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INDEFINITE KAC-MOODY ALGEBRAS OF SPECIAL LINEAR TYPE

GEORGIA BENKART, SEOK-JIN KANG, KAILASH C. MISRA

From the special linear Lie algebra $A_n = s\ell(n+1,\mathbb{C})$ we construct certain indefinite Kac-Moody Lie algebras $IA_n(a,b)$ and then use the representation theory of A_n to determine explicit closed form root multiplicity formulas for the roots α of $IA_n(a,b)$ whose degree satisfies $|deg(\alpha)| \leq 2a+1$. These expressions involve the well-known Littlewood-Richardson coefficients and Kostka numbers. Using the Euler-Poincaré Principle and Kostant's formula, we derive two expressions, one of which is recursive and the other closed form, for the multiplicity of an arbitrary root α of $IA_n(a,b)$ as a polynomial in Kostka numbers.

Introduction.

For Kac-Moody algebras the root multiplicities of only the finite and affine algebras are explicitly known. In this paper, the third of a series of articles on the structure of non-finite, non-affine Kac-Moody algebras, we study certain indefinite Kac-Moody algebras coming from the special linear Lie algebra $A_n = s\ell(n+1,\mathbb{C})$ of traceless $(n+1)\times(n+1)$ complex matrices. The main theme of these articles is that combinatorial results from the representation theory of classical simple Lie algebras can be applied to the problem of determining root multiplicities for Kac-Moody algebras. The starting point is a well-known construction of graded Lie algebras of Kac-Moody type whose ingredients are a Lie algebra G over $\mathbb{C}$, two G-modules V and V', and a G-module homomorphism $\psi : V' \otimes V \longrightarrow G$. The graded Lie algebra $\mathcal{L} = \mathcal{L}(G,V,V',\psi) = \sum_{k\in\mathbb{Z}} \mathcal{L}_k$ built from these components contains no graded ideals which intersect the local part $V \oplus G \oplus V'$ trivially. The algebra G is specialized to be $g\ell(n+1,\mathbb{C}) = s\ell(n+1,\mathbb{C}) \oplus \mathbb{C}I$. The G-module V is assumed to be $V(b\Lambda_1) = V(b\epsilon_1)$, the irreducible G-module with highest weight b times the first fundamental weight Λ_1, or equivalently b times ϵ_1, where ϵ_1 maps a matrix to its $(1,1)$ entry. The homomorphism ψ is the map given by (2.1) below. A certain parameter "a" enters into the definition of ψ. We argue that the algebra $\mathcal{L}(G,V,V^*,\psi)$ is isomorphic to the Kac-Moody algebra having generalized Cartan matrix

$$\mathcal{C} = \begin{pmatrix} 2 & -b & 0 & \cdots & 0 \\ -a & & & & \\ 0 & & C(A_n) & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

where $C(A_n)$ is the Cartan matrix of Lie algebra A_n. For almost all positive integral values of a and b, the matrix $\mathcal{C}$ is of indefinite type, which we denote by $IA_n(a, b)$.

Our investigations in [**BKM2**] focused on the $b = 1$ case where we determined closed form formulas for the multiplicities of roots $\alpha = k\alpha_0 + k_1\alpha_1 + \cdots + k_n\alpha_n$ whose degree k satisfies $-2a \le k \le 2a$. We also considered in that paper the analogous indefinite algebras $IB_n(a, 1)$, $IC_n(a, 1)$, and $ID_n(a, 1)$ constructed from an algebra G which is a central extension of a simple Lie algebra of type B_n, C_n or D_n respectively.

Section 1 of this present work reviews the basic construction and background results. In the next section the construction is specialized and is shown to give the indefinite algebras $IA_n(a, b)$. In the third section we develop closed form multiplicity formulas for the roots of $IA_n(a, b)$ up to degree $2a + 1$, that is, for all roots in the graded components $\mathcal{L}_k$ for $k = 0, \pm 1, \ldots, \pm(2a + 1)$. The multiplicity formulas involve the well-known Littlewood-Richardson coefficients and Kostka numbers and are similar in spirit to the ones found in [**BKM2**] for the case $b = 1$ and in [**BKM1**] for the case $n = 1$. However, there are added complications which must be dealt with here in going from the $b = 1$ case to the general case.

In the final section we use the Euler-Poincaré Principle and Kostant's formula to derive two expressions, one of which is recursive and the other closed form, for the root multiplicities of the Kac-Moody algebras $IA_n(a, b)$. The closed form formula we obtain is related to the Berman-Moody formula [**BM**] in that ours corresponds to a maximal proper subset of the simple roots, while the Berman-Moody formula corresponds to the empty subset. This connection is explained further in [**Kan3**]. These formulas enable us to write the multiplicity of an arbitrary root of $IA_n(a, b)$ as a polynomial in Kostka numbers.

Many interesting Kac-Moody algebras are indefinite Kac-Moody algebras of special linear type. For example, the rank two Kac-Moody algebras are just the algebras $IA_1(a, b)$. The hyperbolic algebra $HA_1^{(1)}$ studied by Feingold and Frenkel [**FF**] and Kang [**Kan1**] [**Kan2**] is $IA_2(2, 2)$ in our notation, and the hyperbolic algebra $HA_2^{(2)}$, whose root multiplicities have been investigated in [**Kan2**], is $IA_2(4, 1)$. Similarly, the hyperbolic algebra $HG_2^{(1)}$

is $IA_3(3,1)$ and $HD_4^{(3)}$ is $IA_3(1,3)$.

1. The Construction.

We begin this section by recalling some necessary background results which can be found in [**BKM1**]. The first is the construction whose basic ingredients are a Lie algebra G, two G-modules V and V', and a G-module homomorphism $\psi : V' \otimes V \longrightarrow G$. Set $\mathcal{F}_0 = G$, $\mathcal{F}_{-1} = V$, and $\mathcal{F}_1 = V'$. Let $\mathcal{F}^- = \sum_{k\geq 1} \mathcal{F}_{-k}$ (resp. $\mathcal{F}^+ = \sum_{k\geq 1} \mathcal{F}_k$) be the free Lie algebra generated by $\mathcal{F}_{-1}$ (resp. $\mathcal{F}_1$). Then $\mathcal{F}_{-k}$ (resp. $\mathcal{F}_k$) for $k > 1$ is the space of all products of k vectors from $\mathcal{F}_{-1}$ (resp. $\mathcal{F}_1$). In particular, the set of elements $[x_1[x_2, \cdots, [x_{k-1}, x_k] \ldots]]$, where the vectors x_i are chosen from a basis for $\mathcal{F}_{-1}$ (resp. $\mathcal{F}_1$), spans $\mathcal{F}_{-k}$ (resp. $\mathcal{F}_k$). There is a Lie bracket for $\mathcal{F} = \mathcal{F}^- \oplus \mathcal{F}_0 \oplus \mathcal{F}^+$ which extends the products in $\mathcal{F}^-$, $\mathcal{F}^+$, and $\mathcal{F}_0$. Thus, for $g \in \mathcal{F}_0$, $v \in \mathcal{F}_{-1}$, and $w \in \mathcal{F}_1$,

$$(1.1) \qquad [g, x] = g \cdot x = -[x, g] \quad \text{if } x = v, w,$$

and

$$(1.2) \qquad [w, v] = \psi(w \otimes v) = -[v, w].$$

Under this bracket, $\mathcal{F} = \sum_{k\in\mathbb{Z}} \mathcal{F}_k$ becomes a graded Lie algebra which is generated by its *local part* $\mathcal{F}_{-1} + \mathcal{F}_0 + \mathcal{F}_1$.

For $k > 1$ define the subspaces

(1.3)
$$J_{\pm k} = \{x \in \mathcal{F}_{\pm k} \mid [y_1, [\cdots [y_{k-1}, x]] \cdots] = 0 \quad \text{for all } y_1, \ldots, y_{k-1} \in \mathcal{F}_{\mp 1}\},$$

and set

$$(1.4) \qquad J^- = \sum_{k>1} J_{-k}, \qquad J^+ = \sum_{k>1} J_k.$$

Then by ([**BKM1**], Proposition 1.7 or [**FF**], Proposition 4.2) J^- and J^+ are ideals of $\mathcal{F}$, and the ideal $J = J^- \oplus J^+$ is the largest graded ideal of $\mathcal{F}$ trivially intersecting $\mathcal{F}_{-1} + \mathcal{F}_0 + \mathcal{F}_1$. Our main object of study is the graded Lie algebra

$$(1.5) \qquad \begin{aligned} \mathcal{L} = \mathcal{L}(G, V, V', \psi) &\stackrel{\text{def}}{=} \mathcal{F}^-/J^- \oplus \mathcal{F}_0 \oplus \mathcal{F}^+/J^+ \\ &= \cdots \oplus \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \end{aligned}$$

where $\mathcal{L}_i = \mathcal{F}_i$ for $i = \pm 1, 0$ and $\mathcal{L}_i = \mathcal{F}_i/J_i$, for $i \neq \pm 1, 0$. The algebra

$\mathcal{L} = \oplus \sum_{k \in \mathbb{Z}} \mathcal{L}_k$ has no graded ideals which trivially intersect the local part $V \oplus G \oplus V'$.

Let $\mathcal{L}^{\pm} = \mathcal{F}^{\pm}/J^{\pm} = \sum_{k \geq 1} \mathcal{L}_{\pm k}$, and for a fixed choice of G, V, V', ψ, let $m \geq 2$ denote the smallest integer such that $J_{-m} \neq (0)$. Then $J_m \neq (0)$ must hold. We will consider cases where $J^- = \sum_{k \geq m} J_{-k}$ (resp. $J^+ = \sum_{k \geq m} J_k$) is the graded ideal of $\mathcal{F}^-$ (resp. $\mathcal{F}^+$) generated by J_{-m} (resp. J_m).

Suppose now that $G = g\ell(n+1, \mathbb{C}) = s\ell(n+1, \mathbb{C}) \oplus \mathbb{C}I$ and let $\mathcal{H}$ be the Cartan subalgebra of diagonal matrices in G. Assume V is a faithful irreducible highest weight G-module relative to $\mathcal{H}$. Then the dual space V^* of V is a lowest weight module for G. The free Lie algebra $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) generated by $\mathcal{F}_1 = V^*$ (resp. $\mathcal{F}_{-1} = V$) is a module for $\mathcal{F}_0 = G$ with finite dimensional weight spaces relative to $\mathcal{H}$, and the multiplicities of those weight spaces can be computed using the following generalization of the Witt formula.

Proposition 1.6 ([Kan1], [Kan2]). *Let $\Omega = \{\omega_1, \omega_2, \dots\}$ be an enumeration of the weights of $\mathcal{F}_{-1} = V$ relative to $\mathcal{H}$. Then for any weight γ of $\mathcal{F}^-$,*

$$\dim \mathcal{F}^-_\gamma = \sum_{\omega | \gamma} \mu\left(\frac{\gamma}{\omega}\right) \frac{\omega}{\gamma} D(\omega) \tag{1.7}$$

where μ denotes the classical Möbius function; $\omega \mid \gamma$ if $\gamma = \kappa\omega$ for some positive integer κ, in which case $\frac{\gamma}{\omega} = \kappa$ and $\frac{\omega}{\gamma} = \frac{1}{\kappa}$; and

$$D(\omega) = \sum_{(t) \in T(\omega)} \frac{((\sum_i t_i) - 1)!}{\prod_i (t_i!)} \prod_i (\dim V_{\omega_i})^{t_i} \tag{1.8}$$

where $T(\omega) = \{(t) = (t_1, t_2, \dots) \mid t_i \in \mathbb{Z}^{\geq 0}$ and $\sum_i t_i \omega_i = \omega\}$.

The algebra $\mathcal{L} = \mathcal{L}(G, V, V^*, \psi)$ has finite dimensional root spaces relative to $\mathcal{H}$, and to compute the multiplicities of those roots we need additional information about the spaces $J_{\pm k}$ in the ideal J. This information comes from considering the homology module $H_3(\mathcal{L}^-)$ which inherits a $\mathbb{Z}$-grading from that of $\mathcal{L}^- = \mathcal{F}^-/J^-$. Suppose that d is the smallest integer with $H_3(\mathcal{L}^-)_{-d} \neq (0)$. As Kang shows in ([**Kan1**], [**Kan2**]), the value of d determines the structure of certain of the homogeneous components of J:

Proposition 1.9. *Let $\mathcal{F}^- = \sum_{k \geq 1} \mathcal{F}_{-k}$ be the free Lie algebra generated by the G-module $\mathcal{F}_{-1} = V$. Let $J^- = \sum_{k \geq m} J_{-k}$ be the ideal of $\mathcal{F}^-$ generated by $J_{-m} \subseteq \mathcal{F}_{-m}$ for some $m \geq 2$. For $\mathcal{L}^- = \mathcal{F}^-/J^-$ let d be the smallest integer such that $H_3(\mathcal{L}^-)_{-d} \neq (0)$. Then*

(1) *for $m \leq j < \min(2m, d)$,*

$$J_{-j} \cong \underbrace{V \otimes \cdots \otimes V}_{(j-m)\ \text{times}} \otimes J_{-m}.$$

(2) *If $d < 2m$, then*

$$J_{-d} \cong \left(\underbrace{V \otimes \cdots \otimes V}_{(d-m)\ \text{times}} \otimes J_{-m} \right) \Big/ H_3(\mathcal{L}^-)_{-d}.$$

In the next section we consider the algebra $\mathcal{L}(G, V, V^*, \psi)$ under more stringent restrictions on the module V.

2. Indefinite Kac-Moody Algebras of Type $\mathbf{IA}_n$(a,b).

The module $V = V(b\Lambda_1) = V(b\epsilon_1)$ for $G = g\ell(n+1, \mathbb{C})$ can be explicitly realized as the $\mathbb{C}$-vector space of homogeneous polynomials of total degree b in the indeterminates $z_1, \ldots, z_{n+1}$. We adopt the shorthand $z^{\underline{r}}$ for the monomial $z_1^{r_1} z_2^{r_2} \cdots z_{n+1}^{r_{n+1}}$ corresponding to the $(n+1)$-tuple $\underline{r} = (r_1, r_2, \ldots, r_{n+1})$, and let ξ_i be the $(n+1)$-tuple with 1 in the ith position and 0 elsewhere. Then the action of the matrix unit $E_{i,j}$ in $g\ell(n+1, \mathbb{C})$ on $z^{\underline{r}}$ is afforded by

$$E_{i,j} z^{\underline{r}} = r_j z^{\underline{r}+\xi_i-\xi_j},$$

where it is understood that $z^{\underline{r}+\xi_i-\xi_j}$ is 0 if any component of the $(n+1)$-tuple $\underline{r}+\xi_i-\xi_j = (r_1, \ldots, r_i+1, \ldots, r_j-1, \ldots, r_{n+1})$ is negative. Thus, $E_{i,j}$ acts as $z_i \partial/\partial_{z_j}$. Assume V^* is the dual space of V, and let $\{\partial^{\underline{s}} \mid \underline{s} = (s_1, \ldots, s_{n+1})$ and $\sum_{i=1}^{n+1} s_i = b\}$ be the dual basis to the basis $\{z^{\underline{r}}\}$ so that $\partial^{\underline{s}}(z^{\underline{r}}) = \delta_{\underline{s},\underline{r}}$. We define

$$\psi(\partial^{\underline{s}} \otimes z^{\underline{r}}) = \frac{-a}{b} \sum_{i,j=1}^{n+1} r_i \delta_{\underline{s},\underline{r}+\xi_j-\xi_i} E_{i,j} + \left(a - \frac{2}{b}\right) \delta_{\underline{s},\underline{r}} I, \tag{2.1}$$

where I is the identity matrix in $G = g\ell(n+1, \mathbb{C})$. Then ψ is a G-module homomorphism, which can be seen by direct computation or by using ([**FF**], Proposition 4.1) coupled with the fact that the basis of matrix units $E_{i,j}$ forms an orthonormal basis for G relative to the trace form $(g, g') = tr(gg')$.

Suppose $\alpha_0 = -b\epsilon_1$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n$, where $\epsilon_i : \mathcal{H} \longrightarrow \mathbb{C}$ is the projection of a matrix in the Cartan subalgebra $\mathcal{H}$ onto its (i,i) entry. Then the monomial $z^{b\xi_1} = z_1^b$ is a maximal vector for V of highest weight

$b\Lambda_1 = b\epsilon_1$ and $\partial^{b\xi_1}$ is a minimal vector of lowest weight $-b\Lambda_1 = -b\epsilon_1$, and the weights of V are given by

$$(2.2) \qquad \{b\Lambda_1 - k_1\alpha_1 - \cdots - k_n\alpha_n \mid b \geq k_1 \geq k_2 \geq \cdots \geq k_n \geq 0\} =$$

$$\{m_1\epsilon_1 + m_2\epsilon_2 + \cdots + m_{n+1}\epsilon_{n+1} \mid m_i \geq 0 \text{ and } m_1 + \cdots + m_{n+1} = b\}.$$

Let $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$, $h_i = E_{i,i} - E_{i+1,i+1}$, where $i = 1, \ldots, n$, denote the canonical generators for $[G,G] = s\ell(n+1,\mathbb{C})$, and let

$$e_0 = \partial^{b\xi_1} \quad \text{and} \quad f_0 = z^{b\xi_1} = z_1^b.$$

Set

$$h_0 = [e_0, f_0] = -aE_{1,1} + \left(a - \frac{2}{b}\right) I.$$

Then it is easy to verify using the maximality of $f_0 = z^{b\xi_1}$ and the minimality of $e_0 = \partial^{b\xi_1}$ that the relations

$$(2.3) \qquad \begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= \alpha_j(h_i)e_j \\ [h_i, f_j] &= -\alpha_j(h_i)f_j \\ [e_i, f_j] &= \delta_{ij}h_i \end{aligned}$$

hold in $\mathcal{L}(G, V, V^*, \psi)$. If $\mathcal{C} = (\mathcal{C}_{i,j})_{i,j=0}^n$ is the matrix whose (i,j) entry is given by $\mathcal{C}_{i,j} = \alpha_j(h_i)$, then as in [**BKM1**] we have,

Theorem 2.4. *The Lie algebra $\mathcal{L}(G, V, V^*, \psi)$ with $G = g\ell(n+1,\mathbb{C})$, $V = V(b\Lambda_1)$ and ψ as in (2.1) is isomorphic to the Kac-Moody algebra $IA_n(a,b)$ with Cartan matrix*

$$(2.5) \qquad \mathcal{C} = \begin{pmatrix} 2 & -b & \cdots & 0 \; 0 \\ -a & & & \\ 0 & & C(A_n) & \\ \vdots & & & \\ 0 & & & \end{pmatrix},$$

where $C(A_n)$ is the Cartan matrix of $A_n = s\ell(n+1,\mathbb{C})$.

3. Root Multiplicities in $\mathbf{IA}_n$(a,b).

An integral combination $\theta = \theta_1\epsilon_1 + \cdots + \theta_{n+1}\epsilon_{n+1}$ of the ϵ_i's is a *dominant weight* of $G = g\ell(n+1, \mathbb{C})$ relative to $\mathcal{H}$ if and only if $\{\theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n+1} \geq 0\}$ determines a partition of $|\,\theta\,| \stackrel{\text{def}}{=} \sum_{i=1}^{n+1} \theta_i$. Thus, if

$$
\begin{aligned}
\alpha &= -(j\alpha_0 + k_1\alpha_1 + \cdots + k_n\alpha_n) \\
&= (jb - k_1)\epsilon_1 + (k_1 - k_2)\epsilon_2 + \cdots + (k_{n-1} - k_n)\epsilon_n + k_n\epsilon_{n+1}
\end{aligned}
$$

is a root of $\mathcal{L} = \mathcal{L}(G, V, V^*, \psi) \cong IA_n(a, b)$ of degree $-j$, then α is a dominant weight if and only if $\{jb - k_1 \geq k_1 - k_2 \geq \cdots \geq k_{n-1} - k_n \geq k_n \geq 0\}$ forms a partition of jb into at most $n+1$ nonzero parts. We identify α with the partition $\{\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n \geq \theta_{n+1} \geq 0\}$ having $\theta_1 = jb - k_1$, and $\theta_i = k_{i-1} - k_i$ for $i \geq 2$ and write $\alpha \vdash jb$ to signify that α determines a partition of jb. It suffices to compute the multiplicities of roots that are dominant weights, for the others are conjugate to those under action of the Weyl group of $\mathcal{L}$. It also suffices to determine the multiplicity mult(α) of α for α a root of $\mathcal{L}^-$ as mult$(-\alpha) =$ mult(α). Now by Section 1,

$$
\begin{aligned}
\text{(3.1)} \qquad \text{mult}(\alpha) &= \dim \mathcal{L}_\alpha \\
&= \dim \mathcal{L}_\alpha^- \\
&= \dim \mathcal{F}_\alpha^- - \dim J_\alpha^-.
\end{aligned}
$$

Thus, our strategy for computing mult(α) is to invoke (1.7) for $\dim \mathcal{F}_\alpha^-$ and to use Proposition 1.9 for $\dim J_\alpha^-$. The latter involves determining the homology $H_3(\mathcal{L}^-)$. Throughout this calculation we use $V(\lambda)$ to denote the irreducible G-module with highest weight λ. In particular, $\mathcal{L}_{-1} = V = V(-\alpha_0) = V(b\epsilon_1)$. Our first result in this direction is

Proposition 3.2. *For $1 \leq j \leq a$, $\mathcal{L}_{-j} = \mathcal{F}_{-j}$, so* mult$(\alpha) = \dim(\mathcal{F}_{-j})_\alpha$ *for all roots of degree $-j$.*

Proof. By the Gabber-Kac Theorem [**GK**], the ideal J^- of $\mathcal{F}^-$ is generated by the element $(ad\, f_0)^{1+a} f_1$, which has degree $-(a+1)$ and weight $-(a+1)\alpha_0 - \alpha_1 = (ab + b - 1)\epsilon_1 + \epsilon_2$. Hence, J^- is generated by the space

$$
\text{(3.3)} \qquad J_{-(a+1)} \cong V(-(a+1)\alpha_0 - \alpha_1) = V((ab + b - 1)\epsilon_1 + \epsilon_2).
$$

The assertions then follow. □

Let $\Delta \subset \mathcal{H}^*$ be the set of roots of $\mathcal{L}$ and let $\alpha_0 = -b\epsilon_1, \alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n$, be the simple roots in Δ. Use Δ^+ (resp. Δ^-) to denote the positive (resp. negative) roots of $\mathcal{L}$ relative to $\alpha_0, \alpha_1, \ldots, \alpha_n$. The Weyl group W of $\mathcal{L}$ is generated by the set $\{s_i \mid i = 0, 1, \ldots, n\}$, where s_i is the reflection $s_i(\gamma) = \gamma - \gamma(h_i)\alpha_i$ determined by the simple root α_i. For $w \in W$, $l(w)$ is the length of w relative to these generators. Let Δ_0^+ (resp. Δ_0^-) denote the set of positive (resp. negative) roots in $\mathcal{L}$ of degree 0, so that every root in $\Delta_0^\pm$ is a combination of $\{\alpha_i \mid i = 1, \ldots, n\}$. Set $\Delta_{\neq 0}^\pm \stackrel{\text{def}}{=} \Delta^\pm \backslash \Delta_0^\pm$, and let

$$W' = \{w \in W \mid w^{-1}\Delta_0^+ \subseteq \Delta^+\},$$

as in ([**GL**], Proposition 8.1). This leads to the following useful lemma:

Lemma 3.4([Kan1], Lemma 4.3). *Suppose $w = w's_j$ and $l(w) = l(w') + 1$. Then $w \in W'$ if and only if $w' \in W'$ and $w'(\alpha_j) \in \Delta_{\neq 0}^+$.*

By Kostant's formula (see Garland and Lepowsky ([**GL**], Theorem 8.6) or Liu [**Li**]), we have for the homology module $H_k(\mathcal{L}^-)$,

$$H_k(\mathcal{L}^-) = \bigoplus \sum_{\substack{w \in W' \\ l(w)=k}} V(w\rho - \rho). \tag{3.5}$$

where $\rho \in \mathcal{H}^*$ satisfies $\rho(h_i) = 1$ for $i = 0, 1, \ldots, n$. Combining these results gives

Proposition 3.6. *Suppose that $a \geq 2$ and $b \geq 1$. Then*

$$H_3(\mathcal{L}^-) = \begin{cases} V(-a(b+1)\alpha_0 - (b+1)\alpha_1) & \text{for} \quad n = 1 \\ V(-a(b+1)\alpha_0 - (b+1)\alpha_1) \oplus V(-(2a+1)\alpha_0 - 2\alpha_1 - \alpha_2) & \text{for} \quad n \geq 2. \end{cases}$$

Proof. By (3.5), $H_3(\mathcal{L}^-) = \oplus \sum_{\substack{w \in W' \\ l(w)=3}} V(w\rho - \rho)$. Using Lemma 3.4 it easy to verify for $n = 1$, that $s_0s_1s_0$ is the only element of length 3 in W' and for $n \geq 2$, the only elements of length 3 in W' are

$$s_0s_1s_0 \qquad \text{and} \qquad s_0s_1s_2.$$

Hence, the result follows since

$$s_0s_1s_0\rho - \rho = -a(b+1)\alpha_0 - (b+1)\alpha_1$$

and

$$s_0s_1s_2\rho - \rho = -(2a+1)\alpha_0 - 2\alpha_1 - \alpha_2.$$

□

Observe from Proposition 3.6 that when $b = 1$, the smallest d with $H_3(\mathcal{L}^-)_{-d} \neq (0)$ is $2a$, while when $b \geq 2$, the smallest value is $2a + 1$. **We assume henceforth that** $b \geq 2$, as the case $b = 1$ can be found in [**BKM2**] or can be gotten by modifying the argument below. Now from Propositions 1.9 and 3.6 we have

Corollary 3.7. *For* $a+1 \leq j \leq 2a$*,*

$$J_{-j} \cong \underbrace{V \otimes \cdots \otimes V}_{j-a-1 \text{ times}} \otimes V(\nu) \tag{3.8}$$

and

$$J_{-(2a+1)} \cong \Big(\underbrace{V \otimes \cdots \otimes V}_{a \text{ times}} \otimes V(\nu)\Big) \Big/ V(\phi), \tag{3.9}$$

where

$$\begin{aligned}\nu &= (ab+b-1)\epsilon_1 + \epsilon_2 \vdash b(a+1)\\ \phi &= (2ab+b-2)\epsilon_1 + \epsilon_2 + \epsilon_3 \vdash b(2a+1).\end{aligned}$$

From Corollary 3.7 we see that determining the structure of J_{-j} for $j = a+2, \ldots, 2a+1$ involves knowing how to decompose the tensor product $V^{\otimes m}$ of $m = j-a-1$ copies of the representation V into irreducible G-summands $V(\lambda)$ and then how to write $V(\lambda) \otimes V(\nu)$ as a sum of irreducible G-modules for λ, ν dominant weights. The first step in this analysis is the following

Proposition 3.10. *Let* V *be the* $g\ell(n+1, \mathbb{C})$*-module* $V(b\Lambda_1) = V(b\epsilon_1)$*. Then*

$$V^{\otimes m} \cong \sum_{\substack{\lambda \vdash mb \\ \ell(\lambda) \leq n+1,\ m}} K_{\lambda, \{b^m\}} V(\lambda)$$

where $\{b^m\}$ *denotes the partition of* mb *having* m *parts equal to* b*, and* $K_{\lambda,\{b^m\}}$ *is the Kostka number.*

Proof. From the Littlewood-Richardson Rule (see for example, the discussion in [**M**], [**BKM2**], or [**BBL**], Chap. 7) we can derive the multiplicity of the irreducible summand $V(\lambda)$ in $\otimes^m V$ as follows: Let $\lambda_1 \vdash b$ denote the partition of b having just one part of size b. Associate to λ_1 its frame, which has just one row with b boxes, and fill in those boxes with "1's". Append b boxes to the frame of λ_1 in such a way that no two lie in the same column and the result is the frame of some partition $\lambda_2 \vdash 2b$. Fill in the adjoined boxes with "2's". Proceed in this fashion to arrive at a partition $\lambda = \lambda_m$ of mb whose frame has been filled with b "i's" for $i = 1, \dots, m$ in such a way that the numbers weakly increase across the rows and strictly increase down the columns. The result is a "semistandard tableau" of shape λ where $\lambda \vdash mb$. The tableau's content is the partition $\{b^m\}$, as it contains b "i's" for $i = 1, \dots, m$. The number of nonzero parts $\ell(\lambda)$ of λ, which is at most m by the construction, must not exceed $n+1$. By the Littlewood-Richardson Rule, the multiplicity of $V(\lambda)$ in $\otimes^m V$ is the number of such semistandard tableaux, which is the Kostka number $K_{\lambda,\{b^m\}}$. □

Theorem 3.11. *Let j be an integer such that $a+1 \le j \le 2a+1$. Assume $\alpha = -(j\alpha_0 + \sum_{i=1}^n k_i\alpha_i) = (jb-k_1)\epsilon_1 + (k_1-k_2)\epsilon_2 + \cdots + (k_{n-1}-k_n)\epsilon_n + k_n\epsilon_{n+1}$ is dominant so that $\alpha \vdash jb$. Then*

$$\begin{aligned}(3.12)\quad \dim J_\alpha^- &= \dim(J_{-j})_\alpha \\ &= \Bigg\{ \sum_{\substack{\pi \vdash jb \\ \ell(\pi) \le n+1,\ j-a+1}} \Bigg(\sum_{\substack{\lambda \vdash (j-a-1)b \\ \ell(\lambda)\le n+1,\ j-a-1}} K_{\lambda,\{b^{j-a-1}\}} c^{\pi}_{\lambda,\nu} \Bigg) K_{\pi,\alpha} \Bigg\} \\ &\qquad - \delta_{2a+1,j} K_{\phi,\alpha},\end{aligned}$$

where

$$\nu = (ab+b-1)\epsilon_1 + \epsilon_2 = \{ab+b-1, 1\} \vdash ab+b$$
$$\phi = (2ab+b-2)\epsilon_1 + \epsilon_2 + \epsilon_3 = \{2ab+b-2, 1^2\} \vdash 2ab+b.$$

Proof. Corollary 3.7 gives

$$J_{-j} \cong V^{\otimes^{j-a-1}} \otimes V(\nu)$$

for $a+1 \le j \le 2a$ and

$$J_{-(2a+1)} \cong \left(V^{\otimes^a} \otimes V(\nu) \right) \Big/ V(\phi)$$

where $\nu = (ab+b-1)\epsilon_1 + \epsilon_2 \vdash ab+b$ and $\phi = (2ab+b-2)\epsilon_1 + \epsilon_2 + \epsilon_3 \vdash 2ab+b$. Therefore by Proposition 3.10, we have

$$J_{-j} \cong \sum_{\substack{\lambda \vdash (j-a-1)b \\ \ell(\lambda) \le n+1,\ j-a-1}} K_{\lambda,\{b^{j-a-1}\}} V(\lambda) \otimes V(\nu),$$

for $a+1 \le j \le 2a$, and

$$J_{-(2a+1)} \cong \Big(\sum_{\substack{\lambda \vdash ab \\ \ell(\lambda) \le n+1,\ a}} K_{\lambda,\{b^a\}} V(\lambda) \otimes V(\nu) \Big) \Big/ V(\phi).$$

Now if $\lambda \vdash j - a - 1$, then since ν has two nonzero parts, we have by the Littlewood-Richardson Rule,

$$V(\lambda) \otimes V(\nu) \cong \sum_{\substack{\pi \vdash jb \\ \ell(\pi) \le n+1,\ j-a+1}} c^{\pi}_{\lambda,\nu} V(\pi),$$

where $c^{\pi}_{\lambda,\nu}$ is the Littlewood-Richardson coefficient (see **[M]**). Since the multiplicity of α in $V(\pi)$ is the Kostka number $K_{\pi,\alpha}$, the assertions in (3.12) follow. $\square$

Remark. For $\lambda \vdash m$, the Kostka number $K_{\lambda,\{1^m\}}$ is just the number of standard tableaux (strictly increasing along each row and column) of shape λ with entries in $\{1, \dots, m\}$. That number equals $m!/h(\lambda)$ where $h(\lambda)$ is the hook length of the partition λ. Thus, (3.12) in the case that $b = 1$ is just (4.11) of [**BKM2**].

Now we need to compute the multiplicity $\dim \mathcal{F}^-_\alpha$ of α in the free part. Recall that the weights of V are

$$Wt = \{m_1\epsilon_1 + \dots + m_{n+1}\epsilon_{n+1} \mid m_i \ge 0 \text{ and } m_1 + \dots + m_{n+1} = b\},$$

and each weight has multiplicity one. As a consequence of Proposition 1.6 we have:

Proposition 3.13. *Assume $\alpha = -(j\alpha_0 + \sum_{i=1}^n k_i\alpha_i) = (jb - k_1)\epsilon_1 + (k_1 - k_2)\epsilon_2 + \dots + (k_{n-1} - k_n)\epsilon_n + k_n\epsilon_{n+1}$ is dominant so that $\alpha \vdash jb$. Let*

$$Wt = \{m_1\epsilon_1 + \dots + m_{n+1}\epsilon_{n+1} \mid m_i \ge 0 \textit{ and } m_1 + \dots + m_{n+1} = b\}.$$

denote the set of weights of $V(b\epsilon_1)$, and let $Wt = \{\omega_1, \omega_2, \dots\}$ be an enumeration of those weights. Then

$$\dim \mathcal{F}^-_\alpha = \sum_{\omega | \alpha} \mu\left(\frac{\alpha}{\omega}\right) \frac{\omega}{\alpha} D(\omega) \tag{3.14}$$

where μ denotes the classical Möbius function; $\omega \mid \alpha$ if $\alpha = \kappa\omega$ for some *positive integer κ, in which case $\frac{\alpha}{\omega} = \kappa$ and $\frac{\omega}{\alpha} = \frac{1}{\kappa}$; and*

$$D(\omega) = \sum_{(t)\in T(\omega)} \frac{((\sum_i t_i) - 1)!}{\prod_i (t_i!)} \tag{3.15}$$

where $T(\omega) = \{(t) = (t_1, t_2, \dots) \mid t_i \in \mathbb{Z}^{\geq 0}$ and $\sum_i t_i\omega_i = \omega\}$.

We close this section by applying the multiplicity formulas derived in this section to calculate root multiplicities for the algebras $IA_n(2,2)$ with $n \geq 3$. In particular, for $IA_3(2,2)$ we compute the multiplicities of the dominant roots $\alpha = -(j\alpha_0 + k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3) = \theta_1\epsilon_1 + \theta_2\epsilon_2 + \theta_3\epsilon_3 + \theta_4\epsilon_4$ where $j = 2, 3, 4, 5 = 2a + 1$. We explicitly exhibit the calculations for one choice of α for each value $j = 3, 4, 5$ and then display the remainder in the tables below. It follows from ([**BKM2**], Prop. 4.12) that the multiplicity of such roots α is the same for all algebras $IA_n(2,2)$ with $n \geq 3$.

To avoid cumbersome notation we adopt the shorthand for the dominant root which illustrates its parts as a partition. Thus, we write $\{4, 1^2\}$ for the root $\alpha = -(3\alpha_0 + 2\alpha_1 + \alpha_2) = 4\epsilon_1 + \epsilon_2 + \epsilon_3$. For the algebra $IA_3(2,2)$ we have $V = V(2\epsilon_1) = V(\{2\})$ and $\alpha_0 = -2\epsilon_1$, and the weights of V are

$$Wt = \{2\epsilon_1, \epsilon_1 + \epsilon_2, 2\epsilon_2, \epsilon_1 + \epsilon_3, \epsilon_2 + \epsilon_3, 2\epsilon_3, \epsilon_1 + \epsilon_4, \epsilon_2 + \epsilon_4, \epsilon_3 + \epsilon_4, 2\epsilon_4\}.$$

Recall by (3.3) that J^- is generated by $J_{-3} \cong V(\nu) = V(5\epsilon_1 + \epsilon_2) = V(\{5, 1\})$. Then by (3.8) and (3.9) and the Littlewood-Richardson rule (see [**BKM2**] or [**BBL**], Chap. 7),

$$\begin{aligned} J_{-4} \cong& V \otimes V(\nu) = V(\{2\}) \otimes V(\{5,1\}) \\ &\cong V(\{7,1\}) \oplus V(\{6,2\}) \oplus V(\{6,1^2\}) \oplus V(\{5,3\}) \oplus V(\{5,2,1\}), \end{aligned}$$

$$\begin{aligned} J_{-5} \cong& \Big(V \otimes V \otimes V(\nu)\Big) \Big/ V(\phi) \\ &= \Big(V(\{2\}) \otimes V(\{2\}) \otimes V(\{5,1\})\Big) \Big/ V(\{8,1^2\}) \\ &\cong V(\{9,1\}) \oplus 2V(\{8,2\}) \oplus V(\{8,1^2\}) \\ &\oplus 3V(\{7,3\}) \oplus 4V(\{7,2,1\}) \oplus V(\{7,1^3\}) \\ &\oplus 2V(\{6,4\}) \oplus 4V(\{6,3,1\}) \\ &\oplus 2V(\{6,2^2\}) \oplus 2V(\{6,2,1^2\}) \oplus V(\{5,5\}) \\ &\oplus 2V(\{5,4,1\}) \oplus 2V(\{5,3,2\}) \oplus V(\{5,3,1^2\}) \oplus V(\{5,2^2,1\}). \end{aligned}$$

Example 1. Consider the root $\alpha = -(3\alpha_0+2\alpha_1+\alpha_2) = 4\epsilon_1+\epsilon_2+\epsilon_3 = \{4,1^2\}$ for $IA_3(2,2)$. Then

$$\begin{aligned}\alpha &= 4\epsilon_1 + \epsilon_2 + \epsilon_3 \\ &= (2\epsilon_1) + (2\epsilon_1) + (\epsilon_2 + \epsilon_3) \\ &= (2\epsilon_1) + (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_3)\end{aligned}$$

so that by (3.14)

$$\dim \mathcal{F}_\alpha^- = \frac{2!}{2!} + \frac{2!}{1!} = 3.$$

Now $\dim J_\alpha^- = \dim (J_{-3})_\alpha = \dim V(\{5,1\})_\alpha = K_{\{5,1\},\alpha} = 2$ by Table 9.12 of [**BBL**]. Therefore $\dim \mathcal{L}_\alpha = 3 - 2 = 1$.

Example 2. Let $\alpha = -(4\alpha_0 + 3\alpha_1 + \alpha_2) = 5\epsilon_1 + 2\epsilon_2 + \epsilon_3 = \{5,2,1\}$ for $IA_3(2,2)$. Then

$$\begin{aligned}\alpha &= 5\epsilon_1 + 2\epsilon_2 + \epsilon_3 \\ &= (2\epsilon_1) + (2\epsilon_1) + (\epsilon_1 + \epsilon_2) + (\epsilon_2 + \epsilon_3) \\ &= (2\epsilon_1) + (2\epsilon_1) + (2\epsilon_2) + (\epsilon_1 + \epsilon_3) \\ &= (2\epsilon_1) + (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_3)\end{aligned}$$

and by (3.14)

$$\dim \mathcal{F}_\alpha^- = \frac{3!}{2!} + \frac{3!}{2!} + \frac{3!}{2!} = 9.$$

Since $\dim J_\alpha^- = \dim (J_{-4})_\alpha = K_{\{7,1\},\alpha} + K_{\{6,2\},\alpha} + K_{\{6,1^2\},\alpha} + K_{\{5,3\},\alpha} + K_{\{5,2,1\},\alpha} = 2 + 2 + 1 + 1 + 1$, we have $\dim \mathcal{L}_\alpha = 9 - 7 = 2$. (We have used the fact (see for example [**S**], Chap. 2) that $K_{\pi,\alpha}$ represents the number of semistandard tableaux of shape π and content α to evaluate $K_{\pi,\alpha}$.)

Example 3. In this final example assume $\alpha = -(5\alpha_0 + 4\alpha_1 + \alpha_2) = 6\epsilon_1 + 3\epsilon_2 + \epsilon_3 = \{6,3,1\}$ for $IA_3(2,2)$. Then

$$\begin{aligned}\alpha &= 6\epsilon_1 + 3\epsilon_2 + \epsilon_3 \\ &= (2\epsilon_1) + (2\epsilon_1) + (2\epsilon_1) + (2\epsilon_2) + (\epsilon_2 + \epsilon_3) \\ &= (2\epsilon_1) + (2\epsilon_1) + (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_2) + (\epsilon_2 + \epsilon_3) \\ &= (2\epsilon_1) + (2\epsilon_1) + (\epsilon_1 + \epsilon_2) + (2\epsilon_2) + (\epsilon_1 + \epsilon_3) \\ &= (2\epsilon_1) + (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_3)\end{aligned}$$

and from (3.14) we obtain,

$$\dim \mathcal{F}_\alpha^- = \frac{4!}{3!} + \frac{4!}{2!\,2!} + \frac{4!}{2!} + \frac{4!}{3!} = 26.$$

By our calculations above

$$\begin{aligned}\dim J_\alpha^- = \dim (J_{-5})_\alpha &= K_{\{9,1\},\alpha} + 2K_{\{8,2\},\alpha} + K_{\{8,1^2\},\alpha} + 3K_{\{7,3\},\alpha} \\ &+ 4K_{\{7,2,1\},\alpha} + K_{\{7,1^3\},\alpha} + 2K_{\{6,4\},\alpha} + 4K_{\{6,3,1\},\alpha} \\ &+ 2K_{\{6,2^2\},\alpha} + 2K_{\{6,2,1^2\},\alpha} + K_{\{5,5\},\alpha} + 2K_{\{5,4,1\},\alpha} \\ &+ 2K_{\{5,3,2\},\alpha} + K_{\{5,3,1^2\},\alpha} + K_{\{5,2^2,1\},\alpha} \\ &= 2+4+1+6+4+0+2+4 \\ &+ 0+0+0+0+0+0+0 \\ &= 23,\end{aligned}$$

so that $\dim \mathcal{L}_\alpha = 26 - 23 = 3$.

Root Multiplicities in $IA_n(2,2)$ for $n \geq 3$.

Dominant Roots: α	$\deg(\alpha)$	$\dim \mathcal{F}_\alpha^-$	$\dim J_\alpha^-$	$\dim \mathcal{L}_\alpha^-$
$\{3,1\}$	-2	1	0	1
$\{2^2\}$	-2	1	0	1
$\{2,1^2\}$	-2	2	0	2
$\{1^4\}$	-2	3	0	3
$\{4,2\}$	-3	2	1	1
$\{3^2\}$	-3	2	1	1
$\{4,1^2\}$	-3	3	2	1
$\{3,2,1\}$	-3	5	2	3
$\{2^3\}$	-3	7	2	5
$\{3,1^3\}$	-3	8	3	5
$\{2^2,1^2\}$	-3	11	3	8
$\{5,3\}$	-4	4	3	1

Root Multiplicities in $IA_n(2,2)$ for $n \geq 3$.

Dominant Roots: α	$\deg(\alpha)$	$\dim \mathcal{F}_\alpha^-$	$\dim J_\alpha^-$	$\dim \mathcal{L}_\alpha^-$
$\{4^2\}$	-4	4	3	1
$\{5,2,1\}$	-4	9	7	2
$\{4,3,1\}$	-4	13	8	5
$\{4,2^2\}$	-4	17	10	7
$\{3^2,2\}$	-4	22	11	11
$\{5,1^3\}$	-4	15	12	3
$\{4,2,1^2\}$	-4	30	17	13
$\{3^2,1^2\}$	-4	31	18	13
$\{3,2^2,1\}$	-4	48	22	26
$\{2^4\}$	-4	69	27	42
$\{6,4\}$	-5	9	8	1
$\{5^2\}$	-5	10	9	1
$\{6,3,1\}$	-5	26	23	3
$\{6,2^2\}$	-5	36	31	5
$\{5,4,1\}$	-5	35	28	7
$\{5,3,2\}$	-5	60	45	15
$\{4^2,2\}$	-5	71	49	22
$\{4,3^2\}$	-5	88	58	30
$\{6,2,1^2\}$	-5	62	42	20
$\{5,3,1^2\}$	-5	104	77	27
$\{5,2^2,1\}$	-5	138	103	35
$\{4^2,1^2\}$	-5	111	84	27
$\{4,3,2,1\}$	-5	158	131	27
$\{4,2^3\}$	-5	282	169	113
$\{3^3,1\}$	-5	240	150	90
$\{3^2,2^2\}$	-5	366	195	171

Remarks. Any partition in the table having only 2 nonzero parts corresponds to a root $-(k_0\alpha_0 + k_1\alpha_1)$, which is a root for all algebras $IA_n(2,2)$ with $n \geq 1$. By ([**BKM2**], Prop. 4.12) its multiplicity in $IA_n(2,2)$ for all $n \geq 1$ is the same as its multiplicity in $IA_1(2,2)$. The algebra $IA_1(2,2)$ is just the affine algebra $A_1^{(1)}$ and all roots for $A_1^{(1)}$ have multiplicity one ([**Kac**], Cor. 7.4), so all partitions with 2 nonzero parts have multiplicity one. Any partition $\theta = \{\theta_1 \geq \theta_2 \geq \theta_3 > 0\}$ with 3 nonzero parts corresponds to a root $-(k_0\alpha_0 + k_1\alpha_1 + k_2\alpha_2)$, where $\theta_1 = 2k_0 - k_1$, $\theta_2 = k_1 - k_2$ and $\theta_3 = k_2$. The multiplicity of θ in $IA_n(2,2)$ for $n \geq 2$ is the same as in $IA_2(2,2)$, which is the hyperbolic Kac-Moody algebra $HA_1^{(1)}$ studied by Feingold and Frenkel [**FF**] and by Kang [**Kan1**] [**Kan2**]. Assume

$$\Gamma(\alpha) = \begin{pmatrix} k_1 - k_2 & k_1 - k_0 \\ k_1 - k_0 & k_2 \end{pmatrix} = \begin{pmatrix} \theta_2 & 1/2(\theta_2 + \theta_3 - \theta_1) \\ 1/2(\theta_2 + \theta_3 - \theta_1) & \theta_3 \end{pmatrix}.$$

Then whenever α is conjugate under the Weyl group to a root having α_2-coefficient equal to -1 or 1, its multiplicity is given by $p(\det(\Gamma(\alpha)) + 1)$ where $p(\cdot)$ is the classical partition function (see [**FF**], p. 117). All the partitions with 3 nonzero parts in the table have such expressions for their multiplicities.

4. Root Multiplicity Formulas.

In this final section we apply the Euler-Poincaré Principle (see [**CE**]) and the extension of Kostant's formula given in [**GL**] to derive two root multiplicity formulas, one of which is recursive and the other closed form, for the algebra $IA_n(a,b) \cong \mathcal{L} = \mathcal{L}(G, V, V^*, \psi)$. Since $\dim \mathcal{L}_\alpha = \dim \mathcal{L}_{-\alpha}$ for any root α, it suffices to determine $\dim \mathcal{L}_\alpha = \dim \mathcal{L}_\alpha^-$ for $\alpha = -(j\alpha_0 + \sum_{i=1}^n k_i\alpha_i)$. Needed in the argument is the construction of the homology $H_*(\mathcal{L}^-) = H_*(\mathcal{L}^-, \mathbb{C})$ with coefficients in the trivial $\mathcal{L}^-$-module $\mathbb{C}$.

Consider the complex $(\wedge^*(\mathcal{L}^-), d_*)$:

$$\cdots \longrightarrow \wedge^{k+1}(\mathcal{L}^-) \xrightarrow{d_{k+1}} \wedge^k(\mathcal{L}^-) \xrightarrow{d_k} \wedge^{k-1}(\mathcal{L}^-) \xrightarrow{d_{k-1}} \cdots$$
$$\xrightarrow{d_2} \wedge^1(\mathcal{L}^-) \xrightarrow{d_1} \wedge^0(\mathcal{L}^-) \xrightarrow{d_0} \mathbb{C} \longrightarrow (0),$$

where d_k is defined by

$$d_k(x_1 \wedge x_2 \wedge \cdots \wedge x_k) = \sum_{1 \le s < t \le k} (-1)^{s+t-1}\big([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \widehat{x}_s \wedge \cdots \wedge \widehat{x}_t \wedge \cdots \wedge x_k\big).$$

Then for all $k \geq 0$, $H_k(\mathcal{L}^-) = (\text{Ker } d_k)/(\text{Im } d_{k+1})$.

By the Euler-Poincaré Principle we have,

$$\sum_{k=0}^{\infty}(-1)^k \operatorname{ch}\big(\wedge^k(\mathcal{L}^-)\big) = \sum_{k=0}^{\infty}(-1)^k \operatorname{ch}\big(H_k(\mathcal{L}^-)\big), \tag{4.1}$$

where for any completely reducible G-module $M = \oplus \sum_{\lambda \in \mathcal{H}^*} M_\lambda$ the formal character is, by definition, the sum

$$\operatorname{ch}(M) = \sum_{\lambda \in \mathcal{H}^*} (\dim M_\lambda) e(\lambda).$$

If $\alpha = -(j\alpha_0 + \sum_{i=1}^n k_i\alpha_i)$ is a root with $j \geq 2$, (the case $j = 1$ corresponds to the weights of V which have multiplicity one), then since $H_0(\mathcal{L}^-) \cong \mathbb{C}$ and $H_1(\mathcal{L}^-) \cong \mathcal{L}^-/[\mathcal{L}^-, \mathcal{L}^-] \cong \mathcal{L}_{-1} = V$, it follows from (4.1) that

$$\begin{aligned} \dim \mathcal{L}^-_\alpha &= \dim(\mathcal{L}_{-j})_\alpha \\ &= \sum_{k=2}^{\infty}(-1)^k \dim \wedge^k(\mathcal{L}^-)_\alpha - \sum_{k=2}^{\infty}(-1)^k \dim H_k(\mathcal{L}^-)_\alpha. \end{aligned} \tag{4.2}$$

If $k > j$, any weight of $\wedge^k(\mathcal{L}^-)$ has degree $\leq -k < -j$. Hence $\dim \wedge^k(\mathcal{L}^-)_\alpha = 0$, as $deg\, \alpha = -j$. Therefore for $k > j$, $\dim H_k(\mathcal{L}^-)_\alpha = \dim(\text{Ker } d_k)_\alpha - \dim(\text{Im } d_{k+1})_\alpha = 0$. Consequently, the sum on the right hand side of (4.2) reduces to

$$\dim \mathcal{L}^-_\alpha = \sum_{k=2}^{j}(-1)^k \dim \wedge^k(\mathcal{L}^-)_\alpha - \sum_{k=2}^{j}(-1)^k \dim H_k(\mathcal{L}^-)_\alpha. \tag{4.3}$$

Kostant's formula (3.5) gives

$$\dim H_k(\mathcal{L}^-)_\alpha = \sum_{\substack{w \in W' \\ l(w)=k}} \dim V(w\rho - \rho)_\alpha,$$

where $V(w\rho-\rho)_\alpha = 0$ unless $deg(w\rho-\rho) = -j = deg(\alpha)$. If $deg(w\rho-\rho) = -j$, then $\dim V(w\rho-\rho)_\alpha = \dim V(w\rho-\rho)_{\overline{\alpha}} = K_{w\rho-\rho,\overline{\alpha}}$, where $\overline{\alpha}$ is the unique dominant weight conjugate to α, and $K_{w\rho-\rho,\overline{\alpha}}$ is the Kostka number. For convenience, let $K_{w\rho-\rho,\alpha} = \dim V(w\rho-\rho)_\alpha = K_{w\rho-\rho,\overline{\alpha}}$. Thus,

$$\dim H_k(\mathcal{L}^-)_\alpha = \sum_{\substack{w\in W' \\ l(w)=k \\ deg(w\rho-\rho)=-j}} K_{w\rho-\rho,\alpha}. \tag{4.4}$$

We also make use of the following total order defined on the root lattice of $\mathcal{L}^-$. If $\alpha = \theta_1\epsilon_1 + \cdots + \theta_{n+1}\epsilon_{n+1}$ and $\beta = \zeta_1\epsilon_1 + \cdots + \zeta_{n+1}\epsilon_{n+1}$, then we say $\alpha < \beta$ iff $\sum_{i=1}^{n+1}\theta_i < \sum_{i=1}^{n+1}\zeta_i$ or $\sum_{i=1}^{n+1}\theta_i = \sum_{i=1}^{n+1}\zeta_i$ and $\theta_i < \zeta_i$ for some i and $\theta_s = \zeta_s$ for $i < s \le n+1$. Then equations (4.3) and (4.4) combine to give the following recursive formula for the root multiplicities.

Theorem 4.5. *Let $IA_n(a,b) \cong \mathcal{L} = \mathcal{L}(G, V, V^*, \psi)$, and for any root β of $\mathcal{L}^-$, let $m_\beta = \dim \mathcal{L}^-_\beta$. If $\alpha = -(j\alpha_0 + \sum_{i=1}^n k_i\alpha_i)$ for some $j \ge 2$, then*

$$\begin{aligned}\dim \mathcal{L}_\alpha = \dim \mathcal{L}^-_\alpha = &\sum_{k=2}^{j}(-1)^k \sum_{\substack{\beta_1<\cdots<\beta_r \\ p_1+\cdots+p_r=k \\ p_1\beta_1+\cdots+p_r\beta_r=\alpha}} \binom{m_{\beta_1}}{p_1}\cdots\binom{m_{\beta_r}}{p_r} \\ &- \sum_{k=2}^{j}(-1)^k \sum_{\substack{w\in W' \\ l(w)=k \\ deg(w\rho-\rho)=-j}} K_{w\rho-\rho,\alpha}.\end{aligned}$$

Next we present an example to illustrate how this recursive formula can be applied in an actual root multiplicity computation.

Example 4.6. Once again we assume the algebra is of type $IA_3(2,2)$, and we let $\alpha = -(6\alpha_0 + 6\alpha_1 + 2\alpha_2 + \alpha_3) = 6\epsilon_1 + 4\epsilon_2 + \epsilon_3 + \epsilon_4 = \{6, 4, 1^2\}$. (This is a root whose multiplicity cannot be computed using the results of Section 3 because its degree $-j = -6$ satisfies $j > 2a + 1 = 5$.) Then by (4.3),

$$\begin{aligned}\dim\ \mathcal{L}_\alpha &= \dim\ \mathcal{L}^-_\alpha \\ &= \dim \wedge^2(\mathcal{L}^-)_\alpha - \dim \wedge^3(\mathcal{L}^-)_\alpha + \dim \wedge^4(\mathcal{L}^-)_\alpha - \dim \wedge^5(\mathcal{L}^-)_\alpha \\ &\quad + \dim \wedge^6(\mathcal{L}^-)_\alpha - \dim H_2(\mathcal{L}^-)_\alpha + \dim H_3(\mathcal{L}^-)_\alpha \\ &\quad - \dim H_4(\mathcal{L}^-)_\alpha + \dim H_5(\mathcal{L}^-)_\alpha - \dim H_6(\mathcal{L}^-)_\alpha.\end{aligned}$$

Now $\dim \wedge^k(\mathcal{L}^-)_\alpha = \dim\{x_1 \wedge x_2 \wedge \cdots \wedge x_k \mid x_i \in \mathcal{L}_{-r_i},\ r_1 + \cdots + r_k = 6;\ wt(x_1) + \cdots + wt(x_k) = \alpha\}$, and we can use the tables in the previous section to compute these dimensions. As a result we obtain

$$\begin{aligned}
\dim \wedge^2(\mathcal{L}^-)_\alpha &= 225\\
\dim \wedge^3(\mathcal{L}^-)_\alpha &= 251\\
\dim \wedge^4(\mathcal{L}^-)_\alpha &= 102\\
\dim \wedge^5(\mathcal{L}^-)_\alpha &= 15\\
\dim \wedge^6(\mathcal{L}^-)_\alpha &= 0.
\end{aligned}$$

For the homology portion, Kostant's formula (3.5) and Lemma 3.4 give:

$$\begin{aligned}
H_2(\mathcal{L}^-) &= \sum_{\substack{w\in W'\\ l(w)=2}} V(w\rho-\rho) = V(s_0s_1\rho-\rho)\\
&= V(-3\alpha_0-\alpha_1) = V(\{5,1\})\\
H_3(\mathcal{L}^-) &= \sum_{\substack{w\in W'\\ l(w)=3}} V(w\rho-\rho) = V(s_0s_1s_0\rho-\rho)\oplus V(s_0s_1s_2\rho-\rho)\\
&= V(-6\alpha_0-3\alpha_1)\oplus V(-5\alpha_0-2\alpha_1-\alpha_2)\\
&= V(\{9,3\})\oplus V(\{8,1^2\})\\
H_4(\mathcal{L}^-) &= \sum_{\substack{w\in W'\\ l(w)=4}} V(w\rho-\rho) = V((s_0s_1)^2\rho-\rho)\oplus V(s_0s_1s_2s_3\rho-\rho)\\
&= V(-10\alpha_0-6\alpha_1)\oplus V(-7\alpha_0-3\alpha_1-2\alpha_2-\alpha_3)\\
&= V(\{14,6\})\oplus V(\{11,1^3\})\\
H_5(\mathcal{L}^-) &= \sum_{\substack{w\in W'\\ l(w)=5}} V(w\rho-\rho) = V((s_0s_1)^2s_0\rho-\rho)\oplus V(s_0s_1s_0s_1s_2\rho-\rho)\\
&= V(-15\alpha_0-10\alpha_1)\oplus V(-8\alpha_0-4\alpha_1-\alpha_2)\\
&= V(\{20,10\})\oplus V(\{12,3,1\})\\
H_6(\mathcal{L}^-) &= \sum_{\substack{w\in W'\\ l(w)=6}} V(w\rho-\rho)\\
&= V((s_0s_1)^3\rho-\rho)\oplus V((s_0s_1)^2s_2s_3\rho-\rho)\oplus V((s_0s_1)^2s_2s_1\rho-\rho)\\
&= V(-21\alpha_0-15\alpha_1)\oplus V(-18\alpha_0-12\alpha_1-2\alpha_2-\alpha_3)\\
&\quad \oplus V(-18\alpha_0-11\alpha_1-2\alpha_2)\\
&= V(\{27,15\})\oplus V(\{24,10,1^2\})\oplus V(\{25,9,2\}).
\end{aligned}$$

From this we see that $\dim H_k(\mathcal{L}^-)_\alpha = 0$ except when $k = 3$ where

$$\dim H_3(\mathcal{L}^-)_\alpha = \dim V(\{9,3\})_\alpha = K_{\{9,3\},\alpha} = 4.$$

Thus,

$$\dim \mathcal{L}_\alpha = \dim \mathcal{L}^-_\alpha = 225 - 251 + 102 - 15 + 4 = 65.$$

With the Euler-Poincaré Principle as a guide, we define M to be the following formal alternating direct sum of finite dimensional modules for $G = g\ell(n+1,\mathbb{C})$:

$$\begin{aligned} M &= \sum_{k=1}^{\infty}(-1)^{k+1} H_k(\mathcal{L}^-) \\ &= \sum_{k=1}^{\infty}(-1)^{k+1} \sum_{\substack{w\in W' \\ l(w)=k}} V(w\rho-\rho) \\ &= \sum_{w\in W'}(-1)^{l(w)+1} V(w\rho-\rho). \end{aligned} \tag{4.7}$$

Then for $\alpha = -(j\alpha_0 + \sum_{i=1}^n k_i\alpha_i)$, we set

$$\dim M_\alpha = \sum_{\substack{w\in W' \\ l(w)\le j \\ deg(w\rho-\rho)=-j}} (-1)^{l(w)+1} K_{w\rho-\rho,\alpha}, \tag{4.8}$$

and define the formal character of M to be

$$\mathrm{ch}(M) = \sum_\alpha (\dim M_\alpha) e(\alpha). \tag{4.9}$$

It may be that $\dim M_\alpha$ is nonpositive for certain values of α. Let $\tau_1, \tau_2, \ldots$ be an enumeration of the weights in $\{\alpha \mid \dim M_\alpha \neq 0\}$ compatible with the total ordering on the root lattice of $\mathcal{L}^-$ given above, and for τ define

$$T(\tau) = \left\{ \underline{t} = (t_1, t_2, \ldots) \mid t_i \in \mathbb{Z}^{\ge 0},\ \sum_i t_i\tau_i = \tau \right\} \tag{4.10}$$

and

$$B(\tau) = \sum_{\underline{t}\in T(\tau)} \frac{((\sum_i t_i) - 1)!}{\prod_i (t_i!)} \prod_i (\dim M_{\tau_i})^{t_i}. \tag{4.11}$$

Then we have

Theorem 4.12. *For any root* $\alpha = -(j\alpha_0 + \sum_{i=1}^n k_i\alpha_i)$ *of* $IA_n(a,b) \cong \mathcal{L} = \mathcal{L}(G, V, V^*, \psi)$,

$$\begin{aligned}\dim \mathcal{L}_\alpha = \dim \mathcal{L}_\alpha^- &= \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau) \\ &= \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} \sum_{\underline{t}\in T(\tau)} \frac{((\sum_i t_i)-1)!}{\prod_i (t_i!)} \\ &\bullet \prod_i \Bigg(\sum_{\substack{w\in W' \\ l(w)\leq -deg(\tau_i) \\ deg(w\rho-\rho)=deg\tau_i}} (-1)^{l(w)+1} K_{w\rho-\rho,\tau_i} \Bigg)^{t_i},\end{aligned}$$

where μ *denotes the classical Möbius function.*

Proof. Since

$$\sum_{k=0}^{\infty} (-1)^k \operatorname{ch}\left(\wedge^k (\mathcal{L}^-)\right) = \prod_{\alpha\in\Delta_{\neq 0}^-} (1-e(\alpha))^{\dim \mathcal{L}_\alpha^-},$$

it follows from (4.1) that

$$\prod_{\alpha\in\Delta_{\neq 0}^-} (1-e(\alpha))^{\dim \mathcal{L}_\alpha^-} = 1 - \operatorname{ch}(M).$$

Thus,

$$\prod_{\alpha\in\Delta_{\neq 0}^-} (1-e(\alpha))^{-\dim \mathcal{L}_\alpha^-} = \frac{1}{1-\operatorname{ch}(M)} = \left(1 - \sum_i (\dim M_{\tau_i}) e(\tau_i)\right)^{-1}.$$

Now taking the formal logarithm of both sides and using the series expansion $\log(1-z) = -\sum_{k=1}^{\infty} \frac{1}{k} z^k$, we obtain

$$-\sum_{\alpha\in\Delta_{\neq 0}^-} (\dim \mathcal{L}_\alpha^-) \log(1-e(\alpha)) = \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{i=1}^{\infty} (\dim M_{\tau_i}) e(\tau_i)\right)^m.$$

Hence,

$$\sum_{\alpha\in\Delta^-_{\neq 0}} (\dim \mathcal{L}^-_\alpha) \sum_{k=1}^{\infty} \frac{1}{k} e(k\alpha) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{\underline{t}=(t_i) \\ m=\sum_i t_i}} \frac{((\sum_i t_i)-1)!}{\prod_i (t_i!)} \cdot \prod_i (\dim M_{\tau_i})^{t_i} e\left(\sum_i t_i\tau_i\right)$$
$$= \sum_{\tau} \sum_{\underline{t}=(t_i)\in T(\tau)} \frac{((\sum_i t_i)-1)!}{\prod_i (t_i!)} \prod_i (\dim M_{\tau_i})^{t_i} e(\tau)$$
$$= \sum_{\tau} B(\tau) e(\tau).$$

Comparison of the coefficient of $e(\tau)$ on both sides gives

$$B(\tau) = \sum_{\alpha|\tau} \frac{\alpha}{\tau} \dim \mathcal{L}^-_\alpha.$$

Therefore, by Möbius inversion we obtain

$$\dim \mathcal{L}^-_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau)$$

as desired. The rest comes from substituting the expression in (4.8). $\square$

To illustrate the result in Theorem 4.12 we calculate the multiplicity of $\alpha = -(5\alpha_0 + 4\alpha_1 + \alpha_2) = 6\epsilon_1 + 3\epsilon_2 + \epsilon_3$ using it. This multiplicity has already been computed in Example 3 of Section 3 by the methods of that section.

The Weyl group of $G = g\ell(n+1, \mathbb{C})$ is the symmetric group $W = S_{n+1}$, which acts on the weights by sending $\theta = \sum_{i=1}^{n+1} \theta_i\epsilon_i$ to $\sum_{i=1}^{n+1} \theta_i\epsilon_{wi}$ for $w \in W$. Thus, each weight θ is conjugate to a unique dominant weight $\overline{\theta}$ whose coefficients relative to the basis $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}$ are in descending order, hence form a partition. If $V(\lambda)$ is the finite dimensional irreducible G-module with highest weight λ, then for any weight θ of $V(\lambda)$, $\dim V(\lambda)_\theta = K_{\lambda,\overline{\theta}}$, the Kostka number.

Example 4.13. Let $\alpha = -(5\alpha_0 + 4\alpha_1 + \alpha_2) = 6\epsilon_1 + 3\epsilon_2 + \epsilon_3$ for $IA_3(2,2)$. First observe that since the coefficient of α_2 is -1, the expression in Theorem 4.12 reduces to

$$\dim \mathcal{L}_\alpha = \dim \mathcal{L}^-_\alpha = B(\alpha) = \sum_{\underline{t}\in T(\alpha)} \frac{((\sum_i t_i)-1)!}{\prod_i (t_i!)} \prod_i (\dim M_{\tau_i})^{t_i} \tag{4.14}$$

where $T(\alpha) = \{\underline{t} = (t_i) \mid t_i \in \mathbb{Z}^{\geq 0}, \ \sum_i t_i \tau_i = \alpha\}$ and $\{\tau_1, \tau_2, \dots\}$ is an enumeration of the weights of $M = \sum_{k=1}^{\infty} (-1)^{k+1} H_k(\mathcal{L}^-)$. Now by the same arguments as in Example 4.6 we see that the terms $H_k(\mathcal{L}^-)$ for $k > 3$ do not contribute to the multiplicity of α since their weights have degree less than $deg(\alpha) = -5$. Recall from Example 4.6 that

$$\begin{aligned} H_1(\mathcal{L}^-) &= V(-\alpha_0) = V(2\epsilon_1) = V(\{2\}) \\ H_2(\mathcal{L}^-) &= V(-3\alpha_0 - \alpha_1) = V(\{5,1\}) \\ H_3(\mathcal{L}^-) &= V(-6\alpha_0 - 3\alpha_1) \oplus V(-5\alpha_0 - 2\alpha_1 - \alpha_2) \\ &= V(\{9,3\}) \oplus V(\{8,1^2\}). \end{aligned}$$

Again from degree considerations, the only modules which can contribute to (4.14) are $V(\{2\})$, $V(\{5,1\})$, and $V(\{8,1^2\})$. The dominant weights of these $g\ell(4)$-modules and their multiplicities are displayed below:

$$V(\{2\}) : \begin{matrix} \{2\} & \{1^2\} \\ 1 & 1 \end{matrix}$$

$$V(\{5,1\}) : \begin{matrix} \{5,1\} & \{4,2\} & \{4,1^2\} & \{3^2\} & \{3,2,1\} & \{3,1^3\} & \{2^3\} & \{2^2,1^2\} \\ 1 & 1 & 2 & 1 & 2 & 3 & 2 & 3 \end{matrix}$$

$$V(\{8,1^2\}) : \begin{matrix} \{8,1^2\} & \{7,2,1\} & \{7,1^3\} & \{6,3,1\} & \{6,2^2\} & \{6,2,1^2\} \\ 1 & 1 & 3 & 1 & 1 & 3 \\ \{5,4,1\} & \{5,3,2\} & \{5,3,1^2\} & \{5,2^2,1\} & \{4^2,2\} & \{4^2,1^2\} \\ 1 & 1 & 3 & 3 & 1 & 3 \\ \{4,3^2\} & \{4,3,2,1\} & \{4,2^3\} & \{3^3,1\} & \{3^2,2^2\} & \\ 1 & 3 & 3 & 3 & 3 & \end{matrix}.$$

These multiplicities can be computed by determining the corresponding Kostka numbers. For $\{2\}$ and $\{5,1\}$ these Kostka numbers can be found in [**M**] or ([**BBL**], Table (9.12)). Alternately, the multiplicities can be gotten by converting the weight to a linear combination of fundamental weights and then by consulting the appropriate table for A_3 in [**BMP**]. The other nondominant weights can be obtained from these by applying permutations. Note that since

$$M = H_1(\mathcal{L}^-) - H_2(\mathcal{L}^-) + H_3(\mathcal{L}^-) - \cdots,$$

for a weight τ of $H_2(\mathcal{L}^-)$ we have $\dim M_\tau = -\dim V(\{5,1\})_\tau = -K_{\{5,1\},\overline{\tau}}$, where $\overline{\tau}$ is the unique dominant conjugate of τ. Keeping this in mind, we proceed with evaluating (4.14). Now the partitions of α in terms of all the weights of M are:

$$\begin{aligned}
\alpha &= (6\epsilon_1 + 3\epsilon_2 + \epsilon_3) \\
&= (5\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_2) + (\epsilon_2 + \epsilon_3) \\
&= (5\epsilon_1 + \epsilon_2) + (2\epsilon_2) + (\epsilon_1 + \epsilon_3) \\
&= (5\epsilon_1 + \epsilon_3) + (\epsilon_1 + \epsilon_2) + (2\epsilon_2) \\
&= (4\epsilon_1 + 2\epsilon_2) + (2\epsilon_1) + (\epsilon_2 + \epsilon_3) \\
&= (4\epsilon_1 + 2\epsilon_2) + (\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_3) \\
&= (4\epsilon_1 + \epsilon_2 + \epsilon_3) + 2(\epsilon_1 + \epsilon_2) \\
&= (3\epsilon_1 + 3\epsilon_2) + (2\epsilon_1) + (\epsilon_1 + \epsilon_3) \\
&= (3\epsilon_1 + 2\epsilon_2 + \epsilon_3) + (2\epsilon_1) + (\epsilon_1 + \epsilon_2) \\
&= (2\epsilon_1 + 3\epsilon_2 + \epsilon_3) + 2(2\epsilon_1) \\
&= 3(2\epsilon_1) + (2\epsilon_2) + (\epsilon_2 + \epsilon_3) \\
&= 2(2\epsilon_1) + 2(\epsilon_1 + \epsilon_2) + (2\epsilon_2) + (\epsilon_1 + \epsilon_3) \\
&= (2\epsilon_1) + 3(\epsilon_1 + \epsilon_2) + (\epsilon_1 + \epsilon_3).
\end{aligned}$$

Hence,

$$\begin{aligned}
\dim \mathcal{L}_\alpha^- &= B(\alpha) \\
&= 1 + \frac{2!}{1!1!1!}(-1)(1)(1) + \frac{2!}{1!1!1!}(-1)(1)(1) + \frac{2!}{1!1!1!}(-1)(1)(1) \\
&\quad + \frac{2!}{1!1!1!}(-1)(1)(1) + \frac{2!}{1!1!1!}(-1)(1)(1)\frac{2!}{1!1!1!}(-2)(1)(1) \\
&\quad + \frac{2!}{1!2!}(-2)(1) + \frac{2!}{1!1!1!}(-1)(1)(1)\frac{2!}{1!1!1!}(-2)(1)(1) \\
&\quad + \frac{2!}{1!2!}(-2)(1) + \frac{4!}{3!1!1!}(1)(1)(1)\frac{4!}{2!2!1!}(1)(1)(1) \\
&\quad + \frac{4!}{2!1!1!1!}(1)(1)(1)(1) + \frac{4!}{1!3!1!}(1)(1)(1)\frac{2!}{1!1!1!}(-2)(1)(1) \\
&= 1 - 2 - 2 - 2 - 2 - 2 - 4 - 2 - 2 - 4 - 2 + 4 + 6 + 12 + 4 \\
&= 3.
\end{aligned}$$

Concluding Remarks. The results of this paper pertain to the algebras $IA_n(a,b)$, but the methods for obtaining the recursive and closed form formulas work in general. The only place where the particular nature of the algebra $IA_n(a,b)$ is used is in evaluating the multiplicities in terms of Kostka numbers.

Although the examples presented are for the indefinite algebra $IA_3(2,2)$, the multiplicities of the roots computed are the same for all algebras $IA_n(2,2)$

with $n \geq 3$ and for the analogous algebras $IB_{n+1}(2,2)$, $IC_{n+1}(2,2)$, and $ID_{n+1}(2,2)$ with $n \geq 3$ (See **[BKM2]**, Sections 4 and 5.). These algebras are constructed by replacing the Cartan matrix $C(A_n)$ with the Cartan matrix corresponding to the simple Lie algebra of type B_{n+1}, C_{n+1} or D_{n+1}.

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UNIVERSITY OF WISCONSIN
MADISON, WI 53706-1388
E-MAIL ADDRESS: BENKART@MATH.WISC.EDU

AND

COLLEGE OF NATURAL SCIENCES
SEOUL NATIONAL UNIVERSITY
SEOUL 151-742, KOREA
E-MAIL ADDRESS: SJKANG@MATH.SNU.AC.KR

AND

NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NC 27695-8205
E-MAIL ADDRESS: MISRA@MATH.NCSU.EDU