GENERALIZATIONS OF HYPERGEODESICS

David B. Dekker

1. Introduction. In a recently published paper [1] the author studied families of hypergeodesic curves on a surface in ordinary euclidean space of three dimensions. Here we wish to define a more general class of curves on a surface which will contain all the hypergeodesics as a subclass and at the same time will possess most of the properties of the subclass of hypergeodesics.

The summation convention of tensor analysis with regard to repeated indices will be observed with Greek letter indices taking on the values 1, 2, and Latin letter indices taking on the values 1, 2, 3. The notation of Eisenhart [2] will be used throughout.

2. The differential equation. Consider a surface S in ordinary space of three dimensions represented by the three parametric equations

$$x^i = x^i(u^1, u^2) \qquad (i = 1,2,3)$$

referred to a rectangular cartesian coordinate system. A family of hypergeodesics on S is defined as the set of all solutions $u^\alpha = u^\alpha(s)$ $(\alpha = 1,2)$ of a differential equation [1]

$$K_g = \Omega_{\alpha\beta\gamma} u'^\alpha u'^\beta u'^\gamma, \tag{2.1}$$

where K_g is the expression for geodesic curvature of a curve C given by $u^\alpha = u^\alpha(s)$ $(\alpha = 1,2)$ in which the parameter s is arc-length, the primes indicate differentiation with respect to s, and the $\Omega_{\alpha\beta\gamma}$ are the covariant components of a tensor of the third order relative to transformations of the surface coordinates u^1 and u^2. If we use the scalar Ω to abbreviate the right member of (2.1), the equation reads $K_g = \Omega$ where Ω is a polynomial homogeneous of degree three in the parameters u'^1 and u'^2 with coefficients as analytic functions of u^1 and u^2. Division by $(u'^1)^3$ and some further simplification reduces this differential equation to a form stating that the second derivative of u^2 with respect to u^1 is equal to a cubic in the first derivative.

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In order to retain most of the properties of hypergeodesics for our generalizations we replace only the polynomial Ω by a rational function U/V with the same homogeneity property. Thus a curve $C: u^{\alpha} = u^{\alpha}(s)$ $(\alpha = 1,2)$ will be called a *supergeodesic* if it satisfies a differential equation of the form

$$K_g = W \tag{2.2}$$

where the scalar W is the quotient U/V of the two scalars

$$U \equiv U_{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_n} u'^{\alpha_1} u'^{\alpha_2} u'^{\alpha_3} \cdots u'^{\alpha_n} \tag{2.3}$$

and

$$V \equiv V_{\beta_1 \beta_2 \beta_3 \cdots \beta_{n-3}} u'^{\beta_1} u'^{\beta_2} u'^{\beta_3} \cdots u'^{\beta_{n-3}} . \tag{2.4}$$

If we divide equation (2.2) by $(u'^1)^3$ it reduces to

$$\frac{d}{du^1}\left(\frac{du^2}{du^1}\right) = \frac{A_0 + A_1 \dfrac{du^2}{du^1} + A_2\left(\dfrac{du^2}{du^1}\right)^2 + \cdots + A_n\left(\dfrac{du^2}{du^1}\right)^n}{B_0 + B_1 \dfrac{du^2}{du^1} + B_2\left(\dfrac{du^2}{du^1}\right)^2 + \cdots + B_{n-3}\left(\dfrac{du^2}{du^1}\right)^{n-3}} ; \tag{2.5}$$

or, if we divide by $(u'^2)^3$, a similar equation is obtained.

It is easy to see that there exists a unique solution [3, p.106] to (2.5) at any point (u^1, u^2) in any direction (u'^1, u'^2) for which V does not vanish. Hence, (2.2) defines a two-parameter family of supergeodesics on the surface S with the property that at any point of S there is a supergeodesic in every direction except those directions in which $V = 0$. It may be said that $V = 0$ defines $n - 3$ one-parameter families of curves which are never tangent to any supergeodesic of the family defined by (2.2) within a region R on S at each point of which U and V considered as polynomials in u'^{α} are relatively prime. (This excludes points of S where U and V have common factors.)

Now W is a polynomial in u'^{α} if and only if the family defined by (2.2) is a family of hypergeodesics. Otherwise, (2.2) represents a more general family of supergeodesics.

3. Supergeodesic curvature. We define the *supergeodesic curvature* of a curve $u^{\alpha} = u^{\alpha}(s)$ at a point P of the curve as the scalar K_s given by

$$K_s \equiv K_g - W. \tag{3.1}$$

The *supergeodesic curvature vector* we define as the vector whose contravariant components λ^δ are given by

$$\lambda^\delta \equiv (K_g - W)\,\mu^\delta \equiv K_s\mu^\delta \tag{3.2}$$

where $\mu^\delta \equiv \epsilon^{\gamma\delta} g_{\gamma\eta} u'^\eta$ is a unit vector which makes a right angle with the unit vector u'^α.

We see from (3.1) and (2.2) that *a curve C is a supergeodesic of the family if and only if the supergeodesic curvature along the curve is identically zero in s.*

4. The elements of the cone related to the supergeodesics through a point. The elements of the cone enveloped by the osculating planes of a family of supergeodesics through a point P of the surface S may be determined by the procedure sketched in Section 4 of [1]. The only change is that the symbol Ω be replaced by W. This replacement is possible since only the homogeneity of Ω was used, and W possesses this same homogeneity property. The direction numbers c^h of the element of the cone corresponding to the supergeodesic in the direction u'^α will have the values

$$c^h = \epsilon^{\alpha\beta} \frac{\partial K_n}{\partial \mu'^\beta} x^h{}_{,\alpha} \qquad (K_n = 0, \quad W \neq 0) \tag{4.1}$$

or

$$c^h = r^\alpha x^h{}_{,\alpha} + X^h \qquad (K_n \neq 0, \quad V \neq 0) \tag{4.2}$$

where

$$r^\alpha \equiv \epsilon^{\beta\alpha} \frac{\partial}{\partial u'^\beta} (W/K_n) \qquad (K_n \neq 0, \quad V \neq 0). \tag{4.3}$$

5. A geometric interpretation of supergeodesic curvature. It can be demonstrated that the supergeodesic curvature of a curve $C: u^\alpha = u^\alpha(s)$ is the curvature of the curve C' which is the projection of the curve C upon the tangent plane at the point P, the lines of projection being parallel to that element of the cone determined by the direction u'^α at P. The proof of this property consists of replacing Ω by W in Section 5 of [1]. Of course at points of C for which $K_n = 0$ or $V = 0$ there can be in general no geometric interpretation of this type since the element lies in the tangent plane or does not exist.

6. Supergeodesic torsion of a curve. The torsion at P of the supergeodesic $u^\alpha = u^\alpha(s)$ of the family having the same direction as a curve C at P (if such a supergeodesic of the family exists) will be called the supergeodesic torsion τ_S of C at the point P. The expression for τ_S as found by the calculations in Section 6 of [1] with Ω replaced by W is

$$\begin{aligned}(6.1)\qquad \tau_s = \frac{\csc^2\phi}{L}\,\epsilon_{\beta\delta}\Big\{ r^\beta\Big[(1/L)_{,\alpha} + \frac{r^\gamma}{L}\,d_{\alpha\gamma}\Big] \\ - \Big[(r^\beta/L)_{,\alpha} - \frac{1}{L}\,d_{\alpha\gamma}\,g^{\gamma\beta}\Big]\Big\}\,u'^\alpha u'^\delta \qquad (K_n \neq 0, \quad W \neq 0),\end{aligned}$$

where ϕ is the angle between the vector c^h as given by (4.2) and the unit tangent vector $x^i_{,\sigma}u'^\sigma$, L is the length of the vector c^h, and r^β is defined by (4.3).

7. A geometric condition that a supergeodesic be a plane curve. If we find the differential equation for the special related intersector net of the complex of cone elements for the family of supergeodesics under consideration, it will be exactly the same as the differential equation

$$(7.1)\qquad \tau_s = 0,$$

as can be verified by simply replacing Ω by W in Section 7 of [1]. Now a curve of the special related intersector net is a curve for which the elements c^h at each point of the curve corresponding to the direction u'^α of the curve form a developable surface. Hence, we may state that *a supergeodesic not in an asymptotic direction is a plane curve if and only if it is a curve of the special intersector net of the complex of cone elements.*

Geometrically speaking the theorem reads: *A supergeodesic not in an asymptotic direction is a plane curve if and only if the one-parameter family of cone elements, which are the elements of contact of the osculating planes of the supergeodesic with the cones, constitutes a developable.*

8. A study of the special related intersector net. If we discard the nonzero multipliers from the left side of the differential equation (7.1) and make use of (4.3), the differential equation reads

$$(8.1)\qquad \epsilon_{\beta\delta}\Big\{\Big[\epsilon^{\gamma\beta}\frac{\partial}{\partial u'^\gamma}\Big(\frac{U}{K_n V}\Big)\Big]_{,\alpha} + \epsilon^{\delta\gamma}\frac{\partial}{\partial u'^\delta}\Big(\frac{U}{K_n V}\Big)\,d_{\alpha\gamma} + d_{\alpha\gamma}\,g^{\gamma\beta}\Big\}\,u'^\alpha u'^\delta = 0.$$

After performing the partial and covariant differentiation in (8.1) and clearing the fractions we see that the equation is of the first order and in general (when U and $K_n V$ are relatively prime) of degree $N = 3n - 1$ where n is the degree of homogeneity of U in the parameters u'^{α}.

If U and $K_n V$ have exactly m common factors then the degree N of (8.1) is $3(n - m) - 1$. For hypergeodesics, V divides U so that N is 8 in general. However, for union curves, $K_n V$ divides U so that N is 2. Now union curves are specializations of hypergeodesics in that K_n divides Ω [1]. A similar specialized class of supergeodesics containing the class of union curves as a subclass is obtained when K_n divides U.

9. **Pangeodesics.** In the case of the family of pangeodesics [4, pp. 203-204] on the surface, it is observed that the differential equation is of the type (2.5) and hence the pangeodesics constitute an example of a family of supergeodesics. For the pangeodesics n is 6 so that in general N is 17.

References

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University of Washington