psi_Q^*(\mathfrak{q}')W_Q(\Psi_Q)T_{\mathfrak{q}'} \end{aligned}$$

for $\mathfrak{q}' \neq \mathfrak{q}$ by [**7**, Proposition 2.4]. □

Thus, $C_{\mathfrak{q}}(\Psi_Q)$ preserves a common eigenspace of $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$ on $\mathcal{S}_k(\mathcal{N}, \Psi)$.

PROPOSITION 2.2. *$C_{\mathfrak{q}}(\Psi_Q)$ commutes with $B_{\mathfrak{L}}$ if $\mathfrak{q} \nmid \mathfrak{L}$.*

Proof. Recall $C_{\mathfrak{q}}(\Psi_Q)$ takes $\mathcal{S}_k(\mathcal{N}, \Psi)$ to $\mathcal{S}_k(\mathcal{N}\mathfrak{L}, \Psi)$. As the definition of $C_{\mathfrak{q}}(\Psi_Q)$ depends only upon the order of $\mathfrak{q}$ dividing $\mathcal{N}$ and the conductor of ψ, we have that $C_{\mathfrak{q}}(\Psi_Q)$ is the same on $\mathcal{S}_k(\mathcal{N}, \Psi)$ as it is on $\mathcal{S}_k(\mathcal{N}\mathfrak{L}, \Psi)$. Thus, we have $C_{\mathfrak{q}}(\Psi_Q)B_{\mathfrak{L}} = B_{\mathfrak{L}}C_{\mathfrak{q}}(\Psi_Q)$ because $T_{\mathfrak{q}}B_{\mathfrak{L}} = B_{\mathfrak{L}}T_{\mathfrak{q}}$ and $W_Q(\Psi_Q)B_{\mathfrak{L}} = \Psi_Q^*(\mathfrak{L})B_{\mathfrak{L}}W_Q(\Psi_Q)$ on $\mathcal{S}_k(\mathcal{N}, \Psi)$ if $\mathfrak{q} \nmid \mathfrak{L}$, by [**7**, Proposition 2.3]. □

The Petersson inner product on $\mathcal{S}_k(\mathcal{N}, \Psi)$ is defined to be $\langle \mathbf{f}, \mathbf{g} \rangle = \sum_\lambda \langle f_\lambda, g_\lambda \rangle$, where the inner product on $M_k(\Gamma_\lambda, \psi, m)$ is given by [**9**, (2.27)]. To gain some insight into how $C_{\mathfrak{q}}(\Psi_Q)$ acts with respect to this inner product, we examine how it acts on component functions. Before we do this, we set some notation. Given $x = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\pi}_{\mathfrak{q}} \end{pmatrix} \in Y(\mathcal{N})$, we can find for each λ an element $a_\lambda \in x_\lambda Y(\mathcal{N}) x_\mu^{-\iota} \cap G_K$ such that $x_\lambda x = a_\lambda x_\mu w$, with $w \in W(\mathcal{N})$. Given a_λ, define the set $\{v_{\lambda j}\}_{j=1}^s \subset x_\lambda Y(\mathcal{N}) x_\mu^{-\iota} \cap G_K$ to be a common set of coset representatives of $\Gamma_\lambda a_\lambda \Gamma_\mu$, i.e., $\Gamma_\lambda a_\lambda \Gamma_\mu = \cup_{j=1}^s \Gamma_\lambda v_{\lambda j} = \cup_{j=1}^s v_{\lambda j} \Gamma_\mu$. With this notation, we can state

PROPOSITION 2.3. *If $\mathbf{f} \in \mathcal{S}_k(\mathcal{N}, \Psi)$, then*

i) $$(\mathbf{f} \mid T_{\mathfrak{q}})_\lambda = N(\mathfrak{q})^{k_0/2-1} \sum_{j=1}^s (\det v_{\lambda j})^{-im} \psi_Y(x_\lambda^{-1} v_{\lambda j} x_\mu)^{-1} f_\mu \| v_{\lambda j}$$

ii) $$(\mathbf{f} \mid W_Q(\Psi_Q) T_{\mathfrak{q}} W_Q^{-1}(\Psi_Q))_\lambda$$
$$= \Psi\overline{\Psi_Q}(\tilde{\pi}_{\mathfrak{q}}) N(\mathfrak{q})^{k_0/2-1} \sum_{j=1}^s (\det v_{\lambda j})^{im} \psi_Y(x_\lambda^{-1} v_{\lambda j} x_\mu) f_\mu \| v_{\lambda j}^\iota.$$

Proof. Part i) is simply a restatement of [**10**, (9.24)]. To prove ii), we first remark that [**9**, (2.10)] and tedious but straightforward manipulations give us

$$\begin{aligned}&\mathbf{f} \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}})T_{\mathfrak{q}}W_{\mathcal{Q}}^{-1}(\Psi_{\mathcal{Q}})(z)\\ &\qquad = N(\mathfrak{q})^{k_0/2-1}\sum_{j=1}^{s}(\psi\overline{\psi}_{\mathcal{Q}}^{2})_Y(x_j)^{-1}\overline{\Psi}_{\mathcal{Q}}(\det x_j)\mathbf{f} \mid y^{-1}x_jy(z),\end{aligned}$$

where $W(\mathcal{N})\left(\begin{smallmatrix}1&0\\0&\pi_{\mathfrak{q}}\end{smallmatrix}\right)W(\mathcal{N}) = \cup_{j=1}^{s}W(\mathcal{N})x_j$, and y is as in the above definition of $W_{\mathcal{Q}}(\Psi_{\mathcal{Q}})$. One can check that $W(\mathcal{N})\left(\begin{smallmatrix}\tilde{\pi}_{\mathfrak{q}}&0\\0&1\end{smallmatrix}\right)W(\mathcal{N}) = \cup_{j=1}^{s}W(\mathcal{N})y^{-1}x_jy = \cup_{j=1}^{s}y^{-1}W(\mathcal{N})x_jy$, and, in addition, we have $W(\mathcal{N})\left(\begin{smallmatrix}\tilde{\pi}_{\mathfrak{q}}&0\\0&1\end{smallmatrix}\right)W(\mathcal{N}) = \cup_{j=1}^{s}W(\mathcal{N})x_\mu^{\iota}(v_{\lambda j}^{\iota})_0x_\lambda^{-\iota}$, by [**8**, Proposition 2.3]. Thus, in the following computations, we can let $\{yx_\mu^{\iota}(v_{\lambda j}^{\iota})_0x_\lambda^{-\iota}y^{-1}\}$ play the role of $\{x_j\}$ in the above.

Let $z \in \mathcal{H}^n$, and let $w_\infty \in GL_2(\mathbb{R})^n$ be such that $w_\infty\mathbf{i} = z$, and let $\mathbf{f}' = (f_1', \ldots, f_h') = \mathbf{f} \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}})T_{\mathfrak{q}}W_{\mathcal{Q}}^{-1}(\Psi_{\mathcal{Q}})$. We then have

$$\begin{aligned}f_\lambda'(z) &= f_\lambda'\|w_\infty(\mathbf{i}) = (\det w_\infty)^{-im}(\mathbf{f} \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}})T_{\mathfrak{q}}W_{\mathcal{Q}}^{-1}(\Psi_{\mathcal{Q}}))(x_\lambda^{-\iota}w_\infty)\\ &= (\det w_\infty)^{-im}\sum_{j=1}^{s}(\overline{\psi}\psi_{\mathcal{Q}}^{2})_Y(yx_\mu^{\iota}(v_{\lambda j}^{\iota})_0x_\lambda^{-\iota}y^{-1})\\ &\qquad\qquad \cdot\overline{\Psi}_{\mathcal{Q}}(\det x_\mu^{\iota}(v_{\lambda j}^{\iota})_0x_\lambda^{-\iota})\mathbf{f} \mid (x_\mu^{\iota}(v_{\lambda j}^{\iota})_0x_\lambda^{-\iota})(x_\lambda^{-\iota}w_\infty)\\ &= \sum_{j=1}^{s}(\det v_{\lambda j})^{-im}(\overline{\psi}\psi_{\mathcal{Q}}^{2})_Y(yx_\mu^{\iota}(v_{\lambda j}^{\iota})_0x_\lambda^{-\iota}y^{-1})\\ &\qquad\qquad \cdot\Psi\overline{\Psi}_{\mathcal{Q}}(\det x_\mu^{\iota}(v_{\lambda j})_0x_\lambda^{-\iota})f_\mu\|v_{\lambda j}^{\iota}(z)\end{aligned}$$

by a series of uncomplicated calculations, and using the fact that all of the above matrices, with the exception of w_∞, have trivial infinite parts.

Let $(v_{\lambda j})_0 = \left(\begin{smallmatrix}\tilde{a}_j&*\\ *&\tilde{d}_j\end{smallmatrix}\right)$. If we let $\mathcal{M} = \mathcal{N}\mathcal{Q}^{-1}$, then the above equation simplifies to

$$\begin{aligned}f_\lambda'(z) = \sum_{j=1}^{s}&\left[\psi_{\mathcal{M}}(\tilde{t}_\mu\tilde{a}_j\tilde{d}_j\tilde{t}_\lambda^{-1} \mod \mathcal{M})\Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{t}_\mu\tilde{a}_j\tilde{d}_j\tilde{t}_\lambda^{-1})\right]\\ &\qquad\qquad\qquad \cdot(\det v_{\lambda j})^{-im}\psi(\tilde{a}_j \mod \mathcal{N})f_\mu\|v_{\lambda j}^{\iota}.\end{aligned}$$

Note that $\psi(\tilde{a}_j \mod \mathcal{N}) = \psi_Y(x_\lambda^{-1}v_{\lambda j}x_\mu)$.

To complete the proof, we need only compute the term in brackets. Since $W(\mathcal{N})\left(\begin{smallmatrix}\tilde{\pi}_{\mathfrak{q}} & 0\\ 0 & 1\end{smallmatrix}\right)W(\mathcal{N}) = \cup_{j=1}^{s} W(\mathcal{N})x_{\mu}^{\iota}(v_{\lambda j}^{\iota})_0 x_{\lambda}^{-\iota}$, there exists, for each $j = 1, \ldots, s$, matrices $w_{1j},\ w_{2j} \in W(\mathcal{N})$ such that $w_{1j}\left(\begin{smallmatrix}\tilde{\pi}_{\mathfrak{q}} & 0\\ 0 & 1\end{smallmatrix}\right)w_{2j} = x_{\mu}^{\iota}(v_{\lambda j}^{\iota})_0 x_{\lambda}^{-\iota}$, and hence

$$\tilde{\pi}_{\mathfrak{q}} = (\det w_{1j}w_{2j})^{-1}\tilde{t}_{\mu}\tilde{a}_j\tilde{d}_j\tilde{t}_{\lambda}^{-1},$$

for each j. As $(\det w_{1j}w_{2j})^{-1} \in K_{\infty}^{\times} \times \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^{\times}$, we have

$$\Psi\overline{\Psi}_{\mathcal{Q}}((\det\ w_{1j}w_{2j})^{-1}) = \psi_{\mathcal{M}}((\det w_{1j}w_{2j})^{-1}),$$

and hence the bracketed term is equal to

$$\overline{\psi}_{\mathcal{M}}(\tilde{\pi}_{\mathfrak{q}})\Psi\Psi_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}}) = \Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}}).$$

This finishes the proof. □

REMARK. Let $\mathbf{f},\ \mathbf{g} \in \mathcal{S}_k(\mathcal{N}, \Psi) \subset \mathfrak{S}_k(\mathcal{N}, \psi, m)$. If ψ is not a character modulo $\mathcal{N}\mathfrak{q}^{-1}$, then, by definition, we have $\mathbf{f} \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}}) = \mathbf{f} \mid T_{\mathfrak{q}}$. Hence, in general, there is no relation between $\langle \mathbf{f} \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}}), \mathbf{g}\rangle$ and $\langle \mathbf{f},\ \mathbf{g} \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}})\rangle$. This is not the case if ψ is a character modulo $\mathcal{N}\mathfrak{q}^{-1}$, as can be seen in

PROPOSITION 2.4. *If* $\mathbf{f},\ \mathbf{g} \in \mathcal{S}_k(\mathcal{N}, \Psi) \subseteq \mathfrak{S}(\mathcal{N}, \psi, m)$, *and* ψ *is character modulo* $\mathcal{N}\mathfrak{q}^{-1}$, *then* $\langle \mathbf{f} \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}}), \mathbf{g}\rangle = \Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}})\langle \mathbf{f}, \mathbf{g} \mid C_{\mathfrak{q}}(\Psi_{\mathcal{Q}})\rangle$, *where* $\langle , \rangle$ *is the Petersson inner product of* [**9**, (2.28)] *on* $\mathcal{S}_k(\mathcal{N}, \Psi)$.

Proof. We first prove that

$$\left\langle \mathbf{f} \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}})T_{\mathfrak{q}}W_{\mathcal{Q}}^{-1}(\Psi_{\mathcal{Q}}),\ \mathbf{g}\right\rangle = \Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}})\langle \mathbf{f},\ \mathbf{g} \mid T_{\mathfrak{q}}\rangle.$$

We have, by definition,

$$\left\langle \mathbf{f} \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}})T_{\mathfrak{q}}W_{\mathcal{Q}}^{-1}(\Psi_{\mathcal{Q}}),\ \mathbf{g}\right\rangle = \sum_{\lambda=1}^{h}\left\langle \left(\mathbf{f} \mid W_{\mathcal{Q}}(\Psi_{\mathcal{Q}})T_{\mathfrak{q}}W_{\mathcal{Q}}^{-1}(\Psi_{\mathcal{Q}})\right)_{\lambda}, g_{\lambda}\right\rangle.$$

Let Γ be a congruence subgroup such that $M_k(\Gamma)$ contains both $\left(\mathbf{f} \mid W_Q(\Psi_Q)T_{\mathfrak{q}}W_Q^{-1}(\Psi_Q)\right)_\lambda$ and g_λ. Then

$$\begin{aligned}
&\mu(\Gamma\backslash\mathcal{H}^n)\left\langle\left(\mathbf{f} \mid W_Q(\Psi_Q)T_{\mathfrak{q}}W_Q^{-1}(\Psi_Q)\right)_\lambda,\ g_\lambda\right\rangle \\
&= \int_{\Gamma\backslash\mathcal{H}^n} \Psi\overline{\Psi}_Q(\tilde{\pi}_{\mathfrak{q}}) \cdot \\
&\quad \sum_{j=1}^{s}(\det\ v_{\lambda j})^{im}\psi_Y(x_\lambda^{-1}v_{\lambda j}x_\mu)f_\mu \| v^{\iota}{}_{\lambda j}(z)\overline{g_\lambda}(z)\ y^k\ d\mu(z) \\
&= \int_{\Gamma\backslash\mathcal{H}^n} \Psi\overline{\Psi}_Q(\tilde{\pi}_{\mathfrak{q}})f_\mu(z) \cdot \\
&\quad \sum_{j=1}^{s}\overline{(\det\ \ v_{\lambda j})^{-im}\psi_Y(x_\lambda^{-1}v_{\lambda j}x_\mu)^{-1}g_\lambda \| v_{\lambda j}(z)}\ y^k\ d\mu(z) \\
&= \mu(\Gamma\backslash\mathcal{H}^n)\Psi\overline{\Psi}_Q(\tilde{\pi}_{\mathfrak{q}})\langle f_\mu,\ (\mathbf{g} \mid T_{\mathfrak{q}})_\mu\rangle.
\end{aligned}$$

Hence, $\langle \mathbf{f} \mid W_Q(\Psi_Q)T_{\mathfrak{q}}W_Q^{-1}(\Psi_Q), \mathbf{g}\rangle = \Psi\overline{\Psi}_Q(\tilde{\pi}_{\mathfrak{q}})\langle \mathbf{f}, \mathbf{g} \mid T_{\mathfrak{q}}\rangle$. Similarly, we can show

$$\langle \mathbf{f} \mid T_{\mathfrak{q}},\ \mathbf{g}\rangle = \Psi\overline{\Psi}_Q(\tilde{\pi}_{\mathfrak{q}})\langle \mathbf{f},\ \mathbf{g} \mid W_Q(\Psi_Q)T_{\mathfrak{q}}W_Q^{-1}(\Psi_Q)\rangle.$$

If $\mathfrak{q}\|\mathcal{N}$ and ψ is a character modulo $\mathcal{N}\mathfrak{q}^{-1}$, then [**7**, Proposition 4.1] implies $\langle \mathbf{f} \mid W_Q(1), \mathbf{g}\rangle = \Psi(\mathfrak{q})\langle \mathbf{f}, \mathbf{g} \mid W_Q(1)\rangle$. This completes the proof. □

An immediate consequence of Proposition 2.4 is the following. The proof is a direct generalization of the proof of [**4**, Corollary 2.5].

COROLLARY 2.5. *$C_{\mathfrak{q}}(\Psi_Q)$ is diagonalizable on $\mathcal{S}_k(\mathcal{N}, \Psi)$.*

PROPOSITION 2.6. *If $\mathbf{f} \in \mathcal{S}_k(\mathcal{N}, \Psi) \subseteq \mathfrak{S}_k(\mathcal{N}, \psi, m)$ is a newform, then $\mathbf{f} \mid C_{\mathfrak{q}}(\Psi_Q) = \mathbf{f} \mid T_{\mathfrak{q}}$.*

Proof. The claim is immediate if ψ is not a character modulo $\mathcal{N}\mathfrak{q}^{-1}$. If ψ is a character modulo $\mathcal{N}\mathfrak{q}^{-1}$, then $\mathbf{f} \mid T_{\mathfrak{q}} = 0 = \mathbf{f} \mid W_Q(\Psi_Q)T_{\mathfrak{q}}$ by [**7**, Theorem 3.3(3)], and hence $\mathbf{f} \mid C_{\mathfrak{q}}(\Psi_Q) = 0$ if $\mathfrak{q}^2 \mid \mathcal{N}$. If ψ is a character modulo $\mathcal{N}\mathfrak{q}^{-1}$ and $\mathfrak{q}\|\mathcal{N}$, then the proposition follows from the fact that $\mathbf{f} \mid W_Q(1)T_{\mathfrak{q}}W_Q^{-1}(1) = C(\mathfrak{q}, \mathbf{f})\mathbf{f}$ by [**7**, Theorem 3.3(1)], and $\mathbf{f} \mid W_Q(1) = -N(\mathfrak{q})^{-k_0/2+1}C(\mathfrak{q}, \mathbf{f})\mathbf{f}$ by a straightforward generalization of [**3**, Theorem 3 iii)]. □

REMARK. The above proposition shows that the substitution of the $C_{\mathfrak{q}}(\Psi_Q)$ operator for the Hecke operator $T_{\mathfrak{q}}$ for $\mathfrak{q} \mid \mathcal{N}$ leaves the newform theory of [**7**] intact, regardless of which choice is made for the Hecke character Ψ_Q.

With Propositions 2.1-2.4, and the newform theory of [**7**], one can emulate the proof of [**4**, Theorem 3.6] to arrive at similar results for the Hilbert modular case. As the proof is long, and no substantially new ideas are introduced, we omit it, and state

THEOREM 2.7. *For each $\mathfrak{q} \mid \mathcal{N}$, let Ψ_Q be a Hecke character extending ψ_Q. Then the space $\mathcal{S}_k(\mathcal{N}, \Psi)$ can be decomposed into a direct sum of common eigenspaces of $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$ and $\{C_{\mathfrak{q}}(\Psi_Q) : \mathfrak{q} \mid \mathcal{N}\}$, each of dimension one. In each common eigenspace, there exists a form $\mathbf{h}$ with Dirichlet series*

$$D(w, \mathbf{h}) = \sum_{\mathfrak{m} \subseteq \mathcal{O}} C(\mathfrak{m}, \mathbf{h}) N(\mathfrak{m})^{-w}$$

in which $C(\mathcal{O}, \mathbf{h}) = 1$, $\mathbf{h} \mid T_{\mathfrak{p}} = C(\mathfrak{p}, \mathbf{h})\mathbf{h}$ for all $\mathfrak{p} \nmid \mathcal{N}$, and $\mathbf{h} \mid C_{\mathfrak{q}}(\Psi_Q) = C(\mathfrak{q}, \mathbf{h})\mathbf{h}$ for all $\mathfrak{q} \mid \mathcal{N}$. In addition, for such $\mathbf{h}$, we have $C(\mathfrak{mn}, \mathbf{h}) = C(\mathfrak{m}, \mathbf{h})C(\mathfrak{n}, \mathbf{h})$ for $(\mathfrak{m}, \mathfrak{n}) = 1$.

REMARK. By Proposition 2.6, the newforms of $\mathcal{S}_k(\mathcal{N}, \Psi)$ are among the above mentioned basis elements for $\mathcal{S}_k(\mathcal{N}, \Psi)$.

3. Eigenvalues of $C_{\mathfrak{q}}(\Psi_Q)$. In this section, we find bounds for the eigenvalues of $C_{\mathfrak{q}}(\Psi_Q)$ on $\mathcal{S}_k(\mathcal{N}, \Psi) \subseteq \mathfrak{S}_k(\mathcal{N}, \psi, m)$. To do so, we follow the methods of [**4**] and restrict our attention to a common eigenspace V of $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$ in $\mathcal{S}_k(\mathcal{N}, \Psi)$. We find a polynomial whose distinct roots consist of the eigenvalues of $C_{\mathfrak{q}}(\Psi_Q)$ on V. By determining bounds on the size of this polynomial's roots, we arrive at the bounds presented in Proposition 3.2.

For the following section, we let V be a common eigenspace of the Hecke operators $\{T_{\mathfrak{p}} : \mathfrak{p} \nmid \mathcal{N}\}$, and let $\mathbf{g} \in \mathcal{S}_k(\mathcal{M}, \Psi)$, where $\mathcal{M} \mid \mathcal{N}$, be the newform such that $\{\mathbf{g} \mid B_{\mathfrak{L}} : \mathfrak{L} \mid \mathcal{N}\mathcal{M}^{-1}\}$ generates V. Fix a prime divisor $\mathfrak{q}$ of $\mathcal{N}$, and a Hecke character Ψ_Q which extends ψ_Q. Let $r(\mathfrak{q}) = \mathrm{ord}_{\mathfrak{q}}(\mathcal{N}\mathcal{M}^{-1})$, $t(\mathfrak{q}) = \mathrm{ord}_{\mathfrak{q}}(\mathcal{M})$, and denote by $\tilde{\mathbf{g}}$ the newform in $\mathcal{S}_k(\mathcal{N}, \Psi\overline{\Psi}_Q^2)$ such that $\mathbf{g} \mid W_Q(\Psi_Q) = \lambda\tilde{\mathbf{g}}$. With $\delta_{i,j}$ the

Kronecker delta, we emulate [**4**, (3.3), (3.4)] and define

$$\begin{aligned}
f_{\mathfrak{q},0}(x) &= 1, \qquad f_{\mathfrak{q},1}(x) = x - C(\mathfrak{q},\mathbf{g})\\
f_{\mathfrak{q},2}(x) &= x f_{\mathfrak{q},1}(x) - \Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}})N(\mathfrak{q})^{k_0}\left(1-\delta_{0,t(\mathfrak{q})}N(\mathfrak{q})^{-1}\right) f_{\mathfrak{q},0}(x)\\
&\qquad \text{if } r(\mathfrak{q}) \geq 3\\
f_{\mathfrak{q},s}(x) &= x f_{\mathfrak{q},s-1}(x) - N(\mathfrak{q})^{k_0}\Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}}) f_{\mathfrak{q},s-2}(x)\\
&\qquad \text{for } 3 \leq s \leq r(\mathfrak{q})-1
\end{aligned}$$

$$f_{\mathfrak{q},r(\mathfrak{q})}(x) = \begin{cases} \left(1-\delta_{0,t(\mathfrak{q})}N(\mathfrak{q})^{-1}\right)^{-1} \\ \quad \cdot\left(x f_{\mathfrak{q},r(\mathfrak{q})-1}(x) - N(\mathfrak{q})^{k_0}\Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}}) f_{\mathfrak{q},r(\mathfrak{q})-2}(x)\right) \\ \qquad \text{if } r(\mathfrak{q}) > 2 \\ \left(1-\delta_{0,t(\mathfrak{q})}N(\mathfrak{q})^{-1}\right)^{-1} x f_{\mathfrak{q},1}(x) \\ \quad -\Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}})N(\mathfrak{q})^{k_0} f_{\mathfrak{q},0}(x) \\ \qquad \text{if } r(\mathfrak{q}) = 2 \end{cases}$$

and

$$F_{\mathfrak{q}}(x) = \begin{cases} (x - C(\mathfrak{q},\tilde{\mathbf{g}}))\, f_{\mathfrak{q},r(\mathfrak{q})}(x) - N(\mathfrak{q})^{k_0} & \\ \quad \cdot\Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}}) f_{\mathfrak{q},r(\mathfrak{q})-1}(x) & \text{if } r(\mathfrak{q}) > 0 \\ (x - C(\mathfrak{q},\mathbf{g})) & \text{if } r(\mathfrak{q}) = 0. \end{cases}$$

Following [**4**], it is a straightforward exercise to show that the roots of $F_{\mathfrak{q}}(x)$ are distinct and are, in fact, the eigenvalues of $C_{\mathfrak{q}}(\Psi_{\mathcal{Q}})$.

We now estimate the size of the roots of $F_{\mathfrak{q}}(x)$. If $r(\mathfrak{q}) = 0$, then the root of $F_{\mathfrak{q}}(x)$ is $C(\mathfrak{q},\mathbf{g})$, and thus we may assume $r(\mathfrak{q}) \geq 1$ in the following. The assumption $r(\mathfrak{q}) \geq 1$ implies that ψ is a character modulo $\mathcal{N}\mathfrak{q}^{-1}$, and hence, by Proposition 2.4, we know $\lambda = \Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}})\overline{\lambda}$ for all eigenvalues λ of $C_{\mathfrak{q}}(\Psi_{\mathcal{Q}})$ on V.

As $C(\mathfrak{q},\mathbf{g})$ and $C(\mathfrak{q},\tilde{\mathbf{g}})$ are integral to the definition of $F_{\mathfrak{q}}(x)$, we break our discussion into the following four cases, which result from [**7**, Lemma 4.3, Proposition 3.3]:

$\mathfrak{q} \mid \mathcal{M}$ and ψ is not a character modulo $\mathcal{M}\mathfrak{q}^{-1}$, so that

(3.1)
$$C(\mathfrak{q},\tilde{\mathbf{g}}) = \Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}})\overline{C(\mathfrak{q},\mathbf{g})} \text{ and } |C(\mathfrak{q},\mathbf{g})| = 0 \text{ or } N(\mathfrak{q})^{(k_0-1)/2},$$

$\mathfrak{q} \| \mathcal{M}$ and ψ is a character modulo $\mathcal{M}\mathfrak{q}^{-1}$, so that

$$C(\mathfrak{q},\tilde{\mathbf{g}}) = C(\mathfrak{q},\mathbf{g}) \text{ and } C(\mathfrak{q},\mathbf{g})^2 = \Psi\overline{\Psi}_{\mathcal{Q}}(\tilde{\pi}_{\mathfrak{q}})N(\mathfrak{q})^{k_0-2}, \tag{3.2}$$

$\mathfrak{q}^2 \mid \mathcal{M}$ and ψ is a character modulo $\mathcal{M}\mathfrak{q}^{-1}$, so that

$$C(\mathfrak{q}, \tilde{\mathbf{g}}) = 0 = C(\mathfrak{q}, \mathbf{g}), \tag{3.3}$$

$\mathfrak{q} \nmid \mathcal{M}$, so that $C(\mathfrak{q}, \tilde{\mathbf{g}}) = C(\mathfrak{q}, \mathbf{g}) = \Psi\overline{\Psi}_Q(\tilde{\pi}_\mathfrak{q})\overline{C(\mathfrak{q}, \mathbf{g})}$ and

$$|C(\mathfrak{q}, \mathbf{g})| \leq 2N(\mathfrak{q})^{(k_0-1)/2+1/5}, \tag{3.4}$$

where the last estimate for $|C(\mathfrak{q}, \mathbf{g})|$ is due to Shahidi [**6**].

In cases (3.1), (3.2), and (3.3), the methods of [**4**] for finding bounds on the roots of $F_\mathfrak{q}(x)$ can be generalized easily to the Hilbert case. In addition, if $r(\mathfrak{q}) = 1$, then the proofs of [**4**] for all four cases can be emulated, to get similar results. We will state these results without proof in the final statements of this section.

In case (3.4), however, the methods of [**4**] rely on the sharp Deligne bound of $2q^{(k-1)/2}$ for the modulus of the q^{th} Fourier coefficient of an elliptic newform. In the case of Hilbert cusp forms, the best bound presently known is Shahidi's bound given above. If one tries to adapt the methods of [**4**] to find a bound on the roots of $F_\mathfrak{q}(x)$ in case (3.4), the difference between Shahidi's bound and Deligne's bound gives rise to complications when dealing with ideals of low norm. It is because of these difficulties that we must implement a significantly different method than [**4**] when examining the roots of $F_\mathfrak{q}(x)$ in case (3.4).

Choose a square root $\Psi\overline{\Psi}_Q(\tilde{\pi}_\mathfrak{q})^{1/2}$ of $\Psi\overline{\Psi}_Q(\tilde{\pi}_\mathfrak{q})$. For a complex number λ, define λ' by $\lambda = \Psi\overline{\Psi}_Q(\tilde{\pi}_\mathfrak{q})^{1/2}N(\mathfrak{q})^{k_0/2}\lambda'$. Now, assume $r(\mathfrak{q}) \geq 2$, and define the polynomials

$$\begin{aligned}
f'_0(x) &= 1, \\
f'_1(x) &= x - C(\mathfrak{q}, \mathbf{g})', \\
f'_2(x) &= x f'_1(x) - (1 - N(\mathfrak{q})^{-1}) f'_0(x) \\
f'_s(x) &= x f'_{s-1}(x) - f'_{s-2}(x) \qquad \text{for } s \geq 3
\end{aligned}$$

and further, we define the polynomials

$$\begin{aligned}
G_0(x) = &-\left(N(\mathfrak{q})^{-1}x - C(\mathfrak{q}, \mathbf{g})'\right)\left(N(\mathfrak{q})^{-1}x - C(\mathfrak{q}, \tilde{\mathbf{g}})'\right) \\
&+ \left(1 - N(\mathfrak{q})^{-1}\right)^2
\end{aligned}$$

$$\begin{aligned}
G_1(x) &= x - (C(\mathfrak{q},\mathfrak{g})' + C(\mathfrak{q},\tilde{\mathfrak{g}})') \\
&\qquad + N(\mathfrak{q})^{-1}\left(C(\mathfrak{q},\mathfrak{g})' + C(\mathfrak{q},\tilde{\mathfrak{g}})' - N(\mathfrak{q})^{-1}x\right) \\
G_2(x) &= (x - C(\mathfrak{q},\mathfrak{g})')\,(x - C(\mathfrak{q},\tilde{\mathfrak{g}})') - \left(1 - N(\mathfrak{q})^{-1}\right)^2 \\
G_s(x) &= (x - C(\mathfrak{q},\tilde{\mathfrak{g}})')\, f'_{s-1}(x) - \left(1 - N(\mathfrak{q})^{-1}\right) f'_{s-2}(x) \quad \text{for } s \geq 3.
\end{aligned}$$

If $\gamma = \Psi\overline{\Psi}_Q(\tilde{\pi}_\mathfrak{q})^{1/2}N(\mathfrak{q})^{k_0/2}$, then it is easy to see that, when we are in case (3.4), we have $f_{\mathfrak{q},s}(x) = f'_s(x')\gamma^s$ for $s < r(\mathfrak{q})$ and that $F_\mathfrak{q}(x) = \gamma^{r(\mathfrak{q})+1}(1-N(\mathfrak{q})^{-1})^{-1}G_{r(\mathfrak{q})+1}(x')$. Hence λ is a root of $F_\mathfrak{q}(x)$ iff λ' is a root of $G_{r(\mathfrak{q})+1}(x)$. Note that $G_s(x) = xG_{s-1}(x) - G_{s-2}(x)$ for $s \geq 2$. If we set $H_s(x) = G_s(x + x^{-1})$ for $s \geq 0$, then, for $x^2 \neq 0, 1$, [**2**, Theorem 6.2.2] tells us that $H_s(x) = a(x)x^s + b(x)x^{-s} = G_s(x + x^{-1})$. We now prove

PROPOSITION 3.1. *Suppose we are in case* (3.4), *and that* $x_0 \in \mathbb{C}$ *is a non-zero root of* $H_s(x)$. *Then* $|x_0| = 1$.

Proof. If $x_0 = \pm 1$, then we are done. Thus, assume $x_0 \neq \pm 1$. Recall that, in case (3.4) we have $C(\mathfrak{q},\mathfrak{g})' = C(\mathfrak{q},\tilde{\mathfrak{g}})' \in \mathbb{R}$, and $|C(\mathfrak{q},\mathfrak{g})'| \leq 2N(\mathfrak{q})^{-1/2+1/5}$. This first identity gives us

$$\begin{aligned}
H_0(x_0) &= a(x_0) + b(x_0) \\
&= (1 + N(\mathfrak{q})^{-1})^2 - [N(\mathfrak{q})^{-1}(x_0 + x_0^{-1}) - C(\mathfrak{q},\mathfrak{g})']^2 \\
H_1(x_0) &= a(x_0)x_0 + b(x_0)x_0^{-1} \\
&= x_0 + x_0^{-1} - 2C(\mathfrak{q},\mathfrak{g})' + 2N(\mathfrak{q})^{-1}C(\mathfrak{q},\mathfrak{g})' \\
&\qquad - N(\mathfrak{q})^{-1}(x_0 + x_0^{-1}).
\end{aligned}$$

Linear elimination and simplication yields

$$\begin{aligned}
H_s(x_0) = \left(x_0^2 - 1\right)^{-1} \Big[&\left(N(\mathfrak{q})x_0^2 - C(\mathfrak{q},\mathfrak{g})'N(\mathfrak{q})x_0 + 1\right)^2 x_0^{s-2} \\
&- \left(x_0^2 - C(\mathfrak{q},\mathfrak{g})'N(\mathfrak{q})x_0 + N(\mathfrak{q})\right)^2 x_0^{-s}\Big].
\end{aligned}$$

As we know $H_s(x_0) = 0$ and as we have assumed $x_0^2 \neq 1$, we have

$$x_0^{2s-2} = \left(\frac{x_0^2 - C(\mathfrak{q},\mathfrak{g})'N(\mathfrak{q})x_0 + N(\mathfrak{q})}{N(\mathfrak{q})x_0^2 - C(\mathfrak{q},\mathfrak{g})'N(\mathfrak{q})x_0 + 1}\right)^2.$$

In order for x_0 to satisfy the above equation, it must necessarily satisfy

(3.5)
$$|x_0|^{2s-2} = \left|\frac{x_0^2 - C(\mathfrak{q},\mathbf{g})'N(\mathfrak{q})x_0 + N(\mathfrak{q})}{N(\mathfrak{q})x_0^2 - C(\mathfrak{q},\mathbf{g})'N(\mathfrak{q})x_0 + 1}\right|^2$$
$$= 1 + \frac{(1-N(\mathfrak{q})^2)(|x_0|^2-1)[(1+N(\mathfrak{q}))(|x_0|^2+1) - 2\,\mathrm{Re}(x_0)C(\mathfrak{q},\mathbf{g})'N(\mathfrak{q})]}{|N(\mathfrak{q})x_0^2 - C(\mathfrak{q},\mathbf{g})'N(\mathfrak{q})x_0 + 1|^2}.$$

Note that $1 - N(\mathfrak{q})^2 < 0$, and, in addition,

$$\begin{aligned}(1+N(\mathfrak{q}))\left(|x_0|^2+1\right) &- 2\,\mathrm{Re}(x_0)C(\mathfrak{q},\mathbf{g})'N(\mathfrak{q}) \\ &\geq (1+N(\mathfrak{q}))\left(|x_0|^2+1\right) - (2|x_0|N(\mathfrak{q}))\,/\left(2N(\mathfrak{q})^{1/2-1/5}\right) \\ &= f\left(|x_0|\right)/N(\mathfrak{q})^{1/2-1/5}\end{aligned}$$

with $f(|x|) = N(\mathfrak{q})^{1/2-1/5}(1+N(\mathfrak{q}))(|x|^2+1) - |x|N(\mathfrak{q})$. Using elementary calculus, we find that the absolute minimum of $f(|x|)$ is positive for any prime $\mathfrak{q}$, and so

$$\left[(N(\mathfrak{q})+1)(|x|^2+1) - 2\,\mathrm{Re}(x)C(\mathfrak{q},\mathbf{g})'N(\mathfrak{q})\right] > 0 \text{ for all } x \in \mathbb{C}.$$

By examining (3.5), we see we must have $|x_0| = 1$, for otherwise one side of equation (3.5) is greater than 1, while the other side is less than 1. This finishes the proof. □

Recall that we are looking for zeroes of $F_{\mathfrak{q}}(x)$ by examining zeroes of $G_{r(\mathfrak{q})+1}(x)$. We have $H_{r(\mathfrak{q})+1}(x) = G_{r(\mathfrak{q})+1}(x+x^{-1})$ and the above theorem tells us that in case (3.4), if $x_0 \neq 0$ is a zero of $H_s(x)$, then $|x_0| = 1$. Thus, if z_0 is a root of $G_{r(\mathfrak{q})+1}(x)$ in case (3.4), then z_0 is of the form $2\cos(\theta)$. We noted before that a complex number z_0 is a zero of $G_{r(\mathfrak{q})+1}$ iff $N(\mathfrak{q})^{k_0/2}(\Psi\overline{\Psi}_Q(\tilde{\pi}_{\mathfrak{q}}))^{1/2}z_0$ is a zero of $F_{\mathfrak{q}}(x)$. Hence, if we incorporate what we have shown with the generalized results of [**4**, Theorem 4.5], we have

THEOREM 3.2. *Let $r(\mathfrak{q}) > 0$. Then the roots of the polynomial $F_{\mathfrak{q}}(x)$ are distinct and of the form $N(\mathfrak{q})^{k_0/2}(\Psi\overline{\Psi}_Q(\tilde{\pi}_{\mathfrak{q}}))^{1/2}\lambda'$ where λ' is as follows: if $r(\mathfrak{q}) = 1$, then $\lambda' = C(\mathfrak{q},\mathbf{g})' \pm 1$ in cases (3.2)–(3.4), and $(1/2)(C(\mathfrak{q},\mathbf{g})' + \overline{C(\mathfrak{q},\mathbf{g})'} \pm \left((C(\mathfrak{q},\mathbf{g})' - \overline{C(\mathfrak{q},\mathbf{g})'})^2 + 4\right)^{1/2}$ in case (3.1); when $r(\mathfrak{q}) \geq 2$, $|\lambda'| \leq 2$.*

Let $\mathbf{h}$ be an a simultaneous eigenfunction of $C_{\mathfrak{q}}(\Psi_Q)$, $\mathfrak{q} \mid \mathcal{N}$ in V with associated Dirichlet series $\sum_{\mathfrak{a}\subseteq\mathcal{O}} C(\mathfrak{a},\mathbf{h})N(\mathfrak{a})^{-w}$ and $C(\mathcal{O},\mathbf{h}) =$

1. Then $C(\mathfrak{q}, \mathbf{h})$ is the eigenvalue of $\mathbf{h}$ for $C_{\mathfrak{q}}(\Psi_Q)$, and thus the above theorem gives a bound for $|C(\mathfrak{q}, \mathbf{h})|$.

COROLLARY 3.3. *If ψ is not a character modulo $\mathcal{N}\mathfrak{q}^{-1}$, then* $|C(\mathfrak{q}, \mathbf{h})| = N(\mathfrak{q})^{(k_0-1)/2}$ *or* 0. *If ψ is a character modulo $\mathcal{N}\mathfrak{q}^{-1}$, then* $|C(\mathfrak{q}, \mathbf{h})| < 2N(\mathfrak{q})^{k_0/2}$, *except for the case when* $\mathrm{ord}_{\mathfrak{q}}(\mathcal{N}) = 1$ *and* $N(\mathfrak{q}) < 11$. *In this last case,* $|C(\mathfrak{q}, \mathbf{h})| \leq N(\mathfrak{q})^{k_0/2}(2N(\mathfrak{q})^{-1/2+1/5} \pm 1)$.

Proof. The first statement is due to [**7**, Theorem 3.3]. The last statement follows from Theorem 3.2 and from the fact that $|2N(\mathfrak{q})^{-1/2+1/5} \pm 1| < 2$ for $N(\mathfrak{q}) \geq 11$. $\square$

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