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A MEAN VALUE INEQUALITY WITH APPLICATIONS TO BERGMAN SPACE OPERATORS

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If u is integrable over the unit disc and $u = Tu$, where T is the Berezin operator then it is known that u must be harmonic. In this paper we give examples to show that the condition $Tu \geq u$ does not imply that u is subharmonic, but we are able to show that the condition $Tu \geq u$ does imply that u must be "almost" subharmonic near the boundary in an appropriate sense. We give two versions of this "almost" subharmonicity, a "pointwise" version and a "weak-star" version. We give applications of these results to hyponormal Toeplitz operators on the Bergman space.

Introduction.

Let D be the open unit disc in the complex plane. We let $H^\infty(D)$ denote the space of bounded holomorphic functions in D and let $B(D)$ denote the Bergman space on D; the set of holomorphic functions f on D such that

$$\int_D |f(z)|^2 dA(z) < \infty$$

where dA denotes planar Lebesgue measure on D. $B(D)$ is a closed subspace of the Hilbert space $L^2(dA)$ and so there is an orthogonal projection $P : L^2(dA) \to B(D)$. If $\varphi \in L^\infty(dA)$ we define the Toeplitz operator $T_\varphi : B(D) \to B(D)$ by $T_\varphi f = P(\varphi f)$. For each $z \in D$ we have the kernel function $k_z(\zeta) = \dfrac{1}{\pi(1-\bar{z}\zeta)^2}$. For each $f \in B(D)$ we have $f(z) = \langle f, k_z\rangle$ where $\langle f, g\rangle$ denotes the inner product in $L^2(dA)$. We use the usual notation of $\|f\|_2^2 = \langle f, f\rangle$ for $f \in L^2(dA)$. Note that $\|k_z\|_2^2 = \langle k_z, k_z\rangle = k_z(z) = \dfrac{1}{\pi(1-|z|^2)^2}$. For each $z \in D$ we have the biholomorphic involution $\varphi_z : D \to D$ given by $\varphi_z(\zeta) = \dfrac{z-\zeta}{1-\bar{z}\zeta}$. With these involutions we can define the Berezin transform Tu of any $u \in L^1(dA)$, by

$$Tu(z) = \frac{1}{\pi}\int_D u \circ \varphi_z dA\,.$$

Equivalently, after a change of variables, we have

$$Tu(z) = \frac{(1-|z|^2)^2}{\pi} \int_D \frac{u(\zeta)}{\left|1-\overline{\zeta}z\right|^4} dA(\zeta) .$$

Finally, if A is a bounded operator on a Hilbert space X, with norm $\|x\|$, we say A is hyponormal if $A^*A \geq AA^*$, or in other words, if

$$\|Ax\| \geq \|A^*x\| \text{ for all } x \in X.$$

It is a simple matter to check that if u is harmonic in D, i.e., $\Delta u(z) = \frac{\partial^2}{\partial z \partial \overline{z}} u(z) \equiv 0$, and $u \in L^1(dA)$, then $Tu(z) = u(z)$ for all $z \in D$. In [**1**], the converse was established, i.e., if $Tu = u$ in D then u must be harmonic. Now if u is subharmonic and in $L^1(dA)$ then it follows easily that $Tu \geq u$ in D. We start Section 1 by showing the converse of this statement to be false, i.e., we show that there exists u (indeed a large class of such u) so that $Tu \geq u$ in D but u is not subharmonic. However in Theorem 2 we show that the condition $Tu \geq u$ in D implies some sort of vestigal subharmonicity near the boundary. We show, under a rather mild integrability condition on Δu, that if $Tu \geq u$ in D then $\overline{\lim}_{z \to \zeta} \Delta u(z) \geq 0$ for all $\zeta \in \partial D$. Actually Theorem 2 gives a more precise "local" theorem. The main tool in the proof is a formula that represents $Tu - u$ as an integral of Δu times a positive kernel. This is the content of Theorem 1.

Our second result of this type says that if $Tu \geq u$ in D and if the measures $\Delta u(re^{i\theta})d\theta$ have a weak-star limit as $r \to 1$ on some interval I, then that limit is a positive measure on I. This is Theorem 3.

In the second section we give two applications of the results of the first section. In [**2**] H. Sadraoui showed that if $f, g \in H^\infty(D)$ and if $T_{f+\overline{g}}$ is hyponormal and if we assume that f', g' both lie in the Hardy class H^2, then $|f'(e^{i\theta})| \geq |g'(e^{i\theta})|$ a.e. on the unit circle. Our first result says that if $f, g \in H^\infty(D)$ and $T_{f+\overline{g}}$ is hyponormal, then $\overline{\lim}_{z \to e^{i\theta}}(|f'(z)| - |g'(z)|) \geq 0$ for all $e^{i\theta}$. Our second result says that if, in addition, there is an arc I on the circle such that $f' \in H^2(I)$, (this is defined precisely in Section 2), then g' has the same property and $|f'(e^{i\theta})| \geq |g'(e^{i\theta})|$ a.e. on I. This last result can be viewed as a local version of Sadraoui's result and it contains his theorem as a special case.

Section 1.

We begin with an example of a function u such that $Tu \geq u$ in D but u is not subharmonic. Note that $\Lambda(a) = \int_D |\varphi_a| \frac{dA}{\pi}$ is continuous and $\Lambda(0) = 2/3$ so there exists $\delta > 0$ such that $\Lambda(a) > \frac{1}{2}$ if $|a| < \delta$. Now let u be any strictly convex function that is continuous and integrable on $[0,1)$ such that $u(0)=u(\alpha)=0$ for some $0<\alpha<\frac{1}{2}$. Then we have $u(r)<0$ for $0<r<\alpha$ and u has a minimum at a unique point β, $0<\beta<\alpha$. We further assume that $\beta<\delta$. We regard u as a radial function on D. We claim any such u satisfies $Tu\geq u$. First suppose $|a|\leq\beta$ then $u(a)=u(|a|)<0$. On the other hand

$$\int|\varphi_a|\frac{dA}{\pi}\geq\frac{1}{2}>\alpha$$

so

$$0<u\left(\int|\varphi_a|\frac{dA}{\pi}\right)\leq\int u\circ\varphi_a\frac{dA}{\pi},$$

the latter inequality is Jensen's. Hence

$$u(a)\leq\int u\circ\varphi_a\frac{dA}{\pi}$$

in this case.

If $|a|>\beta$ we have $a=\int\varphi_a\frac{dA}{\pi}$ and hence $|a|\leq\int|\varphi_a|\frac{dA}{\pi}$ and therefore

$$u(a)\leq u\left(\int|\varphi_a|\frac{dA}{\pi}\right),$$

because u is strictly increasing on $(\beta,1)$. Another application of Jensen's inequality proves that $u(a)\leq\int u\circ\varphi_a\frac{dA}{\pi}$ in this case. Clearly u is not subharmonic since $u(0)=0$ and

$$\frac{1}{2\pi}\int_0^{2\pi}u(re^{i\theta})d\theta=u(r)<0$$

if $0<r<\alpha$.

Suppose $u\in C^2(D)$ and $0<r<1$, then starting from one of Green's identities we obtain the familiar formula

$$u(0)=\frac{1}{2\pi}\int_0^{2\pi}u(re^{i\theta})d\theta+\frac{2}{\pi}\int_{|\zeta|\leq r}\Delta u(\zeta)\log\frac{|\zeta|}{r}dA(\zeta), \tag{1}$$

which we may rewrite as

$$\frac{1}{2\pi}\int_0^{2\pi}u(re^{i\theta})d\theta-u(0)=\frac{2}{\pi}\int_{|\zeta|\leq r}\Delta u(\zeta)\log\frac{r}{|\zeta|}dA(\zeta)\,. \tag{2}$$

Next we multiply both sides of (2) by $2r$ and integrate on r from 0 to 1. We obtain

$$(Tu)(0)-u(0)=\int_{|\zeta|<1}\Delta u(\zeta)K(\zeta)dA(\zeta)\,, \tag{3}$$

where

$$K(\zeta) = \frac{4}{\pi}\int_{|\zeta|}^{1} r\log\frac{r}{|\zeta|}dr = \frac{1}{\pi}\left[\log\frac{1}{|\zeta|^2} - (1-|\zeta|^2)\right]. \tag{4}$$

So far this is a purely formal calculation. To see what conditions are required on u, we look at the kernel K. We let $f(x) = \log\frac{1}{x} - (1-x)$, then an application of Taylor's formula with remainder shows that

$$f(x) = \frac{1}{2t^2}(x-1)^2 \quad \text{where } 0 < x < t < 1. \tag{5}$$

From this we see that $f(x) \geq 0$, $0 < x < 1$ and

$$f(x) \geq \frac{1}{2}(1-x)^2 \text{ for } 0 < x < 1, \text{ and } f(x) \leq 2(1-x)^2 \text{ for } \frac{1}{2} < x < 1. \tag{6}$$

So (3) holds if $u \in C^2(D)$ and if

$$\int_{|\zeta|<1} |u(\zeta)|dA(\zeta) < \infty \quad \text{and} \quad \int_{|\zeta|<1} |\Delta u(\zeta)|(1-|\zeta|^2)^2 dA(\zeta) < \infty.$$

Now we wish to apply (3) not to u but to $u \circ \varphi_z$. This yields

$$Tu(z) - u(z) = \int_{|\zeta|<1} \Delta(u\circ\varphi_z)(\zeta)K(\zeta)dA(\zeta).$$

Recalling that $\Delta(u\circ\varphi_z)(\zeta) = (\Delta u)(\varphi_z(\zeta))|\varphi_z'(\zeta)|^2$and making the change of variables $\omega = \varphi_z(\zeta)$ we arrive at the following

Theorem 1. *Suppose that $u \in C^2(D)$ and that*

$$\int_{|\zeta|<1} |u(\zeta)|dA(\zeta) < \infty$$

and

$$\int_{|\zeta|<1} |\Delta u(\zeta)|(1-|\zeta|^2)^2 dA(\zeta) < \infty.$$

Then

$$Tu(z) - u(z) = \int_{|\zeta|<1} \Delta u(\zeta)K(z,\zeta)dA(\zeta)$$

where

$$K(z,\zeta) = \frac{1}{\pi}\left[\log\frac{1}{|\varphi_z(\zeta)|^2} - (1-|\varphi_z(\zeta)|^2)\right].$$

Moreover the kernel K satisfies:

$$K(z,\zeta) \geq \frac{1}{2\pi}\left[\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{z}\zeta|^2}\right]^2 \quad \textit{for } z,\zeta \in D \tag{7}$$

and

$$K(z,\zeta) \leq \frac{2}{\pi}\left[\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{z}\zeta|^2}\right]^2 \quad \textit{if} \tag{8}$$
$$\frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{z}\zeta|^2} < \frac{1}{2}.$$

Proof. Everything has been proved except (7) and (8) but they follow from (6) and the well-known identity

$$1-|\varphi_z(\zeta)|^2 = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{z}\zeta|^2}.$$

□

The following well-known estimate is proved by a straightforward calculation that we omit.

Lemma 1. *There exists a constant $C_0 > 0$ such that*

$$\int_{|\zeta|<1} \frac{(1-|\zeta|^2)^2}{|1-\bar{z}\zeta|^4}dA(\zeta) \geq C_0 \log\frac{1}{1-|z|}.$$

Theorem 2. *Suppose that $u \in C^2(D)$,*

$$\int_{|\zeta|<1} |u(\zeta)|dA(\zeta) < \infty,$$

$$\int_{|\zeta|<1} |\Delta u(\zeta)|(1-|\zeta|^2)^2 dA(\zeta) < \infty,$$

and that $\overline{\lim}_{z\to\zeta_0}\Delta u(z) < 0$ for some $\zeta_0 \in \partial D$. Then there exists $\delta > 0$ such that $Tu(z) < u(z)$ for all $z \in D$ such that $|z-\zeta_0| < \delta$.

Proof. For convenience we assume that $\zeta_0 = 1$. By assumption there exists $a > 0$ and $\epsilon > 0$ such that if $z \in D$ and $|z-1| < \epsilon$, then $\Delta u(z) \leq -a$. If $D(1,\epsilon)$ denotes the set of points in D with $|z-1| < \epsilon$ and $D(1,\epsilon)'$ the complement of $D(1,\epsilon)$ in D, then we have

$$\int_D \Delta u(\zeta)K(z,\zeta)dA(\zeta) = \int_{D(1,\epsilon)} \Delta u(\zeta)K(z,\zeta)dA(\zeta) + \int_{D(1,\epsilon)'} \Delta u(\zeta)K(z,\zeta)dA(\zeta).$$

We deal with the second integral: if $|z-1|<\epsilon/2$ and $\zeta \in D(1,\epsilon)'$, then $\left|1-\overline{\zeta}z\right|$ is bounded away from 0 and hence

$$\frac{(1-|z|^2)(1-|\zeta|^2)}{\left|1-\overline{\zeta}z\right|^2} \leq C(1-|z|^2) < 1/2$$

if $1-|z|^2$ is sufficiently small, and hence by (8) we have $K(z,\zeta) \leq C(1-|z|^2)^2(1-|\zeta|^2)^2$, so

$$\left|\int_{D(1,\epsilon)'} \Delta u(\zeta)K(z,\zeta)dA(\zeta)\right| \leq C(1-|z|^2)^2 \int_{D(1,\epsilon)'} |\Delta u(\zeta)|(1-|\zeta|^2)^2 dA(\zeta).$$

Note that this is $O\left((1-|z|^2)^2\right)$. Next

$$\begin{aligned}\int_{D(1,\epsilon)} \Delta u(\zeta)K(z,\zeta)dA(\zeta) &\leq -a\int_{D(1,\epsilon)} K(z,\zeta)dA(\zeta)\\ &= -a\int_D K(z,\zeta)dA(\zeta) + a\int_{D(1,\epsilon)'} K(z,\zeta)dA(\zeta)\\ &\leq -\frac{a}{2\pi}\int_D \frac{(1-|z|^2)^2(1-|\zeta|^2)^2}{\left|1-\overline{\zeta}z\right|^4}dA(\zeta)\\ &\quad + \frac{2a}{\pi}\int_{D(1,\epsilon)'} \frac{(1-|z|^2)^2(1-|\zeta|^2)^2}{\left|1-\overline{\zeta}z\right|^4}dA(\zeta)\\ &\leq -C_0 a(1-|z|^2)^2\log\frac{1}{1-|z|} + O\left((1-|z|^2)^2\right).\end{aligned}$$

Here we have used (7) and (8) again as well as Lemma 1. Combining these estimates we have, for $|1-z|<\epsilon/2$,

$$Tu(z)-u(z) \leq -C_0 a(1-|z|^2)^2\log\frac{1}{1-|z|} + O((1-|z|^2)^2),$$

which becomes negative as z approaches 1. □

The next lemma shows that the inequality $Tu \geq u$ is preserved under certain convolutions.

Lemma 2. *Suppose $u \in L^1(D)$ and $Tu \geq u$ in D. Suppose $w \geq 0$ is a bounded measurable function on the circle. Define, for $z \in D$,*

$$U(z) = \frac{1}{2\pi}\int_0^{2\pi} u(ze^{-it})w(e^{it})dt. \tag{9}$$

Then $U \in L^1(D)$ *and*

$$TU \geq U \quad \text{in } D.$$

Proof. Note that if $z = re^{i\theta}$, then

$$\begin{aligned} U(re^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} u\left(re^{i(\theta-t)}\right) w(e^{it})dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(re^{it}) w\left(e^{i(\theta-t)}\right) dt. \end{aligned}$$

By hypothesis,

$$u(re^{it}) \leq \frac{(1-r^2)^2}{\pi}\int_0^1 \rho \int_0^{2\pi} \frac{u\left(\rho e^{i(t-s)}\right)}{|1-r\rho e^{is}|^4} ds d\rho.$$

Since $w \geq 0$ we can multiply both sides of this inequality by $w(e^{i(\theta-t)})$ and integrate on t. After interchanging the order of integration we get

$$\begin{aligned} U(re^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi} u(re^{it}) w\left(e^{i(\theta-t)}\right) dt \\ &\leq \frac{(1-r^2)^2}{\pi}\int_0^1 \rho \int_0^{2\pi} \frac{1}{|1-r\rho e^{is}|^4}\int_0^{2\pi} u\left(\rho e^{i(t-s)}\right) w\left(e^{i(\theta-t)}\right)\frac{dt}{2\pi} ds d\rho \\ &= \frac{(1-r^2)^2}{\pi}\int_0^1 \rho \int_0^{2\pi} \frac{1}{|1-r\rho e^{is}|^4}\int_0^{2\pi} u(\rho e^{it}) w\left(e^{i(\theta-t-s)}\right)\frac{dt}{2\pi} ds d\rho \\ &= \frac{(1-r^2)^2}{\pi}\int_0^1 \rho \int_0^{2\pi} \frac{1}{|1-r\rho e^{is}|^4} U\left(\rho e^{i(\theta-s)}\right) ds d\rho \\ &= (TU)(re^{i\theta}). \end{aligned}$$

□

The next theorem says that if $Tu \geq u$ and Δu has a weak* limit on some interval, that limit is non-negative.

Theorem 3. *Suppose that* $u \in C^2(D) \cap L^1(D)$, *and that* $\int_D |\Delta u(\zeta)|(1-|\zeta|^2)^2 dA(\zeta) < \infty$. *Suppose further that* $Tu \geq u$ *in* D *and that there is a closed arc* I *on the boundary of the unit circle and a finite Borel measure* μ *on* I *such that for all continuous functions* φ *on* I *we have*

$$\lim_{r\to 1}\int_I \Delta u(re^{i\theta})\varphi(e^{i\theta})\frac{d\theta}{2\pi} = \int_I \varphi d\mu,$$

then $\mu \geq 0$ *on* $\overset{\circ}{I}$, *the interior of* I.

Proof. Let $w(e^{-it})$ be a continuous non-negative function with compact support in $\overset{\circ}{I}$, let

$$U(z) = \frac{1}{2\pi}\int_0^{2\pi} u(ze^{-it}) w(e^{it}) dt.$$

From Lemma 2 we know that $TU \geq U$ in D. Since the Laplacian commutes with rotations it follows from (9) that

$$\Delta U(z) = \frac{1}{2\pi}\int_0^{2\pi} (\Delta u)(ze^{-it})w(e^{it})dt, \tag{10}$$

and hence that

$$\int_D |\Delta U(\zeta)|(1-|\zeta|^2)^2 dA(\zeta) < \infty.$$

It follows from Theorem 2 that there exists $r_k \to 1$ and $\theta_k \to 0$ such that $\lim_{k\to\infty} \Delta U(r_k e^{i\theta_k}) \geq 0$. Now it follows from (10) that

$$\Delta U(r_k e^{i\theta_k}) = \frac{1}{2\pi}\int_0^{2\pi} \Delta u(r_k e^{it}) w\left(e^{i(\theta_k - t)}\right) dt.$$

Notice that for all k sufficiently large $w\left(e^{i(\theta_k-t)}\right)$ will have its support in $\mathring{I}$. We have

$$\begin{aligned}
&\int_I w(e^{-it})d\mu(t) - \Delta U(r_k e^{i\theta_k}) \\
&\quad = \int_I w(e^{-it})d\mu(t) - \int_I \Delta u(r_k e^{it}) w(e^{-it})\frac{dt}{2\pi} \\
&\qquad + \int_I \Delta u(r_k e^{it})\left[w(e^{-it}) - w\left(e^{i(\theta_k-t)}\right)\right]\frac{dt}{2\pi}.
\end{aligned}$$

The first difference above goes to 0 as $r_k \to 1$ by hypothesis. The second difference is bounded in modulus by

$$\left(\sup_k \int_I |\Delta u(r_k e^{it})|\frac{dt}{2\pi}\right)\left(\sup_t \left|w(e^{-it}) - w\left(e^{i(\theta_k-t)}\right)\right|\right).$$

The first factor is bounded, by the principle of uniform boundedness and the second goes to zero as $k \to \infty$ by the uniform continuity of w. We have shown that $\int_I w(e^{-it})d\mu(t) \geq 0$ for all non-negative $w(e^{-it})$ continuous with compact support in $\mathring{I}$; the result follows. $\square$

Section 2.

Now suppose that f and g are holomorphic in D and $f + \overline{g} = \varphi$ is bounded. We wish to calculate $\|T_\varphi F\|_2^2$ for $F \in H^\infty(D)$

$$T_\varphi F = P(f+\overline{g})F = fF + P(\overline{g}F),$$

so

$$\begin{aligned}\|T_\varphi F\|_2^2 &= \langle fF + P(\overline{g}F), fF + P(\overline{g}F)\rangle \\ &= \|fF\|_2^2 + \|P\overline{g}F\|_2^2 + \langle P\overline{g}F, fF\rangle + \langle fF, P\overline{g}F\rangle \\ &= \|fF\|_2^2 + \|P\overline{g}F\|_2^2 + \left\langle \overline{f}\overline{g}F, F\right\rangle + \langle fgF, F\rangle,\end{aligned}$$

since P is self-adjoint.

By interchanging the roles of f and g we see that

$$\|T_{\overline{\varphi}}F\|_2^2 = \|gF\|_2^2 + \left\langle \overline{f}\overline{g}F, F\right\rangle + \langle fgF, F\rangle + \left\|P\overline{f}F\right\|_2^2.$$

Hence T_φ is hyponormal if and only if

$$\|fF\|_2^2 + \|P\overline{g}F\|_2^2 \geq \|gF\|_2^2 + \left\|P\overline{f}F\right\|_2^2 \tag{9}$$

for all $F \in H^\infty(D)$.

In particular (9) holds if $F = k_z$ for some $z \in D$. Now it is immediate that $\overline{g}k_z - \overline{g(z)}k_z \perp B(D)$ for any $g \in H^\infty(D)$ and hence that $P(\overline{g}k_z) = \overline{g(z)}k_z$.

Theorem 4. *Suppose that f and g are holomorphic in D, that $f + \overline{g} = \varphi$ is bounded in D and that T_φ is hyponormal, then $Tu \geq u$ in D where $u(z) = |f(z)|^2 - |g(z)|^2$.*

Proof. By the above discussion, if we let $F = k_z$ in (9) we get

$$\|fk_z\|_2^2 + |g(z)|^2\|k_z\|_2^2 \geq \|gk_z\|_2^2 + |f(z)|^2\|k_z\|_2^2. \tag{10}$$

Since $\|k_z\|_2^2 = \dfrac{1}{\pi(1-|z|^2)^2}$, a minor rearrangement of (10) proves the theorem. □

Corollary. *Suppose that f and g are holomorphic in D, that $f + \overline{g} = \varphi$ is bounded in D and that T_φ is hyponormal, then $\overline{\lim}_{z\to\zeta}(|f'(z)|^2 - |g'(z)|^2) \geq 0$ for every $\zeta \in \partial D$. In particular, if f' and g' are continuous at $\zeta \in \partial D$, then $|f'(\zeta)| \geq |g'(\zeta)|$.*

Proof. The proof follows from the theorem and the simple observation that $\Delta|f|^2 = |f'|^2$ for any holomorphic f. □

Suppose that f is holomorphic in an open set of the form

$$\{re^{i\theta} : r_0 < r < 1 \quad \text{and} \quad e^{i\theta} \in I\}$$

where I is some open arc on the boundary of the unit circle. We say that $f \in H^2(I)$ if

(i) f has polynomial growth i.e., there exists $A > 0$ such that $f(re^{i\theta}) = O((1-r)^{-A})$ for all $e^{i\theta} \in I$.

(ii) There exists $r_k \to 1$ such that

$$\int_I |f(r_k e^{i\theta})|^2 d\theta \leq C < \infty, \quad \text{all} \quad k.$$

The next lemma is standard. Since we know of no convenient references we indicate the proof.

Lemma 3. *Suppose $f \in H^2(I)$, then there exists $F \in L^2(I)$ such that $\lim_{r\to 1} f(re^{i\theta}) = F(e^{i\theta})$ a.e. on I and for every compact subinterval $J \subset I$*

$$\lim_{r\to 1} \int_J |f(re^{i\theta}) - F(e^{i\theta})|^2 d\theta = 0.$$

In particular, $\overline{\lim}_{r\to 1} \int_J |f(re^{i\theta})|^2 d\theta < \infty$.

Proof. Pick a compact interval L such that $J \subseteq \mathring{L} \subseteq L \subseteq I$. Let $e^{i\theta_1}, e^{i\theta_2}$ be the end points of L and choose N such that

$$\lim_{r\to 1} [(re^{i\theta} - e^{i\theta_1})(re^{i\theta} - e^{i\theta_2})]^N f(re^{i\theta}) = 0$$

if $\theta = \theta_1$ or θ_2. This is possible by i). Let $g(z) = [(z - e^{i\theta_1})(z - e^{i\theta_2})]^N f(z)$. Let $r_0 < r_1 < 1$ and $\Delta_k = \{re^{i\theta} : r_1 \leq r \leq r_k, e^{i\theta} \in L\}$. Let $\partial\Delta_k = \Gamma_k \cup L_k$ where $L_k = \{r_k e^{i\theta} : e^{i\theta} \in L\}$. If $z \in \mathring{\Delta}_k$ we have

$$g(z) = \frac{1}{2\pi i} \int_{\Gamma_k} \frac{g(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{L_k} \frac{g(\zeta)}{\zeta - z} d\zeta.$$

If we let $k \to \infty$ we get $g(z) = g_1(z) + g_2(z)$ where $g_1(z)$ is holomorphic on $\mathring{L}$ and $g_2(z)$ is the Cauchy integral of an L^2 function on the circle. It follows that the conclusions of the lemma hold for g and hence for f. □

Theorem 5. *Suppose that f and g are holomorphic in D, that $f + \overline{g} = \varphi$ is bounded in D and that T_φ is hyponormal. Suppose further that there is an open interval I such that $f' \in H^2(I)$. Then for any open subinterval $J \subseteq \overline{J} \subseteq I$ $g' \in H^2(J)$ and $|f'(e^{i\theta})| \geq |g'(e^{i\theta})|$ almost everywhere on I.*

Proof. Let $w(e^{-it})$ be a continuous function with compact support in I such that $0 \leq w \leq 1$ and $w(e^{-it}) \equiv 1$ on a neighborhood of $\overline{J}$, combining Theorems 2 and 3 with Lemma 2 we have the existence of $r_k \to 1$ and $\theta_k \to 0$

such that

$$\lim_{k\to\infty} \int_0^{2\pi} (|f'(r_k e^{it})|^2 - |g'(r_k e^{it})|^2) w\left(e^{i(\theta_k - t)}\right) dt \geq 0.$$

Let L be compact interval so that $\overline{J} \subseteq \overset{\circ}{L} \subseteq L \subseteq I$. As before, for large k, $w\left(e^{i(\theta_k - t)}\right)$ has support in L and hence,

$$\int_0^{2\pi} |f'(r_k e^{it})|^2 w\left(e^{i(\theta_k - t)}\right) dt$$
$$\leq \int_L |f'(r_k e^{it})|^2 dt \leq C < \infty, \quad \text{by Lemma 3.}$$

Also, for large k, $w(e^{i(\theta_k - t)}) \equiv 1$ on J from which it follows that

$$\underline{\lim}_{k\to\infty} \int_J |g'(r_k e^{it})|^2 dt \leq C < \infty.$$

Now since $g \in H^\infty(D), g'$ has polynomial growth and hence $g' \in H^2(J)$. It now follows that the measures $(|f'(re^{i\theta})|^2 - |g'(re^{i\theta})|^2)\frac{d\theta}{2\pi}$ have a weak $*$ limit as $r \to 1$, $e^{i\theta} \in J$, and that this limit is $(|f'(e^{i\theta})|^2 - |g'(e^{i\theta})|^2)\frac{d\theta}{2\pi}$. It follows that $|f'(e^{i\theta})| \geq |g'(e^{i\theta})|$ a.e. on J, and hence on I since $J \subseteq \overline{J} \subseteq I$, was arbitrary. $\square$

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