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THE SCHWARTZ SPACE OF A GENERAL SEMISIMPLE LIE GROUP V : SCHWARTZ CLASS WAVE PACKETS

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Suppose G is a connected semisimple Lie group. Then the tempered spectrum of G consists of families of representations induced unitarily from cuspidal parabolic subgroups. In the case that G has finite center, Harish-Chandra used Eisenstein integrals to construct wave packets of matrix coefficients for each series of tempered representations. He showed that these wave packets are Schwartz class functions and that each K-finite Schwartz function is a finite sum of wave packets. Thus he obtained a complete characterization of K-finite functions in the Schwartz space in terms of their Fourier transforms.

Now suppose that G has infinite center. Then every K-compact Schwartz function decomposes naturally as a finite sum of wave packets. A new feature of the infinite center case is that the wave packets into which it decomposes are not necessarily Schwartz class functions. This is because of interference between different series of representations when a principal series representation decomposes as a sum of limits of discrete series. There are matching conditions between the wave packets which are necessary in order that the sum be a Schwartz class function when the individual terms are not. In this paper it is shown that these matching conditions are also sufficient. This gives a complete characterization of K-compact functions in the Schwartz space in terms of their Fourier transforms.

1. Introduction.

Suppose G is a connected semisimple Lie group. Then the tempered spectrum of G consists of families of representations induced unitarily from cuspidal parabolic sub-groups. Each family is parameterized by the unitary characters of a Cartan subgroup. The Plancherel theorem expands Schwartz class functions on G in terms of the distribution characters of these tempered representations. Very roughly, for f in the Schwartz space $\mathcal{C}(G)$, we can write

$$f(x) = \sum_{H \in \mathrm{Car}(G)} f_H(x), x \in G \tag{1.1 a}$$

where Car(G) denotes a complete set of representatives for conjugacy classes of Cartan subgroups of G and

$$f_H(x) = \int_{\widehat{H}} \Theta(H : \chi)(R(x)f)m(H : \chi)d\chi. \tag{1.1 b}$$

Here $\Theta(H : \chi)$ denotes the distribution character of the representation $\pi(H : \chi)$ corresponding to $\chi \in \widehat{H}, R(x)f$ is the right translate of f by $x \in G$, and $m(H : \chi)d\chi$ is the Plancherel measure corresponding to $\pi(H : \chi)$.

Suppose that G has finite center and that $f \in \mathcal{C}(G)$ is K-finite where K is a maximal compact subgroup of G. Fix $H \in \mathrm{Car}(G)$. In [**HC1**, **2**, **3**] Harish-Chandra used Eisenstein integrals to construct wave packets of matrix coefficients of the representations $\pi(H : \chi), \chi \in \widehat{H}$. He showed that these wave packets are Schwartz class functions and that f_H is a finite sum of wave packets. Thus he obtained a complete characterization of K-finite functions in the Schwartz space in terms of their Fourier transforms.

Now suppose that G has infinite center Z_G. (For example, G could be the universal covering group of one of the non-compact simple Lie groups of hermitian type.) Let K be a maximal relatively compact subgroup. That is, $Z_G \subseteq K$ and K/Z_G is a maximal compact subgroup of G/Z_G. Then there are no K-finite functions in $\mathcal{C}(G)$. However the set $\mathcal{C}(G)_K$ of K-compact functions, those with K-types lying in a compact subset of $\widehat{K}$, is dense in $\mathcal{C}(G)$ [**H1**]. Let $H \in \mathrm{Car}(G)$. Then for every $f \in \mathcal{C}(G)_K, f_H$ again decomposes naturally as a finite sum of wave packets. A new feature of the infinite center case is that for $f \in \mathcal{C}(G)_K, f_H$ and the wave packets into which it decomposes are not necessarily Schwartz class functions. This is because of interference between different series of representations when a principal series representation decomposes as a sum of limits of discrete series. When G has infinite center, these limits of discrete series can be actual limits along continuous families of relative discrete series representations, and so occur in a non-trivial way in the Plancherel formula in the terms corresponding to different Cartan subgroups. This means that for $f \in \mathcal{C}(G)$ there are matching conditions between the terms $f_H, H \in \mathrm{Car}(G)$, which are necessary in order that the sum be a Schwartz class function when the individual terms are not. These matching conditions generalize those of H. Kraljević and D. Miličić for the universal covering group of $SL(2, \mathbf{R})$ [**KM**].

In order to obtain a complete characterization of the K-compact Schwartz class functions in this case it is necessary to study elementary mixed wave packets. These are finite sums of wave packets which patch together to form Schwartz class functions. They should be thought of as the basic building blocks from which Schwartz class functions are formed in the infinite center case. Elementary mixed wave packets were defined in [**H3**] and it was shown that every $f \in \mathcal{C}(G)_K$ is a finite sum of elementary mixed wave packets. In this paper we will show that every elementary mixed wave packet is a Schwartz class function. This completes the study of the Plancherel theorem and Schwartz space for general reductive Lie groups which was initiated in [**HW1, 2, 3, 4, 5**] and continued in [**H1, 2, 3**].

In order to explain the results of the paper more precisely and with a minimum of technical notation, we will assume for the remainder of this introduction that G is a simple, simply connected, non-compact real Lie group of hermitian type. Let K be a maximal relatively compact subgroup of G. Then $K = K_1 \times V$ where $K_1 = [K, K]$ is compact and $V \cong \mathbf{R}$ is a one-dimensional vector group in the center of K. Then $\{e^h : h \in i\mathfrak{v}^*\}$ gives a one-parameter family of one-dimensional characters of K. Now let $P = MAN$ be a cuspidal parabolic subgroup of G and $H = TA$ a Cartan subgroup of G with $T \subseteq K$ a maximal relatively compact Cartan subgroup of M. The characters $e^h, h \in i\mathfrak{v}^*$, give characters of T by restriction. Thus each $\chi \in \widehat{T}$ lies in a continuous family of characters of T of the form $\{\chi \otimes e^h : h \in i\mathfrak{v}^*\}$. Each character in the family corresponds to a relative discrete series or limit of discrete series representation $\pi(M : h)$ of M. Note that $\pi(M : h)$ depends only on the restriction of e^h to T, so that these representations may not all be distinct. Let $\lambda(h) \in i\mathfrak{t}^*$ denote the Harish-Chandra parameter of $\pi(M : h)$, let $\mathcal{C}$ be a Weyl chamber of $i\mathfrak{t}^*$, and let $\mathcal{D} = \{h \in i\mathfrak{v}^* : \lambda(h) \in \mathcal{C}\}$. Then $\mathcal{D}$ is an open interval and is unbounded just in case the representations $\pi(M : h), h \in \mathcal{D}$, are holomorphic or anti-holomorphic relative discrete series. Now

$$\{\pi(H : h : \nu) = \mathrm{Ind}_{MAN}^G(\pi(M : h) \otimes e^{i\nu} \otimes 1) : h \in \mathcal{D}, \nu \in \mathfrak{a}^*\}$$

is called a continuous family of representations of G corresponding to H.

Wave packets of Eisenstein integrals corresponding to a continuous family $\{\pi(H : h : \nu) : h \in \mathcal{D}, \nu \in \mathfrak{a}^*\}$ are defined as follows. Fix $\tau_1, \tau_2 \in \widehat{K}$ with the same Z_G character as χ and let W be a finite-dimensional complex vector space on which K acts on the left and right by (τ_1, τ_2). For $h \in i\mathfrak{v}^*$, let $\tau_{i,h} = \tau_i \otimes e^h, i = 1, 2$. In [**HW5**] we defined Eisenstein integrals $E(P) : \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* \times G \to W$ which are holomorphic in h and ν and are $(\tau_{1,h}, \tau_{2,h})$-spherical functions of matrix coefficients of the representations $\pi(H : h : \nu)$ when $h \in \mathcal{D}, \nu \in \mathfrak{a}^*$. Then we defined wave packets of the form

$$\Phi(H : \mathcal{D} : x) = \int_{\mathcal{D} \times \mathfrak{a}^*} E(P : h : \nu : x)\alpha(h : \nu)m(H : h : \nu)d\nu dh \tag{1.2}$$

where $m(H : h : \nu)d\nu dh$ is the Plancherel measure corresponding to $\pi(H : h : \nu)$ and $\alpha : \mathcal{D} \times \mathfrak{a}^* \to \mathbf{C}$ is a jointly smooth function of h and ν which extends smoothly to $\mathrm{cl}(\mathcal{D}) \times \mathfrak{a}^*$ and is rapidly decaying at infinity in both variables. It was proven in [**H1**, **2**, **HW5**] that every K-compact Schwartz function is a finite sum of wave packets of this type and that an individual wave packet $\Phi(H : \mathcal{D})$ is Schwartz class if and only if $\alpha(h : \nu)$ has zeroes of infinite order at the finite endpoints of the interval $\mathcal{D}$ and if $\alpha(h : \nu)m(H : h : \nu)$ is jointly smooth on $\mathcal{D} \times \mathfrak{a}^*$. Finite endpoints of $\mathcal{D}$ correspond to limits of discrete series and points $(h, \nu) \in \mathcal{D} \times \mathfrak{a}^*$ at which $m(H : h : \nu)$ fails to be jointly smooth correspond to reducible principal series representations which decompose into limits of discrete series which are actual limits along continuous families of relative discrete series representations.

Let Φ_M denote the set of roots for $(\mathfrak{m}_\mathbf{C}, \mathfrak{t}_\mathbf{C})$ and choose a set Φ_M^+ of positive roots so that there is a unique non-compact simple root β. We will use $h \leftrightarrow \langle \beta, h \rangle$ to identify $i\mathfrak{v}^* \cong \mathbf{R}$. Fix $\chi \in \widehat{H}$ and let

$$F_0 = \{\alpha \in \Phi_M^+ : \langle \alpha, \lambda(0) \rangle = 0\}$$

where as before, $\lambda(h)$ is the Harish-Chandra parameter of the relative discrete series representation $\pi(M : h)$ of M corresponding to $\chi \otimes e^h$. If there is a compact root $\alpha \in F_0$, then $\langle \lambda(h), \alpha \rangle = \langle \lambda(0), \alpha \rangle = 0$ for all $h \in i\mathfrak{v}^*$ so that the Plancherel function $m(H : h : \nu)$ corresponding to $\pi(H : h : \nu)$ is zero for all $h \in i\mathfrak{v}^*, \nu \in \mathfrak{a}^*$. In this case the family plays no role in the Plancherel formula or the Schwartz space analysis. Thus we assume that F_0 contains no compact roots. Then $\lambda(h)$ is regular for small $h \neq 0$ and so there are Weyl chambers $\mathcal{C}^\pm$ of it^* so that $\lambda(h) \in \mathcal{C}^+$ for small $h > 0$ and $\lambda(h) \in \mathcal{C}^-$ for small $h < 0$. Write $\mathcal{D}^\pm = \{h \in i\mathfrak{v}^* : \lambda(h) \in \mathcal{C}^\pm\}$. (Of course if $F_0 = \emptyset$, then $0 \in \mathcal{D}^+ = \mathcal{D}^-$.)

Now each $F \subseteq F_0$ is a strongly orthogonal family of non-compact roots of M and so corresponds to Cartan subgroups $H_{M,F}$ of M and $H_F = H_{M,F}A = T_F A_F$ of G. We identify roots of H_F with those of H via the Cayley transform c_F corresponding to F. Let $P_F = M_F A_F N_F$ be a cuspidal parabolic subgroup corresponding to H_F. Then for each $F \subseteq F_0, T_F \subseteq T$ and we define $\chi_F \in \widehat{T}_F$ to be the restriction of χ. Let $\pi(F : h)$ be the relative discrete series representation of M_F corresponding to $\chi_F \otimes e^h$ and define

$$(1.3) \quad \pi(F : h : \nu_F) = \mathrm{Ind}_{M_F A_F N_F}^G (\pi(M_F : h) \otimes e^{i\nu_F} \otimes 1), h \in i\mathfrak{v}^*, \nu_F \in \mathfrak{a}_F^*.$$

We call $\{\pi(F : h : \nu_F) : F \subseteq F_0\}$ a matching family of representations.

Now elementary mixed wave packets are defined roughly as follows. (See (2.16) for the precise definition.) Fix a matching family $\{\pi(F : h : \nu_F) : F \subseteq F_0\}$ as in (1.3) such that the Plancherel function $m(H : h : \nu)$ is jointly smooth on $[0, a) \times \mathfrak{a}^*$ and $(-a, 0] \times \mathfrak{a}^*$ for some $a > 0$. Suppose for each $F \subseteq F_0$ we have $\Phi(F) : i\mathfrak{v}^* \times \mathfrak{a}_F^* \times G \to W$ satisfying the following conditions. First, let $\Phi^{\pm}(F)$ denote the restriction of $\Phi(F)$ to $\mathcal{D}^{\pm} \times \mathfrak{a}_F^* \times G$. Then there are finitely many Eisenstein integrals $E_i^{\pm}(P_F)$ corresponding to the family $\{\pi(F : h : \nu_F) : h \in \mathcal{D}^{\pm}, \nu_F \in \mathfrak{a}_F^*\}$ and smooth, rapidly decreasing functions $\alpha_i^{\pm}$ as in (1.2) so that for all $h \in \mathcal{D}^{\pm}, \nu_F \in \mathfrak{a}_F^*, x \in G$,

$$\Phi^{\pm}(F : h : \nu_F : x) = \sum_i \alpha_i^{\pm}(h : \nu_F) E_i^{\pm}(P_F : h : \nu_F : x). \tag{1.4 a}$$

Second, there are a small neighborhood U of $0 \in i\mathfrak{v}^*$ and a compact subset $\omega \subset U$ so that

$$\Phi(F : h : \nu_F : x) = 0 \quad \text{for all } \nu_F \in \mathfrak{a}_F^*, x \in G, \quad \text{if } h \notin \omega. \tag{1.4 b}$$

U must be small enough that $U \subset (-a, a) \cap (\mathcal{D}^+ \cup \mathcal{D}^- \cup \{0\})$. Finally, the functions $\Phi(F)$ must satisfy the same matching conditions as the characters of the representations $\pi(F)$.

These matching conditions can be stated as follows. Fix $E \subseteq F_0$. For every $E \subseteq F \subseteq F_0, \mathfrak{a}_E \subseteq \mathfrak{a}_F$ and we can identify $\mathfrak{a}_F^* \cong \mathfrak{a}_E^* \oplus \mathbf{R}^{|F \backslash E|}$ by $\nu_F \leftrightarrow (\nu_E, (\mu_\alpha)_{\alpha \in F\backslash E})$ where ν_E is the restriction of ν_F to $\mathfrak{a}_E$ and $\mu_\alpha = \langle \nu_F, \alpha \rangle, \alpha \in F\backslash E$. Write $(\nu_E, 0)$ for the element $(\nu_E, (\mu_\alpha)_{\alpha \in F\backslash E})$ with $\mu_\alpha = 0$ for all $\alpha \in F\backslash E$ and define a differential operator on $i\mathfrak{v}^* \times \mathfrak{a}_F^*$ by $D_{F\backslash E} = \partial/\partial h - i\sum_{\alpha\in F\backslash E} \partial/\partial\mu_\alpha$. For $F \subseteq F_0$, let $F^c = F_0 \backslash F$.

Then for all $k \geq 0$,

$$\begin{aligned}
&\lim_{h\downarrow 0} (\partial/\partial h)^k \Phi(E : h : \nu_E : x) + (-1)^{|E^c|+1} \lim_{h \uparrow 0} (\partial/\partial h)^k \Phi(E : h : \nu_E : x) \\
&\quad = \sum_{E \subset F \subseteq F_0} c_{|F\backslash E|} \Big[\lim_{h\downarrow 0} D_{F\backslash E}^k \Phi(F : h : (\nu_E, 0) : x) \\
&\quad\quad + (-1)^{|F^c|} \lim_{h\uparrow 0} D_{F\backslash E}^k \Phi(F : h : (\nu_E, 0) : x) \Big]
\end{aligned} \tag{1.4c}$$

for all $\nu_E \in \mathfrak{a}_E^*, x \in G$. Here for all $p \geq 0, c_p = (d/dx)^p \tanh(x/2)|_{x=0}$.

We say that

$$\Phi(x) = \sum_{F \subseteq F_0} \int_{i\mathfrak{v}^*} \int_{\mathfrak{a}_F^*} \Phi(F : h : \nu_F : x) m(F : h : \nu_F) dh d\nu_F \tag{1.4 d}$$

is an elementary mixed wave packet. If $w^* \in W^*$ we say that

$$\phi(x) = \langle \Phi(x), w^* \rangle \tag{1.4 e}$$

is a scalar-valued elementary mixed wave packet.

Theorem 1.5 ([H3]). *Every $f \in \mathcal{C}(G)_K$ is the sum of finitely many scalar-valued elementary mixed wave packets.*

The main result of this paper is

Theorem 1.6. *Every elementary mixed wave packet is a Schwartz class function.*

The following results from [**H3**] will be used to prove Theorem 1.6. Let $\Phi(x)$ be an elementary mixed wave packet as in (1.4) and write

$$\Phi(h : x) = \sum_{F \subseteq F_0} \int_{\mathfrak{a}_F^*} \Phi(F : h : \nu_F : x) m(F : h : \nu_F) d\nu_F. \tag{1.7}$$

Theorem 1.8 ([**H3**]). *Let $\Phi(x)$ be an elementary mixed wave packet. Then*

$$(h, x) \to \Phi(h : x)$$

is jointly smooth on $i\mathfrak{v}^* \times G$.

Proposition 1.9 ([**H3**]). *Suppose $F : i\mathfrak{v}^* \times G \to W$ is $(\tau_{1,h}, \tau_{2,h})$-spherical and define*

$$F(x) = \int_{i\mathfrak{v}^*} F(h : x) dh.$$

Then $F(x)$ is a Schwartz class function on G if and only if

$$(h, x) \to F(h : x) \quad \text{is jointly smooth on } i\mathfrak{v}^* \times G$$

and

$$\sup_{h \in i\mathfrak{v}^*, a \in \mathrm{cl}(A_0^+)} \Xi(a)^{-1}(1 + \sigma(a))^r (1 + |h|)^r \|F(h; D : D_1 ; a; D_2)\| < \infty$$

for all $r \geq 0$, constant coefficient differential operators D on $i\mathfrak{v}^*$ and $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$.

Thus to complete the proof that elementary mixed wave packets are Schwartz functions we will need to prove the estimate in (1.9). In order to do this we will need an alternate formula for elementary mixed wave packets which was proven in [**H3**] by studying the Plancherel functions. For $h \in i\mathfrak{v}^*, F \subseteq F_0, \nu_F \in \mathfrak{a}_F^*$, write

$$p_F(h : \nu_F) = \prod_{\alpha \in F} (\langle \nu_F, \alpha \rangle + ih). \tag{1.10 a}$$

Then it was proven in [**H3**] that there are functions $g(F : h : \nu_F : x), F \subseteq F_0$, which are jointly smooth on $\mathrm{cl}(\mathcal{D}^{\pm}) \times \mathfrak{a}_F^* \times G$ and satisfy matching conditions similar to those satisfied by the $\Phi(F : h : \nu_F : x), F \subseteq F_0$, so that

$$\Phi(h : x) = \sum_{F \subseteq F_0} (\pi i)^{-|F|} \int_{\mathfrak{a}_F^*} \frac{g(F : h : \nu_F : x)}{p_F(h : \nu_F)} d\nu_F. \tag{1.10 b}$$

Rather than explain how the estimate

$$\sup_{h\in i\mathfrak{v}^*, a\in \text{cl}(A_0^+)} \Xi(a)^{-1}(1+\sigma(a))^r(1+|h|)^r\|\Phi(h;D:D_1;a;D_2)\| < \infty$$

is proven in general, we will specialize further to the simplest example, namely when G is the universal covering group of $SL(2,\mathbf{R})$. In this case we will have non-trivial mixed wave packets just in the case that $M = G$, $H = T$ is the relatively compact Cartan subgroup, and $\lambda(0) = 0$ so that $F_0 = \Phi_M^+ = \{\alpha\}$. In this case $\{\pi(T:h) : h > 0\}$ is the set of holomorphic relative discrete series representations and $\{\pi(T:h) : h < 0\}$ is the set of anti-holomorphic relative discrete series representations. Write $H_0 = H_{F_0}$. It is the Cartan subgroup given by $H_0 = ZA_0$ where Z is the center of G and $A_0 \simeq \mathbf{R}$ is a one-dimensional real vector group. The family $\{\pi(H_0 : h : \nu) : h \in \mathbf{R}, \nu \in \mathfrak{a}_0^* \simeq \mathbf{R}\}$ contains all unitary principal series representations of G. Note that $\pi(H_0 : h+2 : \nu) = \pi(H_0 : h : \nu)$ for all $h, \nu \in \mathbf{R}$ since $e^h|_{\mathbf{Z}} = 1$ for all $h \in 2\mathbf{Z}$. The ones which factor through $SL(2,\mathbf{R})$ are those corresponding to $h = 0$ which in this parameterization are the non-spherical principal series, and those corresponding to $h = 1$ which are the spherical principal series. Thus $\pi(H_0 : 0 : 0)$ is the only reducible one, and it is the direct sum of the two limits of discrete series $\pi(T : \pm : 0)$.

Now an elementary mixed wave packet will have the form

$$\begin{aligned}\Phi(x) = &\int_{-\infty}^{+\infty} \Phi(T:h:x)|h|dh \\ &+ \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(H_0:h:\nu:x)\frac{\nu \sinh \pi\nu}{\cosh \pi\nu - \cos \pi h} d\nu dh.\end{aligned} \tag{1.11}$$

Here for each $h \neq 0, \Phi(T : h : x)$ is a matrix coefficient of the relative discrete series representation $\pi(T : h)$ with K-types $\tau_{1,h}, \tau_{2,h}$. Further, for each $x \in G$, $h \mapsto \Phi(T : h : x)$ is compactly supported in a small neighborhood U of zero and is smooth except at $h = 0$ where it and its derivatives have jump discontinuities. For each $h, \nu, \Phi(H_0 : h : \nu)$ is a matrix coefficient of the principal series representation $\pi(H_0 : h : \nu)$ with K-types $\tau_{1,h}, \tau_{2,h}$. For each $x \in G$, $(h,\nu) \mapsto \Phi(H_0 : h : \nu : x)$ is jointly smooth, an even Schwartz class function of ν, and is non-zero only when $h \in U$. The matching conditions are that for all $k \geq 0, x \in G$, we have

$$\begin{aligned}\lim_{h\downarrow 0}(\partial/\partial h)^k\Phi(T:h:x) &+ \lim_{h\uparrow 0}(\partial/\partial h)^k\Phi(T:h:x) \\ &= (\partial/\partial h - i\partial/\partial\nu)^k\Phi(T:0:0).\end{aligned}$$

Write

$$g(T:h:x) = \Phi(T:h:x)|h|$$

and

$$g(H_0 : h : \nu : x) = \frac{i\pi\nu\Phi(H_0 : h : \nu : x)(\nu + ih)\sinh\pi(\nu + ih)}{(\cosh\pi(\nu + ih) - 1)}.$$

Then formula (1.10b) in this case becomes

$$\Phi(h : x) = g(T : h : x) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(H_0 : h : \nu : x)}{(\nu + ih)} d\nu. \tag{1.12}$$

Now using (1.8) and (1.9), in order to prove that Φ is a Schwartz class function it is sufficient to prove that for all $k, r \geq 0, D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$, we have

$$\sup_{h \in U, a \in \mathrm{cl}(A_0^+)} \Xi(a)^{-1}(1 + \sigma(a))^r |\Phi(h; (\partial/\partial h)^k : D_1 ; a; D_2)| < \infty. \tag{1.13}$$

(Note that the term $(1 + |h|)^r$ is not needed since Φ is compactly supported as a function of h.) The differential operators $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ are handled in the same way as for ordinary wave packets, so in this example we will just look at the case where $D_1 = D_2 = 1$.

Let $H \in \mathfrak{a}_0$ such that $\alpha(H) = 1$ and for $t \in \mathbf{R}$, define $a_t = \exp tH$. Then from the theory of constant terms in **[HW5]** we see that there are functions $c^{\pm}(H_0 : h : \nu)$ which are jointly smooth on $U \times \mathbf{R}$ and Schwartz as functions of ν so that if we write $d(H_0 : h : \nu : a_t) =$

$$g(H_0 : h : \nu : a_t) - e^{(i\nu - 1)t} c^+(H_0 : h : \nu) - e^{(-i\nu - 1)t} c^-(H_0 : h : \nu) \tag{1.14 a}$$

we have the following estimate. Given any $D \in D(\mathbf{R}^2)$, the constant coefficient differential operators in $\mathbf{R}^2$, there are constants C, r_1, and $\epsilon > 0$, so that for $t > 0, h \in U, \nu \in \mathbf{R}$,

$$|d(H_0 : h : \nu; D : a_t)| \leq Ce^{-(1+\epsilon)t}(1 + t)^r. \tag{1.14 b}$$

However, the theory of constant terms from **[HW5]** applied to the function $g(T : h : x)$ just says that given $D \in D(\mathbf{R})$ there are constants C, r_1, and $\epsilon > 0$ so that

$$|g(T : h; D : a_t)| \leq Ce^{-(1+\epsilon|h|)t}(1 + t)^r.$$

This estimate will not be good enough since we are interested in behavior at $h = 0$ where $\epsilon|h| = 0$. In §3 of this paper we will modify the theory of constant terms so that in this case we also obtain functions $c^{\pm}(T : h)$ which are smooth in U except at $h = 0$ so that if we write

$$d(T : h : a_t) = g(T : h : a_t) - e^{(h-1)t} c^+(T : h) - e^{(-h-1)t} c^-(T : h) \tag{1.15 a}$$

we have an estimate similar to the one above. That is, given any $D \in D(\mathbf{R})$ there are constants C, r_1, and $\epsilon > 0$, so that for all $t > 0, h \in U$,

$$|d(T : h; D : a_t)| \leq Ce^{-(1+\epsilon)t}(1+t)^r. \tag{1.15 b}$$

Because of the growth conditions satisfied by $g(T)$, $c^+(T : h) = 0$ if $h > 0$ and $c^-(T : h) = 0$ if $h < 0$. The constant terms $c^{\pm}(H_0 : h : \nu)$ and $c^{\pm}(T : h)$ satisfy the same matching conditions as the original functions $g(H_0 : h : \nu)$ and $g(T : h)$.

Using these constant terms, the Schwartz estimates can be made as follows. Recall that we want to prove that

$$\sup_{h \in U, t \geq 0} \Xi(a_t)^{-1}(1+t)^r |\Phi(h; (\partial/\partial h)^k : a_t)| < \infty.$$

Since $\Phi(h : x)$ is jointly smooth, it is enough to prove that for each of $U^{\pm} = \{h \in U : \pm h > 0\}$ we have

$$\sup_{h \in U^{\pm}, t > 0} \Xi(a_t)^{-1}(1+t)^r |\Phi(h; (\partial/\partial h)^k : a_t)| < \infty.$$

We will do the case that $h \in U^+$ here. The other case is similar.

Recall that

$$\Phi(h : a_t) = g(T : h : a_t) + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{g(H_0 : h : \nu : a_t)}{(\nu + ih)} d\nu.$$

Using (1.15) we have

$$\sup_{h \in U^+, t > 0} \Xi(a_t)^{-1}(1+t)^r \left|d(T : h; (\partial/\partial h)^k : a_t)\right|$$
$$\leq C \sup_{t > 0} \Xi(a_t)^{-1}(1+t)^{r+r_1} e^{-(1+\epsilon)t}.$$

But there are constants $D, q \geq 0$ so that $\Xi(a_t)^{-1} \leq De^t(1+t)^q$ for all $t > 0$. Thus

$$\sup_{t > 0} \Xi(a_t)^{-1}(1+t)^{r+r_1} e^{-(1+\epsilon)t} < \infty.$$

Using a calculus lemma from [**H3**, 7.6], there is a finite subset S of $D(\mathbf{R}^2)$ so that for all $t > 0$,

$$\sup_{h \in U^+} \left|(\partial/\partial h)^k \int_{-\infty}^{+\infty} \frac{d(H_0 : h : \nu : a_t)}{\nu + ih} d\nu\right|$$
$$\leq \sum_{D' \in S} \sup_{h \in U^+, \nu \in \mathbf{R}} |d(H_0 : h : \nu; D' : a_t)|.$$

But using (1.14), for each $D' \in S$ there are $C, r_1, \epsilon > 0$, so that

$$\sup_{h\in U^+,\nu\in\mathbf{R},t>0} \Xi(a_t)^{-1}(1+t)^r|d(H_0 : h : \nu; D' : a_t)| \\ \leq \sup_{t>0} \Xi(a_t)^{-1}(1+t)^{r+r_1}e^{-(1+\epsilon)t} < \infty$$

as above.

Thus it suffices to estimate

$$\sup_{h\in U^+,t>0} \Xi(a_t)^{-1}(1+t)^r \left|(\partial/\partial h)^k \left[e^{(\pm h-1)t}c^{\pm}(T : h) \right.\right. \\ \left.\left. + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{(\pm i\nu -1)t}c^{\pm}(H_0 : h : \nu)}{\nu + ih} d\nu\right]\right|.$$

First, as above, the term $e^{-t}\Xi(a_t)^{-1}$ is of polynomial growth in t, and so it is enough to estimate

$$\sup_{h\in U^+,t>0} (1+t)^r \left|(\partial/\partial h)^k \left[e^{\pm ht}c^{\pm}(T : h) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{\pm i\nu t}c^{\pm}(H_0 : h : \nu)}{\nu + ih} d\nu\right]\right|.$$

First, recall that $c^+(T : h) = 0$. Thus in this case we need to estimate

$$\sup_{h\in U^+,t>0} (1+t)^r \left|(\partial/\partial h)^k \int_{-\infty}^{+\infty} \frac{e^{i\nu t}c^{+}(H_0 : h : \nu)}{\nu + ih} d\nu\right|.$$

Since $c^+(H_0 : h : \nu)$ is Schwartz as a function of ν, and $h, t > 0$, it can be proved using elementary calculus and contour integrals (see (7.5)) that for any $r, k \geq 0$,

$$\sup_{h\in U^+,t>0} (1+t)^r \left|(\partial/\partial h)^k \int_{-\infty}^{+\infty} \frac{e^{i\nu t}c^{+}(H_0 : h : \nu)}{\nu + ih} d\nu\right| < \infty.$$

This lemma does not hold for

$$\int_{-\infty}^{+\infty} \frac{e^{-i\nu t}c^{-}(H_0 : h : \nu)}{\nu + ih} d\nu.$$

For this case we must look at the two constant terms together and use the matching conditions.

It is easy to prove using a contour integral (see (7.1)) that for $h, t > 0$,

$$\mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{e^{-i\nu t}}{\nu + ih} d\nu = -2\pi i e^{-ht}.$$

Thus we can write

$$e^{-ht}c^-(T:h) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-i\nu t}c^-(H_0:h:\nu)}{\nu+ih}d\nu$$
$$= \frac{1}{2\pi i}\,\mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{e^{-i\nu t}[-c^-(T:h)+c^-(H_0:h:\nu)]}{\nu+ih}d\nu.$$

Now the matching conditions satisfied by the constant terms in this case are

$$\lim_{h\downarrow 0}(\partial/\partial h)^k c^-(T:h) = (\partial/\partial h - i\partial/\partial\nu)^k c^-(H_0:0:0)$$

for all $k \geq 0$ since $c^-(T:h) = 0$ for $h < 0$. Equivalently, we can write

$$\lim_{(h,\nu)\to(0,0),h>0}(\partial/\partial h - i\partial/\partial\nu)^k[-c^-(T:h)+c^-(H_0:h:\nu)] = 0$$

for all $k \geq 0$. But this implies (see [**H1**, 10.7]) that

$$\frac{-c^-(T:h)+c^-(H_0:h:\nu)}{\nu+ih}$$

extends to be smooth at $(0,0) \in \mathrm{cl}(U^+) \times \mathbf{R}$. Now another calculus lemma (see (7.6)) gives the required estimate for this term.

The organization of the paper is as follows.

In §2 we review definitions and theorems from [**H1**, **2**, **3**, **HW5**] and derive a consequence of the matching conditions which will be needed in §5 and §6.

In §3 we extend Harish-Chandra's theory of the constant term to obtain more exact asymptotic estimates near the walls of Weyl chambers. As in [**HW5**], it is necessary to study constant terms for a class of functions generalizing Eisenstein integrals and for a class of functions which will contain the functions $g(F)$ used to express the elementary mixed wave packets in (1.10). These new constant terms are a generalization of the constant terms used for the case of relative discrete series matrix coefficients of the universal cover of $SL(2,\mathbf{R})$ in (1.15).

In §4 we use the Casselman-Miličić theory of asymptotics (see [**CM**]) to obtain a meromorphic extension of the constant terms of Eisenstein integrals defined in §3 and study their poles. D. Miličić sketched the theory of the ordinary constant term in this context in a letter to J. Wolf in 1984.

In §5 we use the results of §4 to prove that all constant terms of the functions $g(F)$ are smooth.

In §6 we use the results on asymptotics and constant terms to prove that the elementary mixed wave packets are Schwartz functions.

In §7 we prove some calculus lemmas which are needed for §6.

2. Preliminaries.

Suppose G is a connected reductive Lie group. Fix a Cartan involution θ as in [**W**] and let K denote the fixed point set of θ. Then the center Z_G of G is contained in K, and K is the full inverse image of a maximal compact subgroup of the linear group G/Z_G. Write $K = K_1 \times V$ as in [**HW5**] where K_1 is the unique maximal compact subgroup of K and V is a closed normal vector subgroup of K such that $Z = Z_G \cap V$ is co-compact in both V and Z_G.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the ± 1 eigenspace decomposition under θ. (For any Lie group G we will use the corresponding lower case German letter $\mathfrak{g}$ to denote the real Lie algebra of G.) Choose a maximal abelian subspace $\mathfrak{a}_0 \subset \mathfrak{p}$ and a positive restricted root system $\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{a}_0)$. Let $\rho = 1/2 \sum_{\alpha \in \Phi^+} m(\alpha)\alpha$ where $m(\alpha)$ is the dimension of the root space of $\mathfrak{g}$ corresponding to α. For $x \in G$, define $H(x) \in \mathfrak{a}_0$ using the Iwasawa decomposition, $x \in K \exp(H(x)) N_0$. Then the zonal spherical function on G for $0 \in \mathfrak{a}_0^*$ is

$$\Xi(x) = \int_{K/Z} e^{-\rho(H(xk))} d(kZ). \tag{2.1 a}$$

Now decompose $x \in G$ as $x = v(x) k_1(x) \exp \xi(x)$ where $v(x) \in V, k_1(x) \in K_1$, and $\xi(x) \in \mathfrak{p}$. Polynomial growth in G is determined by the function

$$\widetilde{\sigma}(x) = \sigma_V(x) + \sigma(x) \tag{2.1 b}$$

where $\sigma_V(x) = \|v(x)\|$ and $\sigma(x) = \|\xi(x)\|$. Let W be a Banach space and $f \in C^\infty(G : W)$. If $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ and $r \geq 0$, define

$$_{D_1}\|f\|_{r,D_2} = \sup_{x \in G} (1 + \widetilde{\sigma}(x))^r \Xi(x)^{-1} \|f(D_1 ; x; D_2)\|_W. \tag{2.1 c}$$

The Schwartz space is

$$\mathcal{C}(G : W) = \{f \in C^\infty(G : W) : {}_{D_1}\|f\|_{r,D_2} < \infty \text{ for all } D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}}), r \geq 0\}. \tag{2.1d}$$

We write $\mathcal{C}(G) = \mathcal{C}(G : \mathbf{C})$.

Let $W = W(\tau_1 : \tau_2)$ be a finite-dimensional vector space on which K acts on the left and right by $\tau_1, \tau_2 \in \widehat{K}$. For any $h \in \mathfrak{v}_{\mathbf{C}}^*$, extend h to $\mathfrak{k}$ by making it trivial on $\mathfrak{k}_1$. Then e^h is a one-dimensional character of K which is unitary just in case $h \in i\mathfrak{v}^*$. For any $h \in \mathfrak{v}_{\mathbf{C}}^*$, write $\tau_{i,h} = \tau_i \otimes e^h$. Then $(\tau_{1,h}, \tau_{2,h})$ is a double unitary representation of K on W for all $h \in i\mathfrak{v}^*$. We will say $F : i\mathfrak{v}^* \times G \to W$ is $(\tau_{1,h}, \tau_{2,h})$-spherical if for all $k_1, k_2 \in K, x \in G, h \in i\mathfrak{v}^*$,

$$F(h : k_1 x k_2) = \tau_{1,h}(k_1) F(h : x) \tau_{2,h}(k_2). \tag{2.2}$$

For any finite-dimensional real vector space E, write $D(E)$ for the constant coefficient differential operators on E.

Proposition 2.3 [**H3**, 2.8]. *Suppose $F \in C^\infty(i\mathfrak{v}^* \times G : W)$ is $(\tau_{1,h}, \tau_{2,h})$-spherical, and suppose for all $r \geq 0, D \in D(i\mathfrak{v}^*), D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ that*

$$\sup_{h \in i\mathfrak{v}^*, a \in \mathrm{cl}(A_0^+)} \Xi^{-1}(a)(1+\sigma(a))^r(1+|h|)^r \|F(h; D : D_1 ; a; D_2)\| < \infty.$$

Then if

$$F(x) = \int_{i\mathfrak{v}^*} F(h : x) dh,$$

$F \in \mathcal{C}(G : W)$.

When K is non-compact there are no K-finite functions in $\mathcal{C}(G)$. The appropriate generalization in this case is the notion of a K-compact function defined as follows. For $\tau \in \widehat{K}$, let

$$\delta(\tau) = \deg(\tau^*)\,\mathrm{trace}(\tau^*) \tag{2.4 a}$$

denote the normalized character of the contragredient τ^* of τ. We say $f \in \mathcal{C}(G)$ is K-compact if there is a compact subset Ω of $\widehat{K}$ so that for $\tau \in \widehat{K}$,

$$\delta(\tau^*) *_K f = 0 = f *_K \delta(\tau), \tau \notin \Omega. \tag{2.4 b}$$

It was proven in [**H1**, 2.12] that the space $\mathcal{C}(G)_K$ of K-compact functions is dense in $\mathcal{C}(G)$.

Continuous families of tempered representations of G are defined as follows. Let $H = TA$ be a θ-stable Cartan subgroup of G and let $P = MAN$ be a parabolic subgroup associated to H. Let $\Phi_M = \Phi(\mathfrak{m}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ denote the roots of $\mathfrak{m}_{\mathbf{C}}$ with respect to $\mathfrak{t}_{\mathbf{C}}$, Φ_M^+ a choice of positive roots. Let ρ_M denote the half sum over Φ_M^+. For $h \in i\mathfrak{v}^* = \{h \in i\mathfrak{k}^* : h(\mathfrak{k}_1) = 0\}$, set $h_M(h) = h|_{\mathfrak{t}}$. Let

$$\Phi_{M,1} = \{\alpha \in \Phi_M : \langle \alpha, h_M(h) \rangle = 0 \text{ for all } h \in i\mathfrak{v}^*\}. \tag{2.5 a}$$

Let

$$\Lambda_{M,1} = \{\lambda \in i\mathfrak{t}^* : \lambda - \rho_M \text{ is integral and } \lambda \text{ is } \Phi_{M,1} \text{ non-singular }\}. \tag{2.5 b}$$

For $\lambda \in \Lambda_{M,1}$ set

$$X(\lambda) = \left\{\chi \in Z_M\,(M^0)\widehat{\ }: \chi|_{Z_{M^0}} \text{ is a multiple of } e^{\lambda - \rho_M}|_{Z_{M^0}}\right\} \tag{2.5 c}$$

and let

$$(2.5\,\mathrm{d})\qquad X(T)=\{(\lambda,\chi)\in\Lambda_{M,1}\times Z_M\,(M^0)^{\wedge}:\chi\in X(\lambda)\}\,.$$

Then for $(\lambda,\chi)\in X(T), h\in i\mathfrak{v}^*$, let $\lambda(h)=\lambda+h_M(h)$ and $\chi(h)=\chi\otimes e^h|_{Z_M(M^0)}$. Then if $\lambda(h)$ is regular we will write $\pi(h)$ for the relative discrete series representation of M^0 with Harish-Chandra parameter $\lambda(h)$. For ν in $\mathfrak{a}^*$ we set

$$(2.5\,\mathrm{e})\qquad \pi(H:\lambda:\chi:h:\nu)=\mathrm{Ind}^G_{Z_M(M^0)M^0AN}(\chi(h)\otimes\pi(h)\otimes e^{i\nu}\otimes 1)$$

and let

$$(2.5\,\mathrm{f})\qquad \Theta(H:\lambda:\chi:h:\nu)\text{ be the character of }\pi(H:\lambda:\chi:h:\nu).$$

Z is a central subgroup of $Z_M(M^0)$ so that each $\chi\in Z_M(M^0)^{\wedge}$ has a Z-character $\zeta(\chi)$. Let

$$(2.5\,\mathrm{g})\qquad \widehat{K}(\chi)=\left\{\tau\in\widehat{K}:\tau(kz)=\zeta(\chi:z)\tau(k)\text{ for all }k\in K, z\in Z\right\}.$$

Then all K-types of the representation $\pi(H:\lambda:\chi:h:\nu)$ lie in $\widehat{K}(\chi\otimes e^h)=\left\{\tau_h=\tau\otimes e^h:\tau\in\widehat{K}(\chi)\right\}$.

Holomorphic families of Eisenstein integrals corresponding to a continuous family of tempered representations are defined as follows. Fix $(\lambda,\chi)\in X(T)$ as above and $\tau_1,\tau_2\in\widehat{K}(\chi)$ acting on $W=W(\tau_1:\tau_2)$ on the left and right. Let $\mathcal{D}$ be a connected component of $\{h\in i\mathfrak{v}^*:\langle\lambda(h),\alpha\rangle\neq 0\text{ for all }\alpha\in\Phi_M^+\}$. We first define holomorphic families of spherical functions of matrix coefficients of the representations $\{\chi(h)\otimes\pi(h):h\in\mathcal{D}\}$ of $M^\dagger=Z_M(M^0)M^0$. A construction of these holomorphic families is given in [**HW3**, **4**]. For $h\in\mathcal{D}$, let $S(M^\dagger:W:h)$ be the set of all $\Psi(h):M^\dagger\to W$ such that

$$(2.6\ \mathrm{a})\qquad \Psi(h:k_1xk_2)=\tau_{1,h}(k_1)\Psi(h:x)\tau_{2,h}(k_2)$$
$$\text{for all }k_1,k_2\in K_M^\dagger=K\cap M^\dagger, x\in M^\dagger$$

and for each $w^*\in W^*$,
(2.6 b)
$$x\to\langle\Psi(h:x),w^*\rangle\text{ is a finite sum of matrix coefficients of }\chi(h)\otimes\pi(h).$$

Now let $S\,(M^\dagger:W)=S\,(M^\dagger:\lambda:\chi:\mathcal{D}:W)$ be the set of all $\Psi\in C^\infty(\mathfrak{v}_{\mathbf{C}}^*\times M^\dagger:W)$ such that

$$(2.7\,\mathrm{a})\qquad \Psi(h)\in S\,(M^\dagger:W:h)\ \text{ for all }h\in\mathcal{D},$$

(2.7 b) $h \to \Psi(h : m)$ is holomorphic on $\mathfrak{v}_{\mathbf{C}}^*$ for all $m \in M^\dagger$,

and

(2.7 c) Ψ satisfies a moderate growth condition.

(See [**H1**, 5.2] for the precise growth condition.)

Now for $\Psi \in S(M^\dagger : W)$, we extend Ψ to G by

(2.8 a)
$$\Psi(x) = \tau_{1,h}(\kappa(x))\Psi(h : \mu(x)), x = \kappa(x)\mu(x)\exp(H_P(x))n(x) \in KM^\dagger AN$$

and define the Eisenstein integral $E(P : \Psi) : \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* \times G \to W$ by

(2.8 b)
$$E(P : \Psi : h : \nu : x) = \int_{K/Z} \Psi(h : xk)\tau_{2,h}(k^{-1})e^{(i\nu-\rho_P)H_P(xk)}d(kZ).$$

Let $P(i\mathfrak{v}^* \times \mathfrak{a}^*)$ denote the set of all polynomial coefficient differential operators on $i\mathfrak{v}^* \times \mathfrak{a}^*$. For $\alpha \in C^\infty(\mathrm{cl}(\mathcal{D}) \times \mathfrak{a}^*)$ and $D \in P(i\mathfrak{v}^* \times \mathfrak{a}^*)$ define

(2.9 a)
$$\|\alpha\|_D = \sup_{(h,\nu)\in\mathcal{D}\times\mathfrak{a}^*} |D\alpha(h : \nu)|.$$

Then let

(2.9 b)
$$\mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0 = \{\alpha \in C^\infty(\mathrm{cl}(\mathcal{D}) \times \mathfrak{a}^*) : \|\alpha\|_D < \infty \text{ for all } D \in P(i\mathfrak{v}^* \times \mathfrak{a}^*)\}.$$

Now for any $\Psi \in \mathcal{S}(M^\dagger : W), \alpha \in \mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0$, define

(2.9)
$$\Phi(x) = \int_{\mathcal{D}\times\mathfrak{a}^*} E(P : \Psi : h : \nu : x)\alpha(h : \nu)m(H : h : \nu)d\nu dh$$

where $m(H : h : \nu)d\nu dh$ is the Plancherel measure corresponding to the representation $\pi(H : \lambda : \chi : h : \nu)$. Φ is called a wave packet of Eisenstein integrals.

Before we can define elementary mixed wave packets we must review the definition of matching families of tempered representations from [**H3**, §3]. Let $H = TA$ be a θ-stable Cartan subgroup of G, $P = MAN$ a parabolic subgroup associated to H. Fix $(\lambda, \chi) \in X(T)$ as in (2.5). Let $F_0 = \{\alpha \in \Phi_M^+ : \langle\alpha, \lambda\rangle = 0\}$. Then any subset F of F_0 is a strongly orthogonal system of non-compact roots in Φ_M. Let $H_{M,F}$ denote the corresponding Cartan subgroup of M. That is, the complexified Lie algebra of $H_{M,F}$ is obtained from that of T by Cayley transforms corresponding to the roots in F. Then $H_F = H_{M,F}A = T_F A_F$ is a Cartan subgroup of G. Let $P_F = M_F A_F N_F$ be a parabolic subgroup with split component A_F.

Let $F \subseteq F_0$. Because $T_F \subseteq T$, we can define data for tempered representations of G corresponding to H_F as follows. Let $\lambda_F = \lambda|_{\mathfrak{t}_F}$ and let χ_F be the restriction to $Z_{M_F}(M_F^0)$ of $\chi \otimes e^{\lambda-\rho_M}$. Then $(\lambda_F, \chi_F) \in X(T_F)$. For $h \in i\mathfrak{v}^*$, set $\lambda_F(h) = \lambda_F + h_{M_F}(h), \chi_F(h) = \chi_F \otimes e^h$. Write

$$\pi(F : h : \nu) = \pi(H_F : \chi_F : \lambda_F : h : \nu) \tag{2.10 a}$$

and let

(2.10 b) $\Theta(F : h : \nu)$ be the character of $\pi(F : h : \nu)$.

We call $\{\Theta(F : h : \nu) : F \subseteq F_0\}$ a family of matching characters corresponding to (λ, χ). Fix Cayley transforms $c_F : \mathfrak{h}_\mathbf{C} \to \mathfrak{h}_{F,\mathbf{C}}$. We will use these isomorphisms to identify linear functions on $\mathfrak{h}_{F,\mathbf{C}}$ for any $F \subseteq F_0$.

Given any chamber $\mathcal{C}$ of it^* and $\alpha \in \Phi_M$, set $\epsilon_\alpha(\mathcal{C}) = \text{sign } \langle \tau, \alpha \rangle, \tau \in \mathcal{C}$. Now let $\mathcal{C} \in C(\lambda)$, the set of all chambers with $\lambda \in \text{cl}(\mathcal{C})$. Then for all $\alpha \in \Phi_M^+ \backslash F_0, \epsilon_\alpha(\mathcal{C}) = \text{sign } \langle \lambda, \alpha \rangle$. Thus there is a bijection between $C(\lambda)$ and

$$\Sigma = \{(\epsilon_\alpha)_{\alpha \in F_0} : \epsilon_\alpha = \pm 1 \text{ for all } \alpha \in F_0\} \tag{2.11 a}$$

so that $\epsilon \in \Sigma \leftrightarrow \mathcal{C} = \mathcal{C}(\epsilon)$ if $\epsilon_\alpha = \epsilon_\alpha(\mathcal{C})$ for all $\alpha \in F_0$. Similarly for any $F \subseteq F_0$ there is a unique chamber $\mathcal{C}_F(\epsilon)$ with $\lambda_F \in \text{cl}(\mathcal{C}_F)$ and $\epsilon_\alpha(\mathcal{C}_F) = \epsilon_\alpha$ for all $\alpha \in F_0 \backslash F$. For $\epsilon \in \Sigma$, write

$$\mathcal{D}(\epsilon) = \{h \in i\mathfrak{v}^* : \lambda(h) \in \mathcal{C}(\epsilon)\}. \tag{2.11 b}$$

Let

$$\Sigma_0 = \{\epsilon \in \Sigma : \mathcal{D}(\epsilon) \neq \emptyset\}. \tag{2.11 c}$$

For any $\epsilon \in \Sigma_0, F \subseteq F_0$, set $\mathcal{D}_F(\epsilon) = \{h \in i\mathfrak{v}^* : \lambda_F(h) \in \mathcal{C}_F(\epsilon)\}$.

For any $\alpha \in F_0$, define

$$\mathcal{H}_\alpha = \{h \in i\mathfrak{v}^* : \langle h_M(h), \alpha \rangle = 0\}.$$

Define an equivalence relation on F_0 by $\alpha \sim \beta$ if $\mathcal{H}_\alpha = \mathcal{H}_\beta$. Write $F_0 = F_0^1 \cup F_0^2 \cup ... \cup F_0^m$ where the F_0^i are the distinct equivalence classes. Define

$$\epsilon_\alpha(h) = \text{ sign } \langle h_M(h), \alpha \rangle \in \{1, -1, 0\}.$$

Then by [**H3**, 3.6], the positive system Φ_M^+ can be chosen so that $\epsilon_\alpha(h)$ is independent of $\alpha \in F_0^i$. Write $\epsilon_i(h)$ for this common value and define $\mathcal{H}_i = \mathcal{H}_\alpha, \alpha \in F_0^i$.

Fix $h_i \in i\mathfrak{v}^*$ such that $\alpha(h_i) > 0$ for all $\alpha \in F_0^i$. For any smooth function f on $i\mathfrak{v}^*$, define

$$\partial/\partial h_i f(h) = d/dt|_{t=0} f(h + th_i). \tag{2.12 a}$$

Now for all $\alpha \in F_0^i$, pick $\mu_\alpha \in \mathfrak{a}_{F_0}^*$ such that $\mu_\alpha|_{\mathfrak{a}} = 0, \langle \mu_\alpha, c_{F_0}\alpha \rangle = \langle h_i, \alpha \rangle$, $\langle \mu_\alpha, c_{F_0}\beta \rangle = 0$ for all $\beta \in F_0, \beta \neq \alpha$. For any F we can consider $\mu_\alpha \in \mathfrak{a}_F^*$ by restriction from $\mathfrak{a}_{F_0}$ to $\mathfrak{a}_F$. Now for any smooth function f on $\mathfrak{a}_F^*$, define

$$\partial/\partial \mu_\alpha f(\nu) = d/dt|_{t=0} f(\nu + t\mu_\alpha). \tag{2.12 b}$$

Let $\epsilon \in \Sigma_0, 1 \leq i \leq m$. We will say that $h_0 \in \mathcal{H}_i \cap \mathrm{cl}(\mathcal{D}(\epsilon))$ is semiregular in $h_0 \notin \mathcal{H}_j$ for $1 \leq j \leq m, j \neq i$. We will say that $\mathcal{H}_i$ is a wall of $\mathcal{D}(\epsilon)$ if there are semiregular elements in $\mathcal{H}_i \cap \mathrm{cl}(\mathcal{D}(\epsilon))$. Write Σ_i for the set of all $\epsilon \in \Sigma_0$ such that $\mathcal{H}_i$ is a wall of $\mathcal{D}(\epsilon)$. For any $1 \leq i \leq m, \epsilon \in \Sigma_i$, define $\epsilon^\pm(i) \in \Sigma_i$ by

$$\epsilon^\pm(i)_\alpha = \begin{cases} \epsilon_\alpha, & \text{if } \alpha \in F_0 \backslash F_0^i; \\ \pm 1, & \text{if } \alpha \in F_0^i. \end{cases} \tag{2.13}$$

Now for any $1 \leq i \leq m$ and $\epsilon \in \Sigma_i$, both of $\epsilon^\pm(i) \in \Sigma_i$, ϵ is equal to one of $\epsilon^\pm(i)$, and $\mathcal{D}(\epsilon^+(i))$ and $\mathcal{D}(\epsilon^-(i))$ are separated only by the wall $\mathcal{H}_i$.

Now the matching conditions corresponding to the family $\{\pi(F : h : \nu) : F \subseteq F_0\}$ can be stated as follows. Let W be a finite-dimensional vector space and suppose for each $F \subseteq F_0$ we have

$$g(F) : i\mathfrak{v}^* \times \mathfrak{a}_F^* \to W$$

such that for each $\epsilon \in \Sigma_0$, the restriction $g(F : \epsilon)$ of $g(F)$ to $\mathcal{D}_F(\epsilon) \times \mathfrak{a}_F^*$ extends to be a smooth function on $\mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^*$. Then we say that $\{g(F) : F \subseteq F_0\}$ is a matching family if it satifies the following identities.

Fix $E \subseteq F_0$ and $1 \leq i \leq m, \epsilon \in \Sigma_i$. Write $E(i) = E \cup F_0^i$. For any $\nu_E \in \mathfrak{a}_E^*$ and F such that $E \subseteq F \subseteq E(i)$, define $(\nu_E, 0) \in \mathfrak{a}_F^*$ by $(\nu_E, 0)|_{\mathfrak{a}_E} = \nu_E, \langle (\nu_E, 0), c_F\alpha \rangle = 0$ for all $\alpha \in F \backslash E$. Write $D_{F\backslash E} = \partial/\partial h_i - i\sum_{\alpha \in F\backslash E} \partial/\partial \mu_\alpha$. Then for all $k \geq 0$,

$$\begin{aligned} &(\partial/\partial h_i)^k \left(g(E : \epsilon^+(i) : h_0 : \nu_E) - g(E : \epsilon^-(i) : h_0 : \nu_E)\right) \\ &= \sum_{E \subset F \subseteq E(i)} c_{|F\backslash E|} D_{F\backslash E}^k \\ &\quad \cdot \left(g(F : \epsilon^+(i) : h_0 : (\nu_E, 0)) + g(F : \epsilon^-(i) : h_0 : (\nu_E, 0))\right) \end{aligned} \tag{2.14}$$

for all $\nu_E \in \mathfrak{a}_E^*, h_0 \in \mathcal{H}_i \cap \mathrm{cl}(\mathcal{D}_E(\epsilon))$. Here for all

$$p \geq 0, c_p = (d/dx)^p \tanh(x/2)|_{x=0}.$$

For any $\epsilon \in \Sigma_0, F \subseteq F_0$, $\Theta(F : h : \nu : x)$ extends to a smooth function on $\mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^* \times G$. For $(h, \nu, x) \in \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^* \times G$, write

$$\widetilde{\Theta}(F : \epsilon : h : \nu : x) = \sigma_F(\epsilon)\Theta(F : h : \nu : x) \tag{2.15a}$$

where

$$\sigma_F(\epsilon) = \prod_{\alpha \in F_0 \setminus F} \epsilon_\alpha. \tag{2.15b}$$

Then by [**H3**, 3.11], for every $x \in G$ the family $\{g(F) : F \subseteq F_0\}$ given by $g(F : \epsilon : h : \nu) = \widetilde{\Theta}(F : \epsilon : h : \nu : x)$ satisfies the matching conditions of (2.14).

We are now ready to give the definition of elementary mixed wave packet. Fix $H = TA$ a θ-stable Cartan subgroup and $(\lambda, \chi) \in X(T), \tau_1, \tau_2 \in \widehat{K}(\chi)$. Let $U(0)$ be a neighborhood of 0 in $i\mathfrak{v}^*$ satisfying the conditions of [**H3**, 4.6] and (3.18). We assume that the Plancherel function $m(H : h : \nu)$ corresponding to $\pi(H : \lambda : \chi : h : \nu)$ is jointly smooth as a function of $(h, \nu) \in (U(0) \cap \mathrm{cl}(\mathcal{D})) \times \mathfrak{a}^*$ for every connected component $\mathcal{D}$ of $\{h \in i\mathfrak{v}^* : \langle \lambda(h), \alpha \rangle \neq 0, \alpha \in \Phi_M^+\}$. As in (2.10) we define F_0 and $H_F = T_F A_F, (\lambda_F, \chi_F) \in X(T_F)$ for every $F \subseteq F_0$. Let $\tau_1, \tau_2 \in \widehat{K}(\chi)$ and let $W = W(\tau_1 : \tau_2)$. Suppose for each $F \subseteq F_0$ we have a function

$$\Phi(F) : i\mathfrak{v}^* \times \mathfrak{a}_F^* \times G \to W.$$

Then we will say that

$$\Phi(x) = \sum_{F \subseteq F_0} \int_{i\mathfrak{v}^*} \int_{\mathfrak{a}_F^*} \Phi(F : h : \nu_F : x) m(H_F : \lambda_F : \chi_F : h : \nu_F) d\nu_F dh \tag{2.16 a}$$

is a (W-valued) elementary mixed wave packet if the functions $\Phi(F)$ satisfy the following conditions. First, there is a compact subset $\omega \subset U(0)$ so that for all $F \subseteq F_0, \nu_F \in \mathfrak{a}_F^*, x \in G, h \in i\mathfrak{v}^*$,

$$\Phi(F : h : \nu_F : x) = 0, h \notin \omega. \tag{2.16 b}$$

Second, let $W_F(\lambda, \chi) = \{w \in W(G, H_F) : w\lambda_F = \lambda_F, w\chi_F = \chi_F\}$. Then for all $w \in W_F(\lambda, \chi), \nu_F \in \mathfrak{a}_F^*, x \in G, h \in i\mathfrak{v}^*$,

$$\Phi(F : h : w\nu_F : x) = \Phi(F : h : \nu_F : x). \tag{2.16 c}$$

Third, for each $F \subseteq F_0, \epsilon \in \Sigma_0$, let $\Phi(F : \epsilon)$ denote the restriction of $\Phi(F)$ to $\mathcal{D}_F(\epsilon) \times \mathfrak{a}_F^* \times G$ where $\Sigma_0, \mathcal{D}_F(\epsilon)$ are defined as in (2.11). Then, using the notation of (2.7), (2.8), there are finitely many functions $\Psi_i \in \mathcal{S}(M_F^\dagger : \lambda_F : \chi_F : \mathcal{D}_F(\epsilon) : W), \alpha_i \in \mathcal{C}(\mathcal{D}_F(\epsilon) \times \mathfrak{a}_F^*)_0$ so that

(2.16 d) $$\Phi(F : \epsilon : h : \nu_F : x) = \sum_i \alpha_i(h : \nu_F) E(P_F : \Psi_i : h : \nu_F : x)$$

for all $(h, \nu_F, x) \in \mathcal{D}_F(\epsilon) \times \mathfrak{a}_F^* \times G$. Finally, we require that the functions $\Phi(F : h : \nu_F : x)$ satisfy the same matching conditions as the characters $\Theta(F : h : \nu : x)$. That is, for $F \subseteq F_0, \epsilon \in \Sigma_0$, and $(h, \nu, x) \in \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^* \times G$, define

$$\widetilde{\Phi}(F : \epsilon : h : \nu : x) = \sigma_F(\epsilon) \Phi(F : h : \nu : x).$$

Then for every $x \in G$ the functions

(2.16 e) $\left\{\widetilde{\Phi}(F : \epsilon : h : \nu : x)\right\}$ satisfy the matching conditions of (2.14).

Finally, if Φ is a W-valued elementary mixed wave packet and $w^* \in W^*$, we say that

(2.16 f) $$\phi(x) = \langle \Phi(x), w^* \rangle$$

is a scalar-valued elementary mixed wave packet.

Theorem 2.17 [**H3**, 4.2]. *Every $f \in \mathcal{C}(G)_K$ is the sum of finitely many scalar-valued elementary mixed wave packets.*

The main theorem of this paper is

Theorem 2.18. *Suppose that Φ and ϕ are elementary mixed wave packets defined as in* (2.16a) *and* (2.16f) *respectively. Then $\Phi \in \mathcal{C}(G : W)$ and $\phi \in \mathcal{C}(G)_K$.*

In order to prove (2.18) we will need (2.3) and the following results from [**H3**]. Suppose $\Phi(x)$ is defined as in (2.16a) and for $h \in i\mathfrak{v}^*$ set

(2.19 a) $$\Phi(h : x) = \sum_{F \subseteq F_0} \int_{\mathfrak{a}_F^*} \Phi(F : h : \nu_F : x) m(H_F : \lambda_F : \chi_F : h : \nu_F) d\nu_F.$$

Clearly $\Phi(h : x)$ is $(\tau_{1,h}, \tau_{2,h})$-spherical and

(2.19 b) $$\Phi(x) = \int_{i\mathfrak{v}^*} \Phi(h : x) dh.$$

Theorem 2.20 [**H**, 7.3]. *Let $\Phi(x)$ be a W-valued elementary mixed wave packet . Then $(h, x) \mapsto \Phi(h : x)$ is jointly smooth on $i\mathfrak{v}^* \times G$.*

For any $F \subseteq F_0$, $(h, \nu) \in i\mathfrak{v}^* \times \mathfrak{a}_F^*$, define

(2.21 a) $$p_F(h : \nu) = \prod_{\alpha \in F} (\nu_\alpha + i h_\alpha)$$

where

$$\text{(2.21 b)} \qquad \nu_\alpha = \frac{2\langle \nu, c_F\alpha\rangle}{\langle \alpha,\alpha\rangle} \quad \text{and} \quad h_\alpha = \frac{2\langle h_M(h),\alpha\rangle}{\langle\alpha,\alpha\rangle}.$$

Theorem 2.22 [**H3**, 5.3]. *Suppose for each $F \subseteq F_0$ we have functions*

$$\Phi(F) : i\mathfrak{v}^* \times \mathfrak{a}_F^* \times G \to W$$

satisfying (2.16b-e). *Then for each $F \subseteq F_0$,*

$$\int_{\mathfrak{a}_F^*} \Phi(F : h : \nu : x) m(H_F : \lambda_F : \chi_F : h : \nu) d\nu$$
$$= (\pi i)^{-|F|} \int_{\mathfrak{a}_F^*} \frac{g(F : h : \nu : x)}{p_F(h : \nu)} d\nu$$

where for any $\epsilon \in \Sigma_0, h \in \mathcal{D}_F(\epsilon)$, using the notation of [**H3**, §5],

$$g(F : h : \nu : x) = c\sigma_F(\epsilon)(\pi/2)^{|F|}\Phi(F : h : \nu : x)\pi(F : h : \nu)q(F : h : \nu)$$
$$\cdot \prod_{\alpha\in\Phi'_{F,R}} m_\alpha(F : h : \nu) \sum_{\psi\in\mathcal{T}_F} \epsilon(\psi)t(F : \psi : h : \nu).$$

Further, the functions $g(F)$ have the following properties. For any $\epsilon \in \Sigma_0$,

$$(h,\nu,x) \to g(F : h : \nu : x) \textit{ is jointly smooth on } \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^* \times G.$$

For any $D \in D(i\mathfrak{v}^* \times \mathfrak{a}_F^*), r \geq 0, g_1, g_2 \in \mathcal{U}(\mathfrak{g}_C)$, there are constants $C, s \geq 0$ so that

$$\|g(F : h : \nu; D : g_1; x; g_2)\|(1 + |\nu|)^r \leq C\Xi(x)(1 + \tilde{\sigma}(x))^s$$

for all $x \in G, h \in \mathcal{D}_F(\epsilon), \nu \in \mathfrak{a}_F^*$. Finally, for each $x \in G$, the functions $\{g(F : x) : F \subseteq F_0\}$ satisfy the matching conditions of (2.14).

Suppose that for each $F \subseteq F_0$ we have

$$g(F) : i\mathfrak{v}^* \times \mathfrak{a}_F^* \to W$$

satisfying the matching conditions of (2.14). Fix $\epsilon_0 \in \Sigma_0$. We may as well assume that the ordering of Φ_M^+ was chosen so that $(\epsilon_0)_i = 1$ for all $1 \leq i \leq m$. Now fix $1 \leq j \leq m$. Then if $\epsilon_0 \in \Sigma_j$, so that $\mathcal{H}_j$ is a wall of $\mathcal{D}(\epsilon_0)$, we have matching conditions corresponding to crossing the wall $\mathcal{H}_j$ satisfied at any $h_0 \in \mathrm{cl}(\mathcal{D}(\epsilon_0)) \cap \mathcal{H}_j$. However, we also need matching conditions corresponding to crossing the hyperplane $\mathcal{H}_j$ when $\mathcal{H}_j$ is not a wall of $\mathcal{D}(\epsilon_0)$. These will be a consequence of the basic matching conditions of (2.14). Let

$$I = \{1 \leq i \leq m : h_0 \in \mathcal{H}_i \text{ for all } h_0 \in \mathcal{H}_j \cap \operatorname{cl}(\mathcal{D}(\epsilon_0))\}. \tag{2.23}$$

Then if we define $\mathcal{H}_I = \cap_{i \in I} \mathcal{H}_i$, $\operatorname{cl}(\mathcal{D}(\epsilon_0)) \cap \mathcal{H}_I = \operatorname{cl}(\mathcal{D}(\epsilon_0)) \cap \mathcal{H}_j$. We will say $h_0 \in \operatorname{cl}(\mathcal{D}(\epsilon_0)) \cap \mathcal{H}_I$ is I-semiregular if $h_0 \notin \mathcal{H}_k$ for any $1 \leq k \leq m, k \notin I$. Because of our definition of I, the set of I-semiregular elements in $\operatorname{cl}(\mathcal{D}(\epsilon_0)) \cap \mathcal{H}_I$ is non-empty.

Fix $h_I \in i\mathfrak{b}^*$ such that $\langle \alpha, h_I \rangle > 0$ for all $\alpha \in F_0^I = \cup_{i \in I} F_0^i$. Then in (2.12) we could have chosen $h_i = h_I$ for all $i \in I$. Define $\mu_\alpha, \alpha \in F_0^I$, as in (2.12). Define $\epsilon_0^{\pm}(I) \in \Sigma$ by

$$\epsilon_0^{\pm}(I)_i = \begin{cases} (\epsilon_0)_i, & \text{if } i \notin I; \\ \pm 1, & \text{if } i \in I. \end{cases}$$

Then for any I-semiregular $h_0 \in \operatorname{cl}(\mathcal{D}(\epsilon_0)) \cap \mathcal{H}_I$, we have $h_0 + th_I \in \mathcal{D}(\epsilon_0^+(I))$ and $h_0 - th_I \in \mathcal{D}(\epsilon_0^-(I))$ for $0 < t$ sufficiently small. Note $\epsilon_0^+(I) = \epsilon_0 \in \Sigma_0$ because of our assumption about choice of positive roots, and $\epsilon_0^-(I) \in \Sigma_0$ because by the above, $\mathcal{D}(\epsilon_0^-(I)) \neq \emptyset$. For any $E \subseteq F_0$, define $E(I) = E \cup F_0^I$ and for $E \subseteq F \subseteq E(I)$, define $D_{F \setminus E} = \partial/\partial h_I - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha$.

Lemma 2.24. *Fix $E \subseteq F_0$. Then for all $k \geq 0$,*

$$\begin{aligned}
&(\partial/\partial h_I)^k (g(E : \epsilon_0^+(I) : h_0 : \nu_E) - g(E : \epsilon_0^-(I) : h_0 : \nu_E)) \\
&= \sum_{E \subset F \subseteq E(I)} c_{|F \setminus E|} D_{F \setminus E}^k [g(F : \epsilon_0^+(I) : h_0 : (\nu_E, 0)) \\
&\quad + g(F : \epsilon_0^-(I) : h_0 : (\nu_E, 0))]
\end{aligned}$$

for all $\nu_E \in \mathfrak{a}_E^*, h_0 \in \mathcal{H}_I \cap \operatorname{cl}(\mathcal{D}(\epsilon_0))$.

Proof. Fix $k \geq 0, \nu_E \in \mathfrak{a}_E^*, h_0 \in \mathcal{H}_I \cap \operatorname{cl}(\mathcal{D}(\epsilon_0))$, and for each $E \subseteq F \subseteq E(I)$ and $\epsilon \in \Sigma_0$ such that $h_0 \in \operatorname{cl}(\mathcal{D}(\epsilon))$, write

$$d(F : \epsilon) = D_{F \setminus E}^k g(F : \epsilon : h_0 : (\nu_E, 0)).$$

Then using (2.25) below, it is enough to prove that for all $E \subseteq F \subseteq E(I)$ we have

$$d(F : \epsilon_0^+(I)) = \sum_{F \subseteq F' \subseteq E(I)} d(F' : \epsilon_0^-(I))$$

and

$$d(F : \epsilon_0^-(I)) = \sum_{F \subseteq F' \subseteq E(I)} (-1)^{|F' \setminus F|} d(F' : \epsilon^+(I)).$$

We will prove only the first equality since the second is proved the same way.

For any $i \in I, \epsilon \in \Sigma_i$, define $s_i\epsilon \in \Sigma_i$ by $s_i\epsilon_\alpha = \epsilon_\alpha$ if $\alpha \notin F_0^i$ and $s_i\epsilon_\alpha = -\epsilon_\alpha$ if $\alpha \in F_0^i$. Assume for simplicity of notation that $I = \{1, ..., r\}$ and that the indices are ordered so that $\epsilon_0 \in \Sigma_1$ and for each $1 \le i \le r-1$ we have $\epsilon_i = s_i s_{i-1} ... s_1 \epsilon_0 \in \Sigma_{i+1}$. Then $\epsilon_0^-(I) = \epsilon_r = s_r \epsilon_{r-1}$.

Now since $\epsilon_0 = \epsilon_0^+(1)$ and $\epsilon_1 = s_1\epsilon_0 = \epsilon_0^-(1)$, by the matching conditions and (2.25) we have for any $E \subseteq F \subseteq E(I), j \ge 0$,

$$(\partial/\partial h_I)^j g(F : \epsilon_0 : h_0 : \nu_F) = \sum_{F \subseteq F_1 \subseteq F(1)} D^j_{F_1 \backslash F} g(F_1 : \epsilon_1 : h_0 : \nu_F).$$

But by differentiating with respect to $i \sum_{\alpha \in F\backslash E} \partial/\partial \mu_\alpha$ and evaluating at $\nu_F = (\nu_E, 0)$ we see that

$$d(F : \epsilon_0) = \sum_{F \subseteq F_1 \subseteq F(1)} d(F_1 : \epsilon_1).$$

Similarly, since $\epsilon_1 = \epsilon_1^+(2)$ and $\epsilon_2 = \epsilon_1^-(2)$, we have for each $F \subseteq F_1 \subseteq F(1)$,

$$d(F_1 : \epsilon_1) = \sum_{F_1 \subseteq F_2 \subseteq F_1(2)} d(F_2 : \epsilon_2).$$

Now by an easy induction argument we see that $d(F : \epsilon_0^+(I)) = d(F : \epsilon_0) =$

$$\sum_{F \subseteq F_1 \subseteq F(1)} \sum_{F_1 \subseteq F_2 \subseteq F_1(2)} \cdots \sum_{F_{r-1} \subseteq F_r \subseteq F_{r-1}(r)} d(F_r : \epsilon_r)$$
$$= \sum_{F \subseteq F' \subseteq E(I)} d(F' : \epsilon_0^-(I))$$

since $\epsilon_r = \epsilon_0^-(I)$ and each $F \subseteq F' \subseteq E(I) = F(I)$ occurs exactly once in the above sum corresponding to $F_1 = F \cup (F' \cap F_0^1), F_2 = F_1 \cup (F' \cap F_0^2), ..., F_r = F_{r-1} \cup (F' \cap F_0^r)$. □

Lemma 2.25. *Fix finite sets $E_1 \subseteq E_2$ and suppose for each $E_1 \subseteq F \subseteq E_2$ we have complex numbers $a^\pm(F)$. Then the conditions* (a) *and* (b) *below are equivalent.*
For each $E_1 \subseteq F \subseteq E_2$,

$$\text{(2.25 a)} \qquad a^+(F) - a^-(F) = \sum_{F \subset F' \subseteq E_2} c_{|F' \backslash F|}(a^+(F') + a^-(F')).$$

For each $E_1 \subseteq F \subseteq E_2$,
(2.25 b)
$$a^+(F) = \sum_{F \subseteq F' \subseteq E_2} a^-(F') \quad \textit{and } a^-(F) = \sum_{F \subseteq F' \subseteq E_2} (-1)^{|F' \backslash F|} a^+(F').$$

Proof. The proof that (b) implies (a) is given in [**H3**, 3.20] which is purely combinatorial and hence valid in this general setting. Now assume (a). Using [**H3**, 3.23], which is also purely combinatorial, we know that for all $E_1 \subseteq F_1 \subseteq E_2$ we have

$$\sum_{F_1 \subseteq F' \subseteq E_2} 2^{-|F'|} \left((-1)^{|F' \setminus F_1|} a^+(F') - a^-(F')\right) = 0.$$

Now add these equalities over all $F \subseteq F_1 \subseteq E_2$ to obtain

$$\begin{aligned} 0 &= \sum_{F \subseteq F_1 \subseteq E_2} \sum_{F_1 \subseteq F' \subseteq E_2} 2^{-|F'|} \left((-1)^{|F' \setminus F_1|} a^+(F') - a^-(F')\right) \\ &= \sum_{F \subseteq F' \subseteq E_2} 2^{-|F'|} \left[a^+(F') \left(\sum_{F \subseteq F_1 \subseteq F'} (-1)^{|F' \setminus F_1|} \right) - a^-(F) \left(\sum_{F \subseteq F_1 \subseteq F'} 1 \right) \right]. \end{aligned}$$

But

$$\sum_{F \subseteq F_1 \subseteq F'} (-1)^{|F' \setminus F_1|} = \begin{cases} 0, & \text{if } F' \neq F; \\ 1, & \text{if } F' = F; \end{cases}$$

while $\sum_{F \subseteq F_1 \subseteq F'} 1 = 2^{|F' \setminus F|}$. Thus

$$2^{-|F|} a^+(F) = \sum_{F \subseteq F' \subseteq E_2} 2^{-|F'|} 2^{|F' \setminus F|} a^-(F).$$

This proves the first part of (b). The second part is proved in the same way using

$$0 = \sum_{F \subseteq F_1 \subseteq E_2} (-1)^{|F_1 \setminus F|} \sum_{F_1 \subseteq F' \subseteq E_2} 2^{-|F'|} \left((-1)^{|F' \setminus F_1|} a^+(F') - a^-(F')\right).$$

□

3. Growth estimates.

In this section we will modify the version of the theory of the constant term given in [**HW5**] so that we can give sharper growth estimates near points on the boundary of Weyl chambers.

Fix a minimal parabolic subgroup P_0 of G and let $\Phi^+ = \Delta(P_0, A_0)$ be the roots of A_0 in P_0. Let T_0 be a relatively compact Cartan subgroup of M_0 so that $H_0 = T_0 A_0$ is a θ-stable Cartan subgroup of G. Let P be a parabolic subgroup of G with $P_0 \subseteq P$ so that $A_P \subseteq A_0$ and $\Delta(P, A_0) \subseteq \Phi^+$. Let $\Phi_P^+ = \{\alpha \in \Phi^+ : \alpha|_{\mathfrak{a}_P} = 0\}$. Write $L_P^* = K_P \operatorname{cl}(A_0^+) K_P$ where A_0^+ is the positive Weyl chamber of A_0 with respect to Φ^+ and $W_P = W(\mathfrak{l}_{P,\mathbb{C}}, (\mathfrak{h}_0)_{\mathbb{C}})$.

Note that $L_G^* = G$, but that in general L_P^* is a closed subset of L_P with non-empty interior.

Let $H = TA$ be a θ-stable Cartan subgroup of G and use the notation of (2.5). Fix $(\lambda, \chi) \in X(T), (\tau_1, \tau_2) \in \widehat{K}(\chi)$, $W = W(\tau_1 : \tau_2)$, and a connected component $\mathcal{D}$ of $\{h \in i\mathfrak{t}^* : \langle \lambda(h), \beta \rangle \neq 0 \text{ for all } \beta \in \Phi_M\}$ such that $0 \in \mathrm{cl}(\mathcal{D})$. Let Ω be a relatively compact neighborhood of 0 in $i\mathfrak{t}^*$ and define $\mathcal{D}_{\mathbf{C}} = \{h_R + ih_I : h_R \in \mathcal{D}, h_I \in \Omega\}$. Then for all $(h, \nu) \in \mathfrak{t}_{\mathbf{C}}^* \times \mathfrak{a}^*, \lambda(h) + i\nu \in \mathfrak{h}_{\mathbf{C}}^*$. Let y be a Cayley transform such that $\mathfrak{h}_{\mathbf{C}}^y = \mathfrak{h}_{0,\mathbf{C}}$. For $s \in W_G, (h, \nu) \in \mathfrak{t}_{\mathbf{C}}^* \times \mathfrak{a}^*$, define

$$\Lambda_{h,\nu,s} = s(\lambda(h) + i\nu)^y \in (\mathfrak{h}_0)_{\mathbf{C}}^*.$$

Let $U_{\mathbf{C}}$ be a (relatively open) neighborhood of 0 in $\mathrm{cl}(\mathcal{D}_{\mathbf{C}}), U = U_{\mathbf{C}} \cap \mathrm{cl}(\mathcal{D})$. As in [**HW5**, 7.5] we will write

$J(U_{\mathbf{C}} : L_P^*) = J(U_{\mathbf{C}} : L_P^* : s)$ for the set of all $\phi \in C^\infty(U_{\mathbf{C}} \times \mathfrak{a}^* \times L_P^* : W)$

satisfying the following conditions. First, for all $(\nu, x) \in \mathfrak{a}^* \times L_P^*$,

(3.1 a) $h \mapsto \phi(h : \nu : x)$ is a holomorphic function on $U_{\mathbf{C}} \cap \mathcal{D}_{\mathbf{C}}$.

Next, for all $(h, \nu) \in U_{\mathbf{C}} \times \mathfrak{a}^*$,

(3.1 b) $\phi(h : \nu)$ is a $(\tau_{1,h}|_{K_P}, \tau_{2,h}|_{K_P})$-spherical function on L_P^*

and

(3.1 c) $z\phi(h : \nu) = \mu_P(z : \Lambda_{h,\nu,s})\phi(h : \nu)$ for all $z \in \mathcal{Z}_P$.

Here $\mathcal{Z}_P$ denotes the center of $\mathcal{U}(\mathfrak{l}_{P,\mathbf{C}})$ and $\mu_P : \mathcal{Z}_P \to S(\mathfrak{h}_{0,\mathbf{C}})^{W_P}$ is the canonical isomorphism onto the W_P invariants in $S(\mathfrak{h}_{0,\mathbf{C}})$. Finally, let $\widetilde{\mathcal{L}}_P = \mathcal{P} \otimes \mathcal{U}(\mathfrak{l}_{P,\mathbf{C}})^{(2)}$ where $\mathcal{P} = P(\mathfrak{t}_{\mathbf{C}}^* \times \mathfrak{a}^*)$ is the set of polynomial coefficient differential operators on $\mathfrak{t}_{\mathbf{C}}^* \times \mathfrak{a}^*$. For $D \in \widetilde{\mathcal{L}}_P, r \in \mathbf{R}$, define $T_{D,r}(\phi) =$

(3.1 d) $$\sup_{U_{\mathbf{C}} \times \mathfrak{a}^* \times L_P^*} \|D\phi(h : \nu : x)\| \Xi_P(x)^{-1} |(h, \nu, x)|^{-r} e^{-|h_I|\sigma_V(x)}$$

where for $(h, \nu, x) \in U_{\mathbf{C}} \times \mathfrak{a}^* \times L_P^*$,

(3.1 e) $$|(h, \nu, x)| = (1 + |h|)(1 + |\nu|)(1 + \widetilde{\sigma}(x)).$$

Then we assume that

(3.1 f) for all $D \in \widetilde{\mathcal{L}}_P$, there is $r \geq 0$ so that $T_{D,r}(\phi) < \infty$.

As in [**HW5**, 7.5] we will write $J^0(U : L_P^*) = J^0(U : L_P^* : s)$ for the set of all $\phi \in C^\infty(U \times \mathfrak{a}^* \times L_P^* : W)$ satisfying the following two conditions.

First, there is a finite set of functions $\phi_1, ..., \phi_k \in J(U_{\mathbf{C}} : L_P^*)$ so that for each $(h, \nu) \in U \times \mathfrak{a}^*$ there exist $a_j(h : \nu) \in \mathbf{C}, 1 \leq j \leq k$, such that for all $x \in L_P^*$,

$$\phi(h : \nu : x) = \sum_{j=1}^{k} a_j(h : \nu)\phi_j(h : \nu : x). \tag{3.2 a}$$

Second, for all $D \in \widetilde{\mathcal{L}}_P$, there is $r \geq 0$ so that

$$T^0_{D,r}(\phi) = \sup_{U\times\mathfrak{a}^*\times L_P^*} \|D\phi(h : \nu : x)\|\Xi_P(x)^{-1}(1 + \widetilde{\sigma}(x))^{-r} < \infty. \tag{3.2 b}$$

It was proven in [**H3**, 2.21] that the holomorphic families of Eisenstein integrals defined in (2.8) are elements of $J(\mathrm{cl}(\mathcal{D}_{\mathbf{C}}) : G)$. Further, if $\mathcal{C}(\mathcal{D}\times\mathfrak{a}^*)_0$ is defined as in (2.9), then for all $\phi \in J(\mathrm{cl}(\mathcal{D}_{\mathbf{C}}) : G)$ and $\alpha \in \mathcal{C}(\mathcal{D} \times \mathfrak{a}^*)_0$, we have $\phi \cdot \alpha \in J^0(\mathrm{cl}(\mathcal{D}) : G)$.

Assume that $P_0 \subseteq Q \subseteq P$ are parabolic subgroups of G. Since the results of this section are a technical modification of results in [**HW5**, §7], we will use much of the notation of that section without repeating the definitions. Let ${}^*Q = Q \cap L_P$ and let $\Delta({}^*Q, A_Q)$ denote the roots of A_Q in ${}^*Q$. Let $\mathfrak{a}_Q^+ = (\mathfrak{a}_Q^P)^+$ be the positive chamber of $\mathfrak{a}_Q$ with respect to $\Delta({}^*Q, A_Q)$ and for $H \in \mathfrak{a}_Q$, let $\beta_Q(H) = \inf\{\alpha(H) : \alpha \in \Delta({}^*Q, A_Q)\}$.

Let $s_i, 1 \leq i \leq w$, be representatives for the cosets $W_Q\backslash W_P$ and fix a complex Hilbert space T of dimension w with orthonormal basis $\{e_1, ..., e_w\}$. For $f \in J(U_{\mathbf{C}} : L_P^*) \cup J^0(U : L_P^*)$ and $v \in \mathcal{Z}_Q$, define $\Phi(f)$ and $\Psi_v(f)$ taking values in $W \otimes T$ using the same definition as in [**HW5**, 7.8]. That is, for any $m \in L_Q^* \subseteq L_P^*$,

$$\Phi(f : h : \nu : m) = \sum_{i=1}^{w} d_Q(m)f(h : \nu : m; v_i') \otimes e_i$$

and

$$\Psi_v(f : h : \nu : m) = \sum_{i=1}^{w} d_Q(m)f(h : \nu : m; u_i(v : h : \nu)') \otimes e_i.$$

Next, for $1 \leq i \leq w$ and any f, the functions $\Phi_i(f), \Psi_{v,i}(f)$ are defined by

$$\Phi_i(f : h : \nu : m) = B_1\left(s_i\Lambda^F_{h,\nu,s}\right)\Phi(f : h : \nu : m)$$

and

$$\Psi_{v,i}(f : h : \nu : m) = B_1\left(s_i\Lambda^F_{h,\nu,s}\right)\Psi_v(f : h : \nu : m)$$

using the projections B_1 defined in [**HW5**, 7.13].

Lemma 3.3. (i) *Fix $D \in \mathcal{P}, l_1, l_2 \in \mathcal{U}(\mathfrak{l}_{P,\mathbf{C}})$, and $X \in \mathfrak{n}_Q \cap \mathfrak{l}_P$. Then there is a finite subset F of $\widetilde{\mathcal{L}}_P$ and $r_0 > 0$ such that for all $r \geq 0$,*

$$d_Q(ma)(\|f(h:\nu; D: l_1X ; m\exp H; l_2)\| + \|f(h:\nu; D: l_1 ; m\exp H; \theta(X)l_2)\|)$$
$$\leq \begin{cases} T^0_{F,r}(f)\Xi_Q(m)e^{-\beta_Q(H)}(1+\widetilde{\sigma}(m\exp H))^{r+r_0}, & \text{if } f \in J^0(U : L^*_P); \\ T_{F,r}(f)\Xi_Q(m)e^{-\beta_Q(H)}|(h,\nu,m)|^{r+r_0}(1+|H|)^{r+r_0}e^{|h_I|\sigma_V(m)}, & \text{if } f \in J(U_{\mathbf{C}} : L^*_P); \end{cases}$$

for all $m \in L^*_Q, H \in \mathfrak{a}_Q \cap \mathrm{cl}(\mathfrak{a}_0^+)$.
(ii) *Fix $D \in \mathcal{P}, v \in \mathcal{Z}_Q$, and $b_1, b_2 \in \mathcal{U}(\mathfrak{l}_{Q,\mathbf{C}})$. Then there is a finite subset F of $\widetilde{\mathcal{L}}_P$ and $r_0 > 0$ such that for all $m \in L^*_Q, H \in \mathfrak{a}_Q \cap \mathrm{cl}(\mathfrak{a}_0^+), r \geq 0, 1 \leq i \leq w$,*

$$\|\Psi_{v,i}(f : h:\nu; D: b_1 ; m\exp H; b_2)\|$$
$$\leq \begin{cases} T^0_{F,r}(f)\Xi_Q(m)e^{-\beta_Q(H)}(1+\widetilde{\sigma}(m\exp H))^{r+r_0}, & \text{if } f \in J^0(U : L^*_P); \\ T_{F,r}(f)\Xi_Q(m)e^{-\beta_Q(H)}|(h,\nu,m)|^{r+r_0}(1+|H|)^{r+r_0}e^{|h_I|\sigma_V(m)}, & \text{if } f \in J(U_{\mathbf{C}} : L^*_P); \end{cases}$$

and

$$\|\Phi_i(f : h:\nu; D: b_1 ; m\exp H; b_2)\|$$
$$\leq \begin{cases} T^0_{F,r}(f)\Xi_Q(m)(1+\widetilde{\sigma}(m\exp H))^{r+r_0}, & \text{if } f \in J^0(U : L^*_P); \\ T_{F,r}(f)\Xi_Q(m)|(h,\nu,m)|^{r+r_0}(1+|H|)^{r+r_0}e^{|h_I|\sigma_V(m)}, & \text{if } f \in J(U_{\mathbf{C}} : L^*_P). \end{cases}$$

Proof. The proof is similar to that of [**HW5**, 7.11;7.12], using [**HW5**, 7.13c] to pass from Φ and Ψ_v to Φ_i and $\Psi_{v,i}$. Note that if $m = k_1 a_0 k_2 \in L^*_Q, k_1, k_2 \in K_Q, a_0 \in \mathrm{cl}(A_0^+)$, and $H \in \mathfrak{a}_Q \cap \mathrm{cl}(\mathfrak{a}_0^+)$, then $m\exp H = k_1 a_0 \exp H k_2$ since K_Q centralizes A_Q. Now $a_0 \exp H \in \mathrm{cl}(A_0^+)$ so that $m \exp H \in L^*_Q$. Also, in the notation of [**HW5**, 7.11], $L^*_Q \subseteq L^+_Q$ and $\mathfrak{a}_Q \cap \mathrm{cl}(\mathfrak{a}_0^+) \subseteq \mathrm{cl}(\mathfrak{a}_Q^+)$. □

Lemma 3.4. *Let $b_1, b_2 \in \mathcal{U}(\mathfrak{l}_{Q,\mathbf{C}}), m \in L^*_Q, H \in \mathfrak{a}_Q \cap \mathrm{cl}(\mathfrak{a}_0^+)$. Then for all $T \geq 0, 1 \leq i \leq w$,*

$$\Phi_i(f : h:\nu : b_1 ; m\exp TH; b_2) = e^{Ts_i\Lambda_{h,\nu,s}(H)}\Phi_i(f : h:\nu : b_1; m; b_2)$$
$$+ \int_0^T e^{(T-t)s_i\Lambda_{h,\nu,s}(H)}\Psi_{H,i}(f : h:\nu : b_1 ; m\exp tH; b_2)dt.$$

Proof. The proof is the same as that of [**HW5**, 7.10; 7.14]. □

For $1 \leq i \leq w$, let

$$\lambda_i(h) = \mathrm{Re}\, s_i s\lambda(h)^y, h \in U_{\mathbf{C}},$$

and for $H \in \mathfrak{a}_Q^+$ define

$$U^i_{\mathbf{C}}(H) = \{h \in U_{\mathbf{C}} : \lambda_i(h:H) + \beta_Q(H) > 0\}, U^i(H) = U^i_{\mathbf{C}}(H) \cap U.$$

Lemma 3.5. *Let $D \in \mathcal{P}, b_1, b_2 \in \mathcal{U}(\mathfrak{l}_{Q,\mathbf{C}}), 1 \leq i \leq w, H \in \mathfrak{a}_Q^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$. Then*

$$\int_0^\infty \left\| \Psi_{H,i}\left(f : h : \nu; D \circ e^{-ts_i\Lambda_{h,\nu,s}(H)} : b_1 ; m \exp tH; b_2\right) \right\| dt$$

converges uniformly for ν and m in compact subsets of $\mathfrak{a}^*$ and L_Q^* respectively, and for h in compact subsets of

$$\begin{cases} U_{\mathbf{C}}^i(H), & \text{if } f \in J(U_{\mathbf{C}} : L_P^*); \\ U^i(H), & \text{if } f \in J^0(U : L_P^*). \end{cases}$$

Proof. This follows directly from (3.3). Unlike [**HW5**, 7.17], it holds for all $1 \leq i \leq w$ because we restrict to $m \in L_Q^*$. □

Lemma 3.6. *Let $1 \leq i \leq w, H \in \mathfrak{a}_Q^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$. Then*

$$\Phi_{i,\infty}(f : h : \nu : m : H) = \lim_{T \to +\infty} e^{-Ts_i\Lambda_{h,\nu,s}(H)} \Phi_i(f : h : \nu : m \exp TH)$$

exists and is C^∞ on

$$\begin{cases} U^i(H) \times \mathfrak{a}^* \times L_Q^*, & \text{if } f \in J^0(U : L_P^*); \\ U_{\mathbf{C}}^i(H) \times \mathfrak{a}^* \times L_Q^* \text{ and holomorphic for } h \in U_{\mathbf{C}}^i(H), & \text{if } f \in J(U_{\mathbf{C}} : L_P^*). \end{cases}$$

Further, for all $D \in \mathcal{P}, b_1, b_2 \in \mathcal{U}(\mathfrak{l}_{Q,\mathbf{C}})$,

$$\Phi_{i,\infty}(f : h : \nu; D : b_1 ; m; b_2 : H) = \Phi_i(f : h : \nu; D : b_1 ; m; b_2) + \int_0^\infty \Psi_{H,i}(f : h : \nu; D \circ e^{-ts_i\Lambda_{h,\nu,s}(H)} : b_1 ; m \exp tH; b_2) dt.$$

Proof. Combine (3.4) and (3.5). □

Let $H_1, H_2 \in \mathfrak{a}_Q^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$. For $1 \leq i \leq w$ and

$$h \in \begin{cases} U_{\mathbf{C}}^i(H_1) \cap U_{\mathbf{C}}^i(H_2), & \text{if } f \in J(U_{\mathbf{C}} : L_P^*), \\ U^i(H_1) \cap U^i(H_2), & \text{if } f \in J^0(U : L_P^*), \end{cases}$$

the argument in [**HC1**, §22; Lemma 8] shows that

$$\Phi_{i,\infty}(f : h : \nu : m : H_1) = \Phi_{i,\infty}(f : h : \nu : m : H_2).$$

Thus whenever there is an $H \in \mathfrak{a}_Q^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$ such that $\lambda_i(h : H) + \beta_Q(H) > 0$, we can define

$$\Phi_{i,\infty}(f : h : \nu : m) = \Phi_{i,\infty}(f : h : \nu : m : H), \tag{3.7}$$

and the definition does not depend on the choice of H.

Now as in [**HW5**, 7.15] we can define

(3.8a)
$$I^0 = \{1 \le i \le w : \lambda_i(h : H) = 0 \quad \text{for all } h \in U, H \in \mathfrak{a}_Q\};$$

(3.8b)
$$I^+ = \{1 \le i \le w : \lambda_i(h : H) > 0 \quad \text{for some } h \in U, H \in \mathfrak{a}_Q^+\};$$

(3.8c)
$$I^- = \{1 \le i \le w : \lambda_i(h : H) < 0 \quad \text{for all } h \in U \cap \mathcal{D}, H \in \mathfrak{a}_Q^+\}.$$

Remark. If $i \in I^0 \cup I^+$, the constant terms defined above are the same as those defined in [**HW5**, 7.18].

Define

(3.9) $$I^0(0) = I^0 \cup \{i \in I^- : \lambda_i(0 : H) = 0 \quad \text{for all } H \in \mathfrak{a}_Q\}.$$

Fix $H_0 \in \mathfrak{a}_Q^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$ such that $\beta_Q(H_0) = 1$. Then, for all $i \in I^0(0)$, $\lambda_i(0 : H_0) + \beta_Q(H_0) = 1$, so that there is a (relatively open) neighborhood $U'_{\mathbf{C}}$ of 0 in $U_{\mathbf{C}}$ so that $U'_{\mathbf{C}} \subseteq U^i_{\mathbf{C}}(H_0)$ for all $i \in I^0(0)$. Thus for $i \in I^0(0), \Phi_{i,\infty}$ is defined for all $h \in U'_{\mathbf{C}}$. Redefine

$$\Phi_{i,\infty} = 0 \quad \text{if } i \notin I^0(0).$$

Thus for all $1 \le i \le w$, $\Phi_{i,\infty}$ is defined for all $h \in U'_{\mathbf{C}}$.

Lemma 3.10. *Let $1 \le i \le w, h \in U'_{\mathbf{C}}$. Then*

$$\Phi_{i,\infty}(f : h : \nu : m; v) = \mu_Q(v : s_i\Lambda_{h,\nu,s})\Phi_{i,\infty}(f : h : \nu : m)$$

for all $v \in \mathcal{Z}_Q$ and

$$\Phi_{i,\infty}(f : h : \nu : m \exp H) = e^{s_i\Lambda_{h,\nu,s}(H)}\Phi_{i,\infty}(f : h : \nu : m)$$

for all $m \in L_Q^*, H \in \mathfrak{a}_Q \cap \mathrm{cl}(\mathfrak{a}_0^+)$. Given $b_1, b_2 \in \mathcal{U}(\mathfrak{l}_{Q,\mathbf{C}})$ and $D \in \mathcal{P}$, there exists a finite subset $F \subseteq \widetilde{\mathcal{L}}_P$ and $r_0 > 0$ such that for all $r \ge 0$ there is $C > 0$ so that

$$\|\Phi_{i,\infty}(f : h : \nu; D : b_1 ; m; b_2)\| \le \begin{cases} CT^0_{F,r}(f)\Xi_Q(m)(1+\widetilde{\sigma}(m))^{r+r_0}, & \text{if } f \in J^0(U : L_P^*); \\ CT_{F,r}(f)\Xi_Q(m)|(h,\nu,m)|^{r+r_0}e^{|h_I|\sigma_V(m)}, & \text{if } f \in J(U_{\mathbf{C}} : L_P^*). \end{cases}$$

Proof. The proof is the same as that of [**HW5**, 7.20; 7.27] because for $i \in I^0(0), U'_{\mathbf{C}} \subseteq U^i_{\mathbf{C}}(H_0)$ □

Lemma 3.11. *Let ω be a compact subset of $\mathfrak{a}_Q^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$. There exist a neighborhood $U'_{\mathbf{C}}(\omega)$ of 0 in $U'_{\mathbf{C}}$ and $0 < \delta_0 < \frac{1}{2}$ so that given $D \in \mathcal{P}$ there exists a finite subset $F \subseteq \mathcal{P}$ and $C, r_1 > 0$, so that for all $f \in J(U_{\mathbf{C}} : L_P^*) \cup J^0(U : L_P^*), h \in U'_{\mathbf{C}}(\omega), b_1, b_2 \in \mathcal{U}(\mathfrak{l}_{Q,\mathbf{C}}), m \in L_Q^*, H \in \omega$, and $T \geq 0$,*

$$\begin{aligned}
&\|\Phi_i(f : h : \nu; D : b_1 ; m \exp TH; b_2) - \Phi_{i,\infty}(f : h : \nu; D : b_1 ; m \exp TH; b_2)\| \\
&\leq Ce^{-T\delta_0\beta_Q(H)}(1 + T\|H\|)^{r_1} \sum_{D' \in F} \Big\{ \|\Phi_i(f : h : \nu; D' : b_1 ; m; b_2)\| \\
&\quad + \int_0^\infty \|\Psi_{H,i}(f : h : \nu; D' : b_1 ; m \exp tH; b_2)\| e^{\frac{t\beta_Q(H)}{2}} (1 + t\|H\|)^{r_1} dt \Big\}
\end{aligned}$$

for all $1 \leq i \leq w$. Further, for each $i \in I^- \cap I^0(0)$ there is a continuous piecewise affine function δ_i on $U'_{\mathbf{C}}$ satisfying $0 \leq \delta_i(h) < \frac{1}{2}$ for all $h \in U'_{\mathbf{C}}$ and $\delta_i(h) = 0$ if and only if $\lambda_i(h : H) = 0$ for all $H \in \mathfrak{a}_Q$, so that

$$\begin{aligned}
&\|\Phi_{i,\infty}(f : h : \nu; D : b_1 ; m \exp TH; b_2)\| \\
&\leq Ce^{-T\delta_i(h)\beta_Q(H)}(1 + T\|H\|)^{r_1} \sum_{D' \in F} \Big\{ \|\Phi_i(f : h : \nu; D' : b_1 ; m; b_2)\| \\
&\quad + \int_0^\infty \|\Psi_{H,i}(f : h : \nu; D' : b_1 ; m \exp tH; b_2)\| e^{\frac{t\beta_Q(H)}{2}} (1 + t\|H\|)^{r_1} dt \Big\}.
\end{aligned}$$

Proof. Suppose first that $i \in I^0 \cup I^+$. Then $\Phi_{i,\infty}$ is the same as the constant term defined in [**HW5**, §7], and the argument given in the proof of [**HW5**, 7.21] shows that the inequality is satisfied for any $0 < \delta_0 < \frac{1}{2}$ such that, in the notation of [**HW5**, 7.25], $\delta_0 < \min_{i \in \tilde{I}^+} d_i$.

Now suppose that $i \in I^-$ and $i \notin I^0(0)$. Then, in the notation of [**HW5**, 7.23], $d_i(0) > 0$. Thus there are a neighborhood V of 0 in $\mathfrak{v}_{\mathbf{C}}^*$ and an $\epsilon > 0$ so that $d_i(h) > \epsilon$ for all $h \in V$ and all $i \in I^-$ such that $i \notin I^0(0)$. Now $\lambda_i(h : H) < -\epsilon\beta_Q(H)$ for all $h \in V$. Thus the argument in Case II of [**HW5**, 7.21] works as long as $h \in U'_{\mathbf{C}} \cap V, 0 < \delta_0 < \frac{1}{2}$, and $\delta_0 < \epsilon$.

Finally, suppose that $i \in I^- \cap I^0(0)$. Then $\lambda_i(0 : H) = 0$ for all $H \in \mathfrak{a}_Q$ and there is a neighborhood $V(\omega)$ of 0 in $\mathfrak{v}_{\mathbf{C}}^*$ so that for all $h \in V(\omega), H \in \omega, 0 \leq -\lambda_i(h : H) < \frac{\beta_Q(H)}{2}$. Then, using (3.8) as in Case I of [**HW5**, 7.21], for all $H \in \omega, h \in V(\omega) \cap U'_{\mathbf{C}}$,

$$\|\Phi_i(f : h : \nu; D : b_1 ; m \exp TH; b_2) - \Phi_{i,\infty}(f : h : \nu; D : b_1 ; m \exp TH; b_2)\|$$

$$\leq C \sum_{D' \in F_i} \int_T^\infty (1 + (t-T)\|H\|)^{r_1}$$
$$\cdot \left\| \Psi_{H,i} \left(f : h : \nu; D' \circ e^{-(t-T)\lambda_i(h:H)} : b_1 ; m \exp tH; b_2 \right) \right\| dt$$
$$\leq Ce^{-T\delta_0\beta_Q(H)} \sum_{D' \in F_i} \int_0^\infty (1 + t\|H\|)^{r_1} e^{\frac{t\beta_Q(H)}{2}}$$
$$\cdot \|\Psi_{H,i}(f : h : \nu; D' : b_1 ; m \exp tH; b_2)\| dt$$

since $0 < \delta_0 < \frac{1}{2}$.

Finally, for $i \in I^- \cap I^0(0)$, define $d_i(h), h \in \mathfrak{v}_{\mathbf{C}}^*$, as in [**HW5**, 7.23]. Then $d_i(h) > 0$ for all $h \in U \cap \mathcal{D}$ and for $h \in U \cap \mathrm{cl}(\mathcal{D}), d_i(h) = 0$ if and only if $\lambda_i(h : H) = 0$ for all $H \in \mathfrak{a}_Q$. Now for $h \in U_{\mathbf{C}}$, set $\delta_i(h) = \min\{\mathrm{Re}\ d_i(h),\ \delta_0\}$. Then, as in Case II of [**HW5**, 7.21], we have

$$\|\Phi_i(f : h : \nu; D : b_1 ; m \exp TH; b_2)\|$$
$$\leq Ce^{-T\beta_Q(H)\delta_i(h)}(1 + T\|H\|)^{r_1} \sum_{D' \in F} \left\{ \|\Phi_i(f : h : \nu; D' : b_1 ; m; b_2)\| \right.$$
$$\left. + \int_0^\infty \|\Psi_{H,i}(f : h : \nu; D' : b_1 ; m \exp tH; b_2)\| e^{\frac{t\beta_Q(H)}{2}} dt \right\}.$$

Now, since $\|\Phi_i\|$ and $\|\Phi_i - \Phi_{i,\infty}\|$ both satisfy the desired inequality, so does $\|\Phi_{i,\infty}\|$. □

Recall $\Phi(f)$ and hence all $\Phi_{i,\infty}(f)$ take values in $W \otimes T$ where T has a distinguished basis $e_1, ..., e_w$. Further, $\Phi(f) = \sum_{i=1}^w \det s_i\ \Phi_i(f)$ where $\det s_i = \pm 1$ is defined by $\pi_P(s_i\Lambda_{h,\nu,s}) = \det s_i\ \pi_P(\Lambda_{h,\nu,s})$. For each $1 \leq i \leq w$ we write

(3.12a)
$$\Phi_{i,\infty}(f : h : \nu : m) = \sum_{j=1}^w \phi_{i,j}(f : h : \nu : m) \otimes e_j$$

and write

(3.12b)
$$\tilde{f}_{Q,s_is}(h : \nu : m) = \psi_{f,s_is}(h : \nu : m) = \phi_{i,1}(f : h : \nu : m).$$

Choose $\epsilon > 0$ so that $\beta_Q(H) \geq 2\epsilon$ for all $H \in \omega$. Put $\epsilon_0 = \epsilon\delta_0$ and $\epsilon_i(h) = \epsilon\delta_i(h), i \in I^- \cap I^0(0)$, where δ_0 and $\delta_i(h)$ are defined as in (3.11). Then combining (3.11) with (3.3) and (3.10) we have the following theorem.

Theorem 3.13. *Given $b_1, b_2 \in \mathcal{U}(\mathfrak{l}_{Q,\mathbf{C}})$ and $D \in \mathcal{P}$, there exist a finite subset $F \subseteq \widetilde{\mathcal{L}}_P$ and an $r_1 > 0$ so that for all $r \geq 0$ there is a $C > 0$ so that for all $m \in L_Q^*, H \in \omega, T \geq 0$,*

$$\left\| d_Q(m\exp TH)\pi_P(\Lambda_{h,\nu,s})f(h:\nu;D:b_1';m\exp TH;b_2') - \sum_{i=1}^{w} det(s_i)\psi_{f,s_is}(h:\nu;D:b_1;m\exp TH;b_2)\right\| \leq$$

$$\begin{cases} CT^0_{F,r}(f)e^{-\epsilon_0 T}\Xi_Q(m)(1+\widetilde{\sigma}(m\exp TH))^{r+r_1}, & \text{if } f \in J^0(U:L_P^*), h \in U'(\omega); \\ CT_{F,r}(f)e^{-\epsilon_0 T}\Xi_Q(m)|(h,\nu,m\exp TH)|^{r+r_1}e^{|h_I|\sigma_V(m)}, & \text{if } f \in J(U_{\mathbf{C}}:L_P^*), h \in U'_{\mathbf{C}}(\omega); \end{cases}$$

and for $i \in I^- \cap I^0(0), h \in U'_{\mathbf{C}}(\omega)$,

$$\|\psi_{f,s_is}(h:\nu;D:b_1;m\exp TH;b_2)\|$$
$$\leq \begin{cases} CT^0_{F,r}(f)e^{-\epsilon_i(h)T}\Xi_Q(m)(1+\widetilde{\sigma}(m\exp TH))^{r+r_1}, & \text{if } f \in J^0(U:L_P^*); \\ CT_{F,r}(f)e^{-\epsilon_i(h)T}\Xi_Q(m)|(h,\nu,m\exp TH)|^{r+r_1}e^{|h_I|\sigma_V(m)}, & \text{if } f \in J(U_{\mathbf{C}}:L_P^*). \end{cases}$$

Further, for all $1 \leq i \leq w$,

$$\psi_{f,s_is} \in \begin{cases} J(U'_{\mathbf{C}}:L_Q^*:s_is), & \text{if } f \in J(U_{\mathbf{C}}:L_P^*:s); \\ J^0(U':L_Q^*:s_is), & \text{if } f \in J^0(U:L_P^*:s). \end{cases}$$

Given $D \in \widetilde{\mathcal{L}}_Q$, there are a finite subset $F \subseteq \widetilde{\mathcal{L}}_P$ and an $r_1 \geq 0$ such that for all $r \geq 0$ there is a $C > 0$ so that

$$\begin{cases} T_{D,r+r_1}(\psi_{f,s_is}) \leq CT_{F,r}(f), & \text{if } f \in J(U_{\mathbf{C}}:L_P^*); \\ T^0_{D,r+r_1}(\psi_{f,s_is}) \leq CT^0_{F,r}(f), & \text{if } f \in J^0(U:L_P^*). \end{cases}$$

Corollary 3.14. *Let $H \in \mathfrak{a}_Q^+$. Then there are $\epsilon > 0$ and a neighborhood $U'_{\mathbf{C}}(H)$ of 0 in $U'_{\mathbf{C}}$ so that for all $m \in L_Q^*, h \in U'_{\mathbf{C}}(H)$ (respectively $U'_{\mathbf{C}}(H) \cap$ $\mathrm{cl}(\mathcal{D})$), $\nu \in \mathfrak{a}^*, f \in J(U_{\mathbf{C}}:L_P^*)$ (respectively $J^0(U:L_P^*)$),*

$$\lim_{t\to+\infty} e^{\epsilon t}\left[d_Q(m\exp tH)\pi_P(\Lambda_{h,\nu,s})f(h:\nu:m\exp tH) - \sum_{i=1}^{w}\det s_i\ \psi_{f,s_is}(h:\nu:m\exp tH)\right] = 0.$$

Assume that $P_0 \subseteq Q' \subseteq Q \subseteq P$. Let $s_i, 1 \leq i \leq w$, denote coset representatives for $W_Q\backslash W_P$ and let $u_j, 1 \leq j \leq p$, denote coset representatives for $W_{Q'}\backslash W_Q$. Then we can take $u_js_i, 1 \leq i \leq w, 1 \leq j \leq p$, as coset representatives for $W_{Q'}\backslash W_P$. Let $f \in J(U_{\mathbf{C}} : L_P^* : s)$ (respectively $J^0(U : L_P^* : s)$) and $1 \leq i \leq w$. Then, as in [**HW5**, 7.3.2], $g^i(h : \nu) = \pi_Q(\Lambda_{h,\nu,s_is})^{-1}\psi_{f,s_is}(h : \nu) \in J(U'_{\mathbf{C}} : L_Q^* : s_is)$ (respectively $J^0(U' : L_Q^* : s_is)$). Now there is a neighborhood $U''_{\mathbf{C}}$ of 0 in $U'_{\mathbf{C}}$ so that for each $1 \leq j \leq p$ we can define

$$\psi_{g^i,u_js_is} \in J(U''_{\mathbf{C}} : L_{Q'}^* : u_js_is)(\text{ respectively } J^0(U'' : L_{Q'}^* : u_js_is)).$$

Write

$$\psi'_{f,u_js_is} \in J(U'''_{\mathbf{C}} : L_{Q'}^* : u_js_is)(\text{ respectively } J^0(U''' : L_{Q'}^* : u_js_is))$$

for the constant term of f with respect to Q' corresponding to the coset representative u_js_i.

Lemma 3.15. *There is a neighborhood $\widetilde{U}_{\mathbf{C}}$ of 0 in $U''_{\mathbf{C}} \cap U'''_{\mathbf{C}}$ so that for all $(h,\nu,m) \in \widetilde{U}_{\mathbf{C}} \times \mathfrak{a}^* \times L_{Q'}^*$ (respectively $\widetilde{U} \times \mathfrak{a}^* \times L_{Q'}^*$),*

$$\psi'_{f,u_js_is}(h : \nu : m) = \psi_{g^i,u_js_is}(h : \nu : m).$$

Proof. Suppose $f \in J(U_{\mathbf{C}} : L_P^* : s)$ and use the notation above. (If $f \in J^0(U : L_P^* : s)$, the proof is the same.) For any pair $Q \subseteq P$ of parabolic subgroups we will write $\left(\mathfrak{a}_Q^P\right)^+$ for the positive Weyl chamber of $\mathfrak{a}_Q$ with respect to the roots $\Delta(L_P \cap Q, A_Q)$ and write $d_Q^P(ma) = e^{\rho}(a), m \in M_Q, a \in A_Q$, where $\rho = 1/2\sum m(\alpha)\alpha, \alpha \in \Delta(L_P \cap Q, A_Q)$. Now for $Q' \subseteq Q \subseteq P$ as above, choose $H_1 \in \left(\mathfrak{a}_Q^P\right)^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$ and $H_2 \in \left(\mathfrak{a}_{Q'}^Q\right)^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$. Then $H_0 = H_1 + H_2 \in \left(\mathfrak{a}_{Q'}^P\right)^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$ and every $H_0 \in \left(\mathfrak{a}_{Q'}^P\right)^+ \cap \mathrm{cl}(\mathfrak{a}_0^+)$ can be decomposed in this way. Now, using the fact that for all $m \in L_{Q'}^*$ we have $\psi_{f,s_is}(h : \nu : m) = \pi_Q(\Lambda_{h,\nu,s_is})g^i(h : \nu : m)$ and $d_{Q'}^P(m) = d_Q^P(m)d_{Q'}^Q(m)$, for all $(h,\nu,m) \in U''_{\mathbf{C}} \times \mathfrak{a}^* \times L_{Q'}^*$ we have

$$\Bigg\| d_{Q'}^P(m\exp(tH_0))\pi_P(\Lambda_{h,\nu,s})f(h : \nu : m\exp(tH_0))$$

$$- \sum_{1\leq i\leq w, 1\leq j\leq p} \det(u_js_i)\psi_{g^i,u_js_is}(h : \nu : m\exp(tH_0))\Bigg\|$$

$$\leq d_{Q'}^Q(m\exp(tH_0))\Bigg\| d_Q^P(m\exp(tH_0))\pi_P(\Lambda_{h,\nu,s})f(h : \nu : m\exp(tH_0))$$

$$- \sum_{1\leq i\leq w} \det s_i\ \psi_{f,s_is}(h : \nu : m\exp(tH_0))\Bigg\|$$

$$+ \sum_{1 \le i \le w} \Bigg\| d_{Q'}^{Q}(m \exp(tH_0)) \pi_Q(\Lambda_{h,\nu,s_i s}) g^i(h : \nu : m \exp(tH_0))$$

$$- \sum_{1 \le j \le p} \det u_j \; \psi_{g^i, u_j s_i s}(h : \nu : m \exp(tH_0)) \Bigg\|.$$

Now by applying Theorem 3.13 to the pair (P, Q) with $H = H_1$ we see that there are a neighborhood V_1 of 0 in $U''_{\mathbf{C}}$, $\epsilon_1 > 0, r \ge 0$, and a constant C so that for all $(h, \nu, m) \in V_1 \times \mathfrak{a}^* \times L^*_{Q'}$, $t \ge 0$,

$$d_{Q'}^{Q}(m \exp(tH_0)) \Bigg\| d_Q^P(m \exp(tH_0)) \pi_P(\Lambda_{h,\nu,s}) f(h : \nu : m \exp(tH_0))$$

$$- \sum_{1 \le i \le w} \det s_i \; \psi_{f, s_i s}(h : \nu : m \exp(tH_0)) \Bigg\|$$

$$\le C e^{-\epsilon_1 t} d_{Q'}^{Q}(m \exp(tH_0)) \Xi_Q(m \exp(tH_2)) |(h, \nu, m \exp(tH_0))|^r e^{|h_I| \sigma_V(m)}.$$

Now as in [**HW5**, 7.11], there are constants $c > 0, r_0 \ge 0$ so that

$$d_{Q'}^{Q}(m \exp(tH_0)) \Xi_Q(m \exp(tH_2)) \le c \Xi_{Q'}(m)(1 + \sigma(m \exp(tH_0)))^{r_0}.$$

Thus for fixed m, ν, h there is a polynomial $p(t)$ so that

$$d_{Q'}^{Q}(m \exp(tH_0)) \Bigg\| d_Q^P(m \exp(tH_0)) \pi_P(\Lambda_{h,\nu,s}) f(h : \nu : m \exp(tH_0))$$

$$- \sum_{1 \le i \le w} \det s_i \; \psi_{f, s_i s}(h : \nu : m \exp(tH_0)) \Bigg\| \le |p(t)| e^{-\epsilon_1 t}$$

for all $t \ge 0$. Thus for any $0 < \epsilon < \epsilon_1$ we have

$$\lim_{t \to +\infty} e^{\epsilon t} d_{Q'}^{Q}(m \exp(tH_0)) \Bigg[d_Q^P(m \exp(tH_0)) \pi_P(\Lambda_{h,\nu,s}) f(h : \nu : m \exp(tH_0))$$

$$- \sum_{1 \le i \le w} \det s_i \; \psi_{f, s_i s}(h : \nu : m \exp(tH_0)) \Bigg] = 0.$$

Next, applying Theorem 3.13 to the pair (Q, Q') with $H = H_2$ we see that there are a neighborhood V_2 of 0 in $U''_{\mathbf{C}}$, $\epsilon_2 > 0, r \ge 0$, and a constant C so that for all $(h, \nu, m) \in V_2 \times \mathfrak{a}^* \times L^*_{Q'}, 1 \le i \le w$,

$$\Bigg\| d_{Q'}^{Q}(m \exp(tH_0)) \pi_Q(\Lambda_{h,\nu,s_i s}) g^i(h : \nu : m \exp(tH_0))$$

$$- \sum_{1\le j\le p} \det u_j \ \psi_{g^i,u_js_is}(h:\nu:m\exp(tH_0))\Bigg\|$$
$$\le Ce^{-\epsilon_2 t}\Xi_{Q'}(m)|(h,\nu,m\exp(tH_0))|e^{|h_I|\sigma_V(m)}.$$

Thus if $0<\epsilon<\epsilon_2$ we have

$$\lim_{t\to+\infty} e^{\epsilon t}\Bigg[d^Q_{Q'}(m\exp(tH_0))\pi_Q(\Lambda_{h,\nu,s_is})g^i(h:\nu:m\exp(tH_0)) - \sum_{1\le j\le p} \det u_j \ \psi_{g^i,u_js_is}(h:\nu:m\exp(tH_0))\Bigg]=0.$$

Combining the above, we see that there is $\epsilon>0$ so that for all $(h,\nu,m)\in V_1\cap V_2\times\mathfrak{a}^*\times L^*_{Q'}$

$$\lim_{t\to+\infty} e^{\epsilon t}\Bigg[d^P_{Q'}(m\exp(tH_0))\pi_P(\Lambda_{h,\nu,s})f(h:\nu:m\exp(tH_0)) - \sum_{1\le i\le w, 1\le j\le p} \det(u_js_i)\psi_{g^i,u_js_is}(h:\nu:m\exp(tH_0))\Bigg]=0.$$

But we also have $\epsilon'>0$ and a neighborhood V' of 0 so that for all $(h,\nu,m)\in V'\times\mathfrak{a}^*\times L^*_{Q'}$,

$$\lim_{t\to+\infty} e^{\epsilon' t}\Bigg[d^P_{Q'}(m\exp(tH_0))\pi_P(\Lambda_{h,\nu,s})f(h:\nu:m\exp(tH_0)) - \sum_{1\le i\le w, 1\le j\le p} \det(u_js_i)\psi'_{f,u_js_is}(h:\nu:m\exp(tH_0))\Bigg]=0.$$

Thus we have a neighborhood V of h_0 in $U''_{\mathbf{C}}$ and $\epsilon>0$ so that for all $(h,\nu,m)\in V\times\mathfrak{a}^*\times L^*_{Q'}$,

$$\lim_{t\to 0} e^{\epsilon t}\Bigg[\sum_{1\le i\le w, 1\le j\le p} \det(u_js_i)\psi_{g^i,u_js_is}(h:\nu:m\exp(tH_0)) - \sum_{1\le i\le w, 1\le j\le p} \det(u_js_i)\psi'_{f,u_js_is}(h:\nu:m\exp(tH_0))\Bigg]=0.$$

But for each i,j we have

$$\psi_{g^i,u_js_is}(h:\nu:m\exp(tH_0)) = \psi_{g^i,u_js_is}(h:\nu:m)\exp(tu_js_is\Lambda_{h,\nu}(H_0))$$

and

$$\psi'_{f,u_js_is}(h:\nu:m\exp(tH_0)) = \psi'_{f,u_js_is}(h:\nu:m)\exp(tu_js_is\Lambda_{h,\nu}(H_0)).$$

Let $S = \{(i,j) : \psi_{g^i,u_js_is}$ or ψ'_{f,u_js_is} is not identically zero$\}$. Then for all $(i,j) \in S$ we have $\operatorname{Re} u_js_is\Lambda_{h,\nu}(H_0) \leq 0$ for all $(h,\nu) \in V \times \mathfrak{a}^*$, and $\operatorname{Re} u_js_is\Lambda_{0,\nu}(H_0) = 0$ for all $\nu \in \mathfrak{a}^*$. By shrinking V if necessary we can assume that for all non-zero terms we have $-\epsilon < \operatorname{Re} u_js_is\Lambda_{h,\nu}(H_0) \leq 0$ for all $(h,\nu) \in V \times \mathfrak{a}^*$. Further, there is a dense subset W of $V \times \mathfrak{a}^*$ so that for $(h,\nu) \in W$ the complex numbers $w_{i,j} = u_js_is\Lambda_{h,\nu}(H_0)$ are all distinct. Thus for fixed $(h,\nu,m) \in W \times L^*_{Q'}$ there are distinct complex numbers $w_{i,j}$ with $0 \leq \operatorname{Re} w_{i,j} < \epsilon$ and complex numbers

$$c_{i,j} = \det(u_js_i)(\psi_{g^i,u_js_is}(h:\nu:m) - \psi'_{f,u_js_is}(h:\nu:m))$$

so that

$$\sum_{1\leq i\leq w, 1\leq j\leq p} \det(u_js_i)\psi_{g^i,u_js_is}(h:\nu:m\exp(tH_0))$$
$$- \sum_{1\leq i\leq w, 1\leq j\leq p} \det(u_js_i)\psi'_{f,u_js_is}(h:\nu:m\exp(tH_0)) = \sum_{(i,j)\in S} c_{i,j}e^{-tw_{i,j}}.$$

It now follows from (3.16) below that $c_{i,j} = 0$ for all $(i,j) \in S$. Thus $\psi_{g^i,u_js_is}(h:\nu:m) = \psi'_{f,u_js_is}(h:\nu:m)$ for all $(h,\nu,m) \in W \times L^*_{Q'}$. But since the constant terms are continuous functions we have $\psi_{g^i,u_js_is}(h:\nu:m) = \psi'_{f,u_js_is}(h:\nu:m)$ for all $(h,\nu,m) \in V \times \mathfrak{a}^* \times L^*_{Q'}$. □

Lemma 3.16. *Suppose $\epsilon > 0, w_1, ..., w_k$ are distinct complex numbers with $0 \leq Re\ w_i < \epsilon$ for $1 \leq i \leq k, p_i(t), 1 \leq i \leq k$, are polynomials, and*

$$\lim_{t\to+\infty} e^{\epsilon t}\sum_{i=1}^{k} p_i(t)e^{-tw_i} = 0.$$

Then $p_1 = ... = p_k = 0$.

Proof. The proof is by induction on the number of elements in the sum. Suppose that $k = 1$ and that $w_1 = x + iy, x, y \in \mathbf{R}$. Then $0 \leq x < \epsilon$ implies that

$$0 = \lim_{t\to+\infty} e^{xt}p_1(t)e^{-w_1t} = \lim_{t\to+\infty} p_1(t)e^{-iyt}.$$

This implies that $p_1 = 0$.

Now suppose that $k \geq 2$ and that the result is true for sums with less that k terms. Let $x = \min\{Re\ w_i : 1 \leq i \leq k\}$. Then

$$0 = \lim_{t\to+\infty} e^{xt}\sum_{i=1}^{k} p_i(t)e^{-tw_i} = \lim_{t\to+\infty}\sum_{i=1}^{k} p_i(t)e^{-t(w_i-x)}.$$

But for all i such that $Re\ w_i > x, \lim_{t\to+\infty} p_i(t)e^{-t(w_i-x)} = 0$. Thus

$$\lim_{t\to+\infty} \sum_{1\le i\le k, Re\ w_i=x} p_i(t)e^{-it\ Im\ w_i} = 0$$

so that as in [**HC1**, 21.3], $p_i = 0$ for all i such that $Re\ w_i = x$. But now

$$\lim_{t\to+\infty} e^{\epsilon t} \sum_{1\le i\le k, Re\ w_i>x} p_i(t)e^{-tw_i} = 0$$

and by the induction hypothesis, $p_i = 0$ for all such i also. □

Fix $H = TA$ a θ-stable Cartan subgroup and $(\lambda,\chi) \in X(T), \tau_1, \tau_2 \in \widehat{K}(\chi), W = W(\tau_1 : \tau_2)$. As in (2.10) we define F_0 and $H_F = T_F A_F, (\lambda_F, \chi_F) \in X(T_F)$ for every $F \subseteq F_0$. Fix a Cayley transform y with $\mathfrak{h}_{\mathbf{C}}^y = \mathfrak{h}_{0,\mathbf{C}}$. Then for each $s \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{0,\mathbf{C}}), F \subseteq F_0, (h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_F^*$, we can define

$$\Lambda_{h,\nu,s}^F = s(\lambda_F(h) + i\nu)^{c_F^{-1}y} \in \mathfrak{h}_{0,\mathbf{C}}^*.$$

Let $U_{\mathbf{C}}$ be a neighborhood of 0 in $\mathfrak{v}_{\mathbf{C}}^*$ and for each $F \subseteq F_0, \epsilon \in \Sigma_0$, let $U_{F,\mathbf{C}}(\epsilon) = U_{\mathbf{C}} \cap \mathrm{cl}(\mathcal{D}_{F,\mathbf{C}}(\epsilon)), U_F(\epsilon) = U_{F,\mathbf{C}}(\epsilon) \cap i\mathfrak{v}^*$.

Let $P_0 \subseteq P$ be a standard parabolic subgroup of G. Then for all $F \subseteq F_0, \epsilon \in \Sigma_0, s \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{0,\mathbf{C}})$ we can define the subset

(3.17 a) $$J_F(U_{\mathbf{C}} : \epsilon : L_P^* : s) = J(U_{F,\mathbf{C}}(\epsilon) : L_P^* : s)$$

of $C^\infty(U_{F,\mathbf{C}}(\epsilon) \times \mathfrak{a}_F^* \times L_P^* : W)$ as in (3.1) using $\Lambda_{h,\nu,s}^F$. Similarly, we define the subset

(3.17 b) $$J_F^0(U : \epsilon : L_P^* : s) = J^0(U_F(\epsilon) : L_P^* : s)$$

of $C^\infty(U_F(\epsilon) \times \mathfrak{a}_F^* \times L_P^* : W)$ as in (3.2). Now assume that $P_0 \subseteq Q \subseteq P$. Then for each fixed $F \subseteq F_0, \epsilon \in \Sigma_0, 1 \le i \le w$, we can define the constant term $\psi_{f,s_i s}(F : \epsilon)$ of $f(F : \epsilon)$ as in the first part of this section.

Let Θ be the set of simple roots in $\Delta(P_0, A_0)$. Then there is a bijection between standard parabolic subgroups of G and subsets of Θ so that $P \leftrightarrow \Theta_P$ if $\mathfrak{a}_P = \{H \in \mathfrak{a}_0 : \alpha(H) = 0 \text{ for all } \alpha \in \Theta_P\}$. Since we assume that our Cartan involution is chosen so that $Z_G \subseteq K$, we have $\mathfrak{a}_G = \{0\}$. Given $P_0 \subseteq Q \subseteq P \subseteq G$, define $H = H(P,Q) \in \mathrm{cl}(\mathfrak{a}_0^+)$ by $\alpha(H) = 0$ for all $\alpha \in \Theta_Q \cup (\Theta\backslash\Theta_P)$ and $\alpha(H) = 1$ for all $\alpha \in \Theta_P\backslash\Theta_Q$. Then $H(P,Q) \in (\mathfrak{a}_Q^P)^+ = \{H \in \mathfrak{a}_Q : \alpha(H) > 0 \text{ for all } \alpha \in \Phi_P^+\}$.

Lemma 3.18. *There is a neighborhood U of 0 in $i\mathfrak{v}^*$ so that for any $F \subseteq F_0, \epsilon \in \Sigma_0, s \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{0,\mathbf{C}}), P_0 \subseteq Q \subseteq P \subseteq G$, if $f \in J_F^0(U : \epsilon : L_P^* : s)$,* *then all constant terms* $\psi_{f,s_is} \in J^0_F(U : \epsilon : L^*_Q : s_is)$ *and the estimates of Theorem* 3.13 *hold for all* $h \in U_F(\epsilon)$ *when* $H = H(P,Q)$.

Proof. For fixed F, ϵ, s, P, Q as above, how small $U' \subseteq U$ has to be to make Theorem 3.13 valid depends on λ_F and $H = H(P,Q)$, but is independent of f. Since there are only finitely many possibilities for F, ϵ, s, P, Q, a neighborhood U can be found which works for all. □

Fix a neighborhood U of 0 in $i\mathfrak{v}^*$ as in (3.18). We will define $J^1_F(U : \epsilon : L^*_P : s)$ to be the set of all $f \in J^0_F(U : \epsilon : L^*_P : s)$ such that there is a compact subset ω of $U_F(\epsilon)$ so that $\operatorname{supp} f \subseteq \omega$ and such that for all $P_0 \subseteq Q \subseteq P$ we have

$$f_{Q,s_is}(h : \nu : m) = \pi_P\left(\Lambda^F_{h,\nu,s}\right)^{-1} \psi_{f,s_is}(h : \nu : m)$$

extends to a C^∞ function on $U_F(\epsilon) \times \mathfrak{a}^*_F \times L^*_Q$ for all i.

Let $P_0 \subseteq Q \subseteq P$ and $F \subseteq F_0, \epsilon \in \Sigma_0$. Define ϵ_0 and $\epsilon_i(h), i \in I^- \cap I^0(0)$ as in (3.13) for $\omega = \{H = H(P,Q)\}$.

Theorem 3.19. *Given* $b_1, b_2 \in \mathcal{U}(\mathfrak{l}_{Q,\mathbf{C}})$ *and* $D \in \mathcal{P}$, *there exist a finite subset* $E \subseteq \widetilde{\mathcal{L}}_P$ *and an* $r_1 > 0$ *so that for all* $r \geq 0$ *there is a* $C > 0$ *so that for all* $m \in L^*_Q, h \in U_F(\epsilon), T \geq 0, f \in J^1_F(U : \epsilon : L^*_P : s)$,

$$\begin{aligned}&\Bigg\| d_Q(m \exp TH) f(h : \nu; D : b_1' ; m \exp TH; b_2') \\ &\qquad - \sum_{i=1}^{w} \det(s_i) f_{Q,s_is}(h : \nu; D : b_1 ; m \exp TH; b_2)\Bigg\| \\ &\leq C T^0_{E,r}(f) e^{-\epsilon_0 T} \Xi_Q(m)(1 + \tilde{\sigma}(m \exp TH))^{r+r_1}\end{aligned}$$

and for each $i \in I^- \cap I^0(0)$,

$$\begin{aligned}&\|f_{Q,s_is}(h : \nu; D : b_1 ; m \exp TH; b_2)\| \\ &\leq C T^0_{E,r}(f) e^{-\epsilon_i(h)T} \Xi_Q(m)(1 + \tilde{\sigma}(m \exp TH))^{r+r_1}.\end{aligned}$$

Further, for all $1 \leq i \leq w$,

$$f_{Q,s_is} \in J^1_F(U : \epsilon : L^*_Q : s_is).$$

Given $D \in \widetilde{\mathcal{L}}_Q$, *there are a finite subset* $E \subseteq \widetilde{\mathcal{L}}_P$ *and an* $r_1 > 0$ *such that for all* $r \geq 0$ *there is a* $C > 0$ *so that*

$$T^0_{D,r+r_1}(f_{Q,s_is}) \leq C T^0_{F,r}(f).$$

Proof. All the estimates and the fact that $f_{Q,s_is} \in J_F^0(U : \epsilon : L_Q^* : s_is)$ follow from (3.13), using [**H2**, 6.8]. Further, $\operatorname{supp} f_{Q,s_is} \subseteq \operatorname{supp} f \subseteq \omega$. Finally, for $P_0 \subseteq Q' \subseteq Q$, using (3.15) we see that $(f_{Q,s_is})_{Q',u_js_is} = f_{Q',u_js_is}$ for all j. Thus $(f_{Q,s_is})_{Q',u_js_is}$ extends to be smooth. □

Now suppose for each $F \subseteq F_0, \epsilon \in \Sigma_0$, we have

$$f(F : \epsilon) \in J_F(U_{\mathbf{C}} : \epsilon : L_P^* : s)$$

(or in $J_F^1(U : \epsilon : L_P^* : s)$) satisfying the matching conditions of (2.14) for each $x \in L_P^*$. Let $P_0 \subseteq Q \subseteq P$.

Theorem 3.21. *Suppose $\{f(F : \epsilon)\}$ is a matching collection of functions in $J_F(U_{\mathbf{C}} : \epsilon : L_P^* : s)$. Then for each $1 \leq i \leq w$, the collection*

$$\{\psi_{f,s_is}(F : \epsilon)\}_{F \subseteq F_0, \epsilon \in \Sigma_0}$$

satisfies the matching conditions of (2.14) for each $x \in L_Q^*$. If $\{f(F : \epsilon)\}$ is a matching collection of functions in $J_F^1(U : \epsilon : L_P^* : s)$, then for each $1 \leq i \leq w$, the collection

$$\{f_{Q,s_is}(F : \epsilon)\}_{F \subseteq F_0, \epsilon \in \Sigma_0}$$

satisfies the matching conditions of (2.14) for each $x \in L_Q^*$.

In order to prove (3.21) we will need the following lemma. Recall for each $s \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{0,\mathbf{C}}), F \subseteq F_0, (h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_F^*$, we have defined $\Lambda_{h,\nu,s}^F \in \mathfrak{h}_{0,\mathbf{C}}^*$. For each $1 \leq i \leq m, E \subseteq F \subseteq E(i)$, define $D_{F\setminus E} = \partial/\partial h_i - i \sum_{\alpha \in F \setminus E} \partial/\partial \mu_\alpha$.

Lemma 3.22. *Let $E \subseteq F_0, 1 \leq i \leq m, \nu_E \in \mathfrak{a}_E^*, h_0 \in \mathcal{H}_i$. Then for any $E \subseteq F \subseteq E(i), k \geq 0, H \in \mathfrak{h}_{0,\mathbf{C}}$,*

$$D_{F\setminus E}^k \Lambda_{h_0,(\nu_E,0),s}^F(H) = D_{E\setminus E}^k \Lambda_{h_0,\nu_E,s}^E(H).$$

Proof. For $(h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_F^*, H \in \mathfrak{h}_{0,\mathbf{C}}$,

$$\Lambda_{h,\nu,s}^F(H) = s(\lambda_F(h) + i\nu)^{c_F^{-1}y}(H) = (\lambda_F(h) + i\nu)((s^{-1}H)^{y^{-1}c_F})$$

where $(s^{-1}H)^{y^{-1}} \in \mathfrak{h}_{\emptyset,\mathbf{C}}$. Now in [**H3**, 5.16] it was proven that for every $\beta \in \Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\emptyset,\mathbf{C}}), k \geq 0$, $D_{F\setminus E}^k \langle \lambda_F(h_0) + i(\nu_E, 0), \beta^{c_F} \rangle$ is independent of $E \subseteq F \subseteq E(i)$. Using exactly the same proof as that in [**H3**, 5.16] we see that for all $H_1 \in \mathfrak{h}_{\emptyset,\mathbf{C}}, k \geq 0$,

$$D_{F\setminus E}^k(\lambda_F(h_0) + i(\nu_E, 0))(H_1^{c_F})$$

is independent of $E \subseteq F \subseteq E(i)$. Now take $H_1 = (s^{-1}H)^{y^{-1}}$ to see that for all $k \geq 0$, $D^k_{F\backslash E}\Lambda^F_{h_0,(\nu_E,0),s}(H)$ is independent of $E \subseteq F \subseteq E(i)$. □

Proof of Theorem 3.21. We start with the family of functions $\{f(F : \epsilon)\}$ satisfying the matching conditions of (2.14). The first step in defining the constant terms is to define associated functions $\Phi(f(F : \epsilon)), \Psi_v(f(F : \epsilon)), v \in \mathcal{Z}_Q$, taking values in $W \otimes T$. Recall that for any $f, \Phi(f)$ and $\Psi_v(f)$ are defined by

$$\Phi(f : h : \nu : m) = \sum_{i=1}^{w} d_Q(m) f(h : \nu : m; v_i') \otimes e_i$$

and

$$\Psi_v(f : h : \nu : m) = \sum_{i=1}^{w} d_Q(m) f(h : \nu : m; u_i(v : h : \nu)') \otimes e_i$$

where $u_i(v : h : \nu) = \sum_{j=1}^{w} \mu_{PQ}\left(z_{vij} - \mu_P\left(z_{vij} : \Lambda^F_{h,\nu,s}\right)\right) v_j$, the $v_i \in \mathcal{Z}_Q, z_{vij} \in \mathcal{Z}_P$, and $e_1, ..., e_w$ is a basis for T.

Now since the $f(F : \epsilon : h : \nu : m)$ satisfy the matching conditions for all $m \in L^*_P$, the functions $f(F : \epsilon : h : \nu : m; v)$ will satisfy the matching conditions for any $m \in L^*_Q, v \in \mathcal{Z}_Q$. Thus it is clear that the functions $\Phi(f(F : \epsilon))$ satisfy the matching conditions. But using (3.22), for all $u \in S(\mathfrak{h}_{0,\mathbf{C}}), k \geq 0$, $D^k_{F\backslash E}\Lambda^F_{h_0,(\nu_E,0),s}(u)$ is independent of $E \subseteq F \subseteq E(i)$. Thus for all $k \geq 0$, $D^k_{F\backslash E}\mu_P\left(z_{vij} : \Lambda^F_{h_0,(\nu_E,0),s}\right)$ is independent of $E \subseteq F \subseteq F_0$. Thus the functions $\Psi_v(f(F : \epsilon))$ also satisfy the matching conditions of (2.14).

Next, for $1 \leq i \leq w$ and any f, the functions $\Phi_i(f), \Psi_{v,i}(f)$ are defined by

$$\Phi_i(f : h : \nu : m) = B_1\left(s_i\Lambda^F_{h,\nu,s}\right)\Phi(f : h : \nu : m)$$

and

$$\Psi_{v,i}(f : h : \nu : m) = B_1\left(s_i\Lambda^F_{h,\nu,s}\right)\Psi_v(f : h : \nu : m)$$

where B_1 is an $r \times r$ matrix with entries in $S(\mathfrak{h}_{0,\mathbf{C}})^{W_Q}$. Thus again using (3.22), if b is any matrix entry of B_1, for all $k \geq 0$, $D^k_{F\backslash E}b(s_i\Lambda^F_{h_0,(\nu_E,0),s})$ is independent of $E \subseteq F \subseteq F_0$. Thus the functions $\Phi_i(f(F : \epsilon))$ and $\Psi_{v,i}(f(F : \epsilon))$ also satisfy the matching conditions of (2.14).

By (3.6), for any f, D we have

$$\Phi_{i,\infty}(f : h : \nu; D : m) = \Phi_i(f : h : \nu; D : m)$$

$$+\int_0^\infty \Psi_{H_0,i}\left(f:h:\nu; D\circ e^{-ts_i\Lambda^F_{h,\nu,s}(H_0)}:m\exp tH_0\right)dt.$$

But for all $k\geq 0$, $D^k_{F\setminus E}\left(s_i\Lambda^F_{h_0,(\nu_E,0),s}\right)(H_0)$ is independent of $E\subseteq F\subseteq F_0$. Thus, using the matching conditions for the functions $\Phi_i(f(F:\epsilon))$ and $\Psi_{v,i}(f(F:\epsilon))$, we obtain the matching conditions for the functions $\Phi_{i,\infty}(f(F:\epsilon))$ and hence $\psi_{f,s_is}(F:\epsilon)$.

Finally, in the case that $f\in J^1_F(U:\epsilon:L^*_P:s)$, we have

$$\psi_{f,s_is}(F:\epsilon:h:\nu)=\pi_P\left(\Lambda^F_{h,\nu,s}\right)f_{Q,s_is}(F:\epsilon:h:\nu).$$

Fix $E\subseteq F_0, 1\leq i\leq m$, $\epsilon\in\Sigma_i$, $x\in L^*_Q$, $\nu_E\in\mathfrak{a}^*_E$, $h_0\in\mathcal{H}_i\cap U_E(\epsilon)$, and for each $E\subseteq F\subseteq E(i), k\geq 0$, write

$$a^\pm(k:F)=D^k_{F\setminus E}f_{Q,s_is}(F:\epsilon^\pm(i):h_0:(\nu_E,0):x);$$
$$b^\pm(k:F)=D^k_{F\setminus E}\psi_{f,s_is}(F:\epsilon^\pm(i):h_0:(\nu_E,0):x);$$
$$c(k:F)=D^k_{F\setminus E}\pi_P\left(\Lambda^F_{h_0,(\nu_E,0),s}\right).$$

As in (3.22), $c(k:F)=c(k:E)$ is independent of F. Further, since $D_{E\setminus E}=\partial/\partial h_i$ and $\pi_P\left(\Lambda^E_{h,\nu_E,s}\right)$ is a polynomial in h and ν_E, there is k_0 so that $c(k_0:E)\neq 0$ as long as ν_E is in a dense subset of regular elements. Assume that ν_E is regular.

Now for any $k\geq 0$ we have

$$b^\pm(k:F)=\sum_{j=0}^{k}\binom{k}{j}c(k-j:E)a^\pm(j:F).$$

Thus using the fact that the $\psi_{f,s_is}(F:\epsilon)$ satisfy the matching conditions, we have for all $E\subseteq F\subseteq E(i), k\geq 0$,

$$\sum_{j=0}^{k}\binom{k}{j}c(k-j:E)\Bigg[a^+(k:E)-a^-(k:E)$$
$$-\sum_{E\subset F\subseteq E(i)}c_{|F\setminus E|}(a^+(k:F)+a^-(k:F))\Bigg]=0.$$

Now as in [**H1**, 10.10], it is easy to use the fact that $c(k_0:E)\neq 0$ and induction to prove that

$$a^+(k:E)-a^-(k:E)-\sum_{E\subset F\subseteq E(i)}c_{|F\setminus E|}(a^+(k:F)+a^-(k:F))=0$$

for all k.

Now since the matching conditions are satisfied for ν_E in the dense subset of regular elements, and the functions $f_{Q,s_is}(F:\epsilon^\pm(i):h:\nu:x)$ are jointly smooth, the matching conditions will be satisfied for any ν_E. □

4. Asymptotic expansions.

In this section we will obtain the constant terms of §3 in a different way, via the Casselman-Miličić theory of asymptotic expansions. From this point of view it is easy to show that the constant terms have meromorphic continuations and get some information on possible poles which will be needed in §5. In [**HW4**] we used the theory of asymptotic expansions to study holomorphic families of matrix coefficients of relative discrete series representations. We considered only the case in which the group G is simple, simply connected, with infinite center so that $\mathfrak{v}_{\mathbf{C}}^* \cong \mathbf{C}$. In order to do this we had to extend the Casselman-Miličić theory of [**CM**] to include dependence on a complex parameter $h \in \mathfrak{v}_{\mathbf{C}}^* \cong \mathbf{C}$. The results proven in [**HW4**, §7] extend easily to the general case when $h \in \mathfrak{v}_{\mathbf{C}}^* \cong \mathbf{C}^n$. Thus we will use these results without reproving them for this case. We will change the notation used in [**CM**, **HW4**] slightly so that we study asympotics in the positive Weyl chamber rather than the negative Weyl chamber. This is so that the constant terms obtained from the asymptotic expansions can be easily compared with those obtained in §3 using the techniques of Harish-Chandra.

Let $H = TA$ be a θ-stable Cartan subgroup of $G, (\lambda, \chi) \in X(T), \tau_1, \tau_2 \in \widehat{K}(\chi), W = W(\tau_1 : \tau_2)$. Let $\mathcal{D}$ be a connected component of $\{h \in i\mathfrak{v}^* : \langle \lambda(h), \alpha \rangle \neq 0$ for all $\alpha \in \Phi_M^+\}$ such that $0 \in \mathrm{cl}(\mathcal{D})$. Let

$$F \in C^\infty(\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* \times G : W)$$

such that for all $x \in G$,

$$\text{(4.1 a)} \qquad (h, \nu) \to F(h : \nu : x) \quad \text{is a holomorphic function on } \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*;$$

for all $(h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$,

$$\text{(4.1 b)} \qquad F(h : \nu) \quad \text{is a } (\tau_{1,h}, \tau_{2,h}) - \text{ spherical function on } G;$$

for all $(h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$, using the notation of (3.1),

$$\text{(4.1 c)} \qquad zF(h : \nu) = \mu_G(z : \Lambda_{h,\nu}) F(h : \nu) \quad \text{for all } z \in \mathcal{Z}_G;$$

and for all $(h, \nu) \in \mathcal{D} \times \mathfrak{a}^*, g_1, g_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$, there is an $r > 0$ so that

$$\text{(4.1 d)} \qquad \sup_{x \in G} \|F(h : \nu : g_1; x; g_2)\| \Xi(x)^{-1} (1 + \tilde{\sigma}(x))^{-r} < \infty.$$

Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}$ such that $A \subseteq A_0$ and let A_0^+ be the positive Weyl chamber of A_0 with respect to a choice $\Phi_{\mathfrak{a}_0}^+ = \Phi^+(\mathfrak{g}, \mathfrak{a}_0)$ of positive restricted roots, so that $G = K\,\mathrm{cl}(A_0^+)K$ is the Cartan decomposition of G. Let Δ be the set of simple (multiplicative) roots for the **negative** roots of A_0. Thus $\Delta = \{e^\beta : \beta \in \Delta_{\mathfrak{a}_0}\}$ where $\Delta_{\mathfrak{a}_0}$ is the set of simple roots in $\Phi^-_{\mathfrak{a}_0} = -\Phi^+_{\mathfrak{a}_0}$. Note that although G is not necessarily semisimple, our Cartan involution θ is chosen so that $Z_G \subseteq K$ where Z_G is the center of G. Thus $\Delta_{\mathfrak{a}_0}$ is a basis for $\mathfrak{a}_0^*$. Let $\underline{\alpha} : A_0 \to \mathbf{C}^\Delta$ be the embedding given by

$$\underline{\alpha}(a) = (\alpha(a) : \alpha \in \Delta), a \in A_0. \tag{4.2 a}$$

In these coordinates A_0^+ corresponds to $(0,1)^\Delta$ and and $\mathrm{cl}(A_0^+)$ corresponds to $(0,1]^\Delta$.

For $s \in \mathbf{C}^\Delta, m \in \mathbf{Z}_+^\Delta$, and $z \in \mathbf{C}^\Delta$, set

$$z^s = \prod_{\alpha\in\Delta} z_\alpha^{s_\alpha}, \qquad \log^m z = \prod_{\alpha\in\Delta} (\log z_\alpha)^{m_\alpha}. \tag{4.2 b}$$

For $\Theta \subseteq \Delta$ we regard $\mathbf{C}^{\Delta\backslash\Theta}$ as a subset of $\mathbf{C}^\Delta$, and we say $s,t \in \mathbf{C}^{\Delta\backslash\Theta}$ are $(\Delta\backslash\Theta)$-integrally equivalent if $t - s \in \mathbf{Z}^{\Delta\backslash\Theta}$. Let $C(\Theta)$ be the domain in $\mathbf{C}^\Delta$ containing $[0,1)^{\Delta\backslash\Theta} \times (0,1]^\Theta$ which is defined in [**CM**, pp. 895-896]. Then for each $(h,\nu) \in \mathfrak{v}_\mathbf{C}^* \times \mathfrak{a}_\mathbf{C}^*$, there exists a finite set $S_{\Delta\backslash\Theta}(h:\nu)$ of mutually $(\Delta\backslash\Theta)$-integrally inequivalent elements of $\mathbf{C}^{\Delta\backslash\Theta}$ satisfying the following. For each $s \in S_{\Delta\backslash\Theta}(h:\nu)$ there is a finite set $\bar{F}_{s,m}^{\Delta\backslash\Theta}(h:\nu), m \in \mathbf{Z}_+^{\Delta\backslash\Theta}$, of holomorphic functions on $C(\Theta)$ such that on each of the coordinate hyperplanes $\mathcal{H}_\alpha = \{z \in C(\Theta) : z_\alpha = 0\}, \alpha \in \Delta\backslash\Theta$, at least one of them is not identically zero, so that for all $a \in A_0^+(\Theta) = \{a \in A_0 : \alpha(a) \leq 1, \alpha \in \Theta; \alpha(a) < 1, \alpha \in \Delta\backslash\Theta\}$,

$$F(h:\nu:a) = \sum_{s\in S_{\Delta\backslash\Theta}(h:\nu)} \sum_m \bar{F}_{s,m}^{\Delta\backslash\Theta}(h:\nu:\underline{\alpha}(a))\underline{\alpha}^s(a) \log^m \underline{\alpha}(a). \tag{4.2 c}$$

We will define a uniform asymptotic expansion as follows. Let $\{\mu_\alpha : \alpha \in \Delta\}$ be the dual basis in $\mathfrak{a}_0^*$ to $\{\log\alpha : \alpha \in \Delta\}$. Let W be the Weyl group of $\Phi(\mathfrak{g}_\mathbf{C}, (\mathfrak{h}_0)_\mathbf{C})$ and let $\rho_{\mathfrak{a}_0} = 1/2\sum_{\beta\in\Phi^+_{\mathfrak{a}_0}} \beta$. For each $w \in W$ define $s(w:h:\nu) \in \mathbf{C}^\Delta$ by

$$s(w:h:\nu)_\alpha = \langle w\Lambda_{h,\nu} - \rho_{\mathfrak{a}_0}, \mu_\alpha\rangle, \quad \alpha \in \Delta \tag{4.3 a}$$

so that

$$\underline{\alpha}^{s(w:h:\nu)}(a) = e^{w\Lambda_{h,\nu} - \rho_{\mathfrak{a}_0}}(a), \quad a \in A_0. \tag{4.3 b}$$

For $w \in W$, define $b(w), a(w:h:\nu) \in \mathbf{C}^\Delta$ by

$$b(w)_\alpha = \langle w\Lambda_{0,0} - \rho_{\mathfrak{a}_0}, \mu_\alpha\rangle \tag{4.3 c}$$

and

(4.3 d) $$a(w:h:\nu)_\alpha = \langle w(h_M(h)+i\nu)^y, \mu_\alpha\rangle,\ \alpha\in\Delta, (h,\nu)\in\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}},$$

so that each $a(w)_\alpha$ is a linear functional on $\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}}$ and

(4.3 e) $$s(w:h:\nu) = b(w) + a(w:h:\nu).$$

Fix $\Theta\subseteq\Delta$. We will define an equivalence relation on W by $w\sim_{\Delta\backslash\Theta} w'$ if and only if

(4.4 a) $$s(w:h:\nu)_\alpha - s(w':h:\nu)_\alpha\in\mathbf{Z}\quad\text{for all } \alpha\in\Delta\backslash\Theta, (h,\nu)\in\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}}.$$

Thus $w\sim_{\Delta\backslash\Theta} w'$ if and only if

(4.4 b) $$b(w)_\alpha - b(w')_\alpha\in\mathbf{Z}\quad\text{and}\quad a(w)_\alpha = a(w')_\alpha\quad\text{for all } \alpha\in\Delta\backslash\Theta.$$

For each $\alpha\in\Delta\backslash\Theta, w, w'\in W$ such that $w\not\sim_{\Delta\backslash\Theta} w'$, define

(4.4 c) $$L_{\alpha,w,w'} = \{(h,\nu)\in\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}} : s(w:h:\nu)_\alpha - s(w':h:\nu)_\alpha\in\mathbf{Z}\}$$

and set

(4.4 d) $$L_{w,w'} = \cap_{\alpha\in\Delta\backslash\Theta}L_{\alpha,w,w'},\qquad L = \cup_{w\not\sim_{\Delta\backslash\Theta}w'}L_{w,w'}.$$

Then L is a closed subset of $\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}}$ which is a countable union of affine subspaces of co-dimension at least one. For $(h,\nu)\in(\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}})\backslash L, w, w'\in W, s(w:h:\nu)$ is $\Delta\backslash\Theta$-integrally equivalent to $s(w':h:\nu)$ if and only if $w\sim_{\Delta\backslash\Theta} w'$. Further, for each $(h,\nu)\in(\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}})\backslash L$ and $s\in S_{\Delta\backslash\Theta}(h:\nu)$ there is an equivalence class U of W so that $s_\alpha\in\{s(w:h:\nu)_\alpha : w\in U\}$ for all $\alpha\in\Delta\backslash\Theta$. Let U be an equivalence class of W and for $\alpha\in\Delta\backslash\Theta$ define $I(U)_\alpha = \{w\in U : s(w:h:\nu)_\alpha = s_\alpha$ for some $s\in S_{\Delta\backslash\Theta}(h:\nu), (h,\nu)\in(\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}})\backslash L\}$. Let U be an equivalence class of W such that $I(U)_\alpha\neq\emptyset$ for all $\alpha\in\Delta\backslash\Theta$. Then for $\alpha\in\Delta\backslash\Theta, (h,\nu)\in\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}}$, define

(4.4e) $$a(U:h:\nu)_\alpha = a(w:h:\nu)_\alpha, w\in U;$$

(4.4f) $$b(U)_\alpha = \min\{b(w)_\alpha : w\in I(U)_\alpha\};$$

(4.4g) $$s(U:h:\nu)_\alpha = a(U:h:\nu)_\alpha + b(U)_\alpha.$$

Write $S_{\Delta\backslash\Theta}$ for the set of all $s(U):\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}}\to\mathbf{C}^{\Delta\backslash\Theta}$ defined above. The elements of $S_{\Delta\backslash\Theta}$ are $\Delta\backslash\Theta$- mutually inequivalent for $(h,\nu)\notin L$.

For each $(h,\nu)\in(\mathfrak{v}^*_{\mathbf{C}}\times\mathfrak{a}^*_{\mathbf{C}})\backslash L, s(U)\in S_{\Delta\backslash\Theta}, m\in\mathbf{Z}_+^{\Delta\backslash\Theta}, z\in C(\Theta)$, define

(4.5a)
$$F^{\Delta\backslash\Theta}_{s(U),m}(h:\nu:z) =$$

$$\begin{cases} z^{s-s(U:h:\nu)}\bar{F}^{\Delta\backslash\Theta}_{s,m}(h:\nu:z), & \text{if there is } s \in S_{\Delta\backslash\Theta}(h:\nu) \text{ with } s - s(U:h:\nu) \in \mathbf{Z}_+^{\Delta\backslash\Theta};\\ 0, & \text{otherwise.}\end{cases}$$

Each $F^{\Delta\backslash\Theta}_{s(U),m}(h:\nu)$ is holomorphic on $C(\Theta)$ and for each $s(U) \in S_{\Delta\backslash\Theta}, \alpha \in \Delta\backslash\Theta$, there are an $m \in \mathbf{Z}_+^{\Delta\backslash\Theta}, (h,\nu) \in (\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}})\backslash L$, so that $F^{\Delta\backslash\Theta}_{s(U),m}(h:\nu)$ is not identically zero on $\mathcal{H}_\alpha$. For $(h,\nu) \in (\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}})\backslash L$ we now have a uniform asymptotic expansion for $a \in A_0^+(\Theta)$ given by

$$F(h:\nu:a) = \sum_{s \in S_{\Delta\backslash\Theta}} \sum_m F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}(a))\underline{\alpha}^s(a)\log^m \underline{\alpha}(a). \tag{4.5 b}$$

Now that we have the uniform asymtotic expansions the following two results can be proven in the same way as the corresponding results in [**HW4**, 8.5, 8.6]. For each $s \in S_{\Delta\backslash\Theta}$, let $M(s) = \Big\{m \in \mathbf{Z}_+^{\Delta\backslash\Theta} : F^{\Delta\backslash\Theta}_{s,m}(h:\nu)$ is not identically zero for some $(h,\nu) \in \mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}}\backslash L\Big\}$.

Lemma 4.6. $[M(s)] < \infty$ *for all* $s \in S_{\Delta\backslash\Theta}$.

Theorem 4.7. *For each* $s \in S_{\Delta\backslash\Theta}, m \in M(s), F^{\Delta\backslash\Theta}_{s,m}(h:\nu:z)$ *is jointly holomorphic on* $(\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}})\backslash L \times C(\Theta)$ *and jointly meromorphic on* $\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}} \times C(\Theta)$. *In fact, there is a holomorphic function* g *on* $\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}}$ *so that* g *has no zeroes on* $(\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}})\backslash L$ *and* $(h,\nu,z) \to g(h:\nu)F^{\Delta\backslash\Theta}_{s,m}(h:\nu:z)$ *is jointly holomorphic on* $\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}} \times C(\Theta)$ *for all* $s \in S_{\Delta\backslash\Theta}, m \in M(s)$.

Lemma 4.8. *For all* $s \in S_{\Delta\backslash\Theta}, \alpha \in \Delta\backslash\Theta$ *and* $(h,\nu) \in \mathcal{D} \times \mathfrak{a}^*$,

$$\operatorname{Re} s(h:\nu)_\alpha \geq -\langle \rho_{\mathfrak{a}_0}, \mu_\alpha\rangle.$$

Proof. Suppose there are $t \in S_{\Delta\backslash\Theta}, \alpha \in \Delta\backslash\Theta$, and $(h_0,\nu_0) \in \mathcal{D} \times \mathfrak{a}^*$ such that $\operatorname{Re} t(h_0:\nu_0)_\alpha < -\langle \rho_{\mathfrak{a}_0}, \mu_\alpha\rangle$. Then there is a non-empty open subset U of $\mathcal{D} \times \mathfrak{a}^*$ such that $\operatorname{Re} t(h:\nu)_\alpha < -\langle \rho_{\mathfrak{a}_0}, \mu_\alpha\rangle$ for all $(h,\nu) \in U$. Let $U' = \{(h,\nu) \in U : (h,\nu) \notin L\}$ and fix $(h,\nu) \in U'$. Since $F(h:\nu)$ satisfies the weak inequality of (4.1d), using [**CM**, 7.5] we have $\operatorname{Re} s_\alpha \geq -\langle \rho_{\mathfrak{a}_0}, \mu_\alpha\rangle$ for all $s \in S_{\Delta\backslash\Theta}(h:\nu)$. Thus $\operatorname{Re} s_\alpha - \operatorname{Re} t(h:\nu)_\alpha > 0$ for all $s \in S_{\Delta\backslash\Theta}(h:\nu)$. In particular, if there is $s \in S_{\Delta\backslash\Theta}(h:\nu)$ with $s - t(h:\nu) \in \mathbf{Z}_+^{\Delta\backslash\Theta}$, then $s_\alpha - t(h:\nu)_\alpha = n$ for some $n > 0$ in $\mathbf{Z}$. We now see from (4.5a) that $F^{\Delta\backslash\Theta}_{t,m}(h:\nu:z) = 0$ for all $m \in M(t), z \in \mathcal{H}_\alpha$. But this holds for all $(h,\nu) \in U'$ and $(h,\nu) \to F^{\Delta\backslash\Theta}_{t,m}(h:\nu:z)$ is meromorphic for $(h,\nu) \in \mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}}$ by Theorem (4.7). Thus $F^{\Delta\backslash\Theta}_{t,m}(h:\nu:z)$ is identically zero for $z \in \mathcal{H}_\alpha$ for all $m \in M(t), (h,\nu) \in (\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}})\backslash L$. This contradicts the remark following (4.5a). □

Let

$$(4.9\,\mathrm{a})\qquad S^0_{\Delta\backslash\Theta} = \{s \in S_{\Delta\backslash\Theta} : s(0:0)_\alpha = -\langle \rho_{\mathfrak{a}_0}, \mu_\alpha\rangle \quad \text{for all } \alpha \in \Delta\backslash\Theta\}.$$

For all $a \in \mathrm{cl}(A_0^+)$, define $\underline{\alpha}^\Theta(a) \in \mathbf{C}^\Delta$ by

$$(4.9\,\mathrm{b})\qquad \underline{\alpha}^\Theta(a)_\alpha = \begin{cases} \alpha(a), & \text{if } \alpha \in \Theta; \\ 0, & \text{if } \alpha \in \Delta\backslash\Theta. \end{cases}$$

Note that for all $a \in \mathrm{cl}(A_0^+), \underline{\alpha}^\Theta(a) \in C(\Theta)$. We now define the constant term of F with respect to Θ for $a \in \mathrm{cl}(A_0^+)$ by

$$(4.9\mathrm{c})\quad F_\Theta(h:\nu:a) = e^{\rho_{\mathfrak{a}_0}}(a) \sum_{s\in S^0_{\Delta\backslash\Theta}} \sum_{m\in M(s)} F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}^\Theta(a))\underline{\alpha}^{s(h:\nu)}(a)\log^m \underline{\alpha}(a).$$

Theorem 4.10. *Suppose $H_0 \in \mathfrak{a}_0$ such that $\log\alpha(H_0) = 0$ for all $\alpha \in \Theta$ and $\log\alpha(H_0) < 0$ for all $\alpha \in \Delta\backslash\Theta$. Then there exists an $\epsilon > 0$ and a neighborhood $U(0)$ of 0 in $\mathrm{cl}(\mathcal{D})$ such that for all $(h,\nu,a) \in U(0)\times\mathfrak{a}^*\times\mathrm{cl}(A_0^+)$ such that $(h,\nu) \notin L$,*

$$\lim_{t\to+\infty} e^{\epsilon t}\left(e^{\rho_{\mathfrak{a}_0}}(a\exp tH_0)F(h:\nu:a\exp tH_0) - F_\Theta(h:\nu:a\exp tH_0)\right) = 0.$$

Proof. Fix $a \in \mathrm{cl}(A_0^+)$ and write $a\exp tH_0 = a_t, t > 0$. Note that $\underline{\alpha}(a_t) \in C(\Theta)$. For each $\alpha \in \Delta\backslash\Theta$, write $\epsilon_\alpha = -\log\alpha(H_0) > 0$. Then

$$\underline{\alpha}(a_t) = ((\alpha(a))_{\alpha\in\Theta}, (\alpha(a)e^{-t\epsilon_\alpha})_{\alpha\in\Delta\backslash\Theta})$$

and

$$\underline{\alpha}^\Theta(a_t) = \underline{\alpha}^\Theta(a) = ((\alpha(a))_{\alpha\in\Theta}, (0)_{\alpha\in\Delta\backslash\Theta}).$$

Thus

$$\lim_{t\to+\infty} \underline{\alpha}(a_t) = \underline{\alpha}^\Theta(a)$$

and for all $s \in S_{\Delta\backslash\Theta}, m \in M(s), (h,\nu) \notin L$,

$$\lim_{t\to+\infty} F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}(a_t)) = F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}^\Theta(a)).$$

For $s \in S_{\Delta\backslash\Theta}, \alpha \in \Delta\backslash\Theta$, define $s'(h:\nu)_\alpha = s(h:\nu)_\alpha + \langle\rho_{\mathfrak{a}_0}, \mu_\alpha\rangle$. Then, for all $s \in S_{\Delta\backslash\Theta}, m \in M(s)$,

$$e^{\rho_{\mathfrak{a}_0}}(a_t)\underline{\alpha}(a_t)^{s(h:\nu)} = \underline{\alpha}(a_t)^{s'(h:\nu)} = \underline{\alpha}(a)^{s'(h:\nu)} \prod_{\alpha\in\Delta\backslash\Theta} e^{-t\epsilon_\alpha s'(h:\nu)_\alpha}$$

and

$$\log^m \underline{\alpha}(a_t) = \prod_{\alpha\in\Delta\backslash\Theta} (\log \alpha(a) - t\epsilon_\alpha)^{m_\alpha}.$$

Note that using (4.7) we have $\left|\underline{\alpha}(a)^{s'(h:\nu)}\right| \le 1$ for all $a \in \mathrm{cl}(A_0^+)$ and $\left|e^{-t\epsilon_\alpha s'(h:\nu)_\alpha}\right| \le 1$ for all $\alpha \in \Delta\backslash\Theta, t > 0$.

Now use (4.5b) to write

$$\begin{aligned} &e^{\rho_{\mathfrak{a}_0}}(a_t)F(h:\nu:a_t) - F_\Theta(h:\nu:a_t) \\ &= \sum_{s\in(S_{\Delta\backslash\Theta})\backslash(S^0_{\Delta\backslash\Theta})} \sum_{m\in M(s)} F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}(a_t))\underline{\alpha}(a_t)^{s'(h:\nu)} \log^m \underline{\alpha}(a_t) \\ &\quad + \sum_{s\in S^0_{\Delta\backslash\Theta}} \sum_{m\in M(s)} \Big(F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}(a_t)) \\ &\qquad - F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}^\Theta(a_t))\Big)\, \underline{\alpha}(a_t)^{s'(h:\nu)} \log^m \underline{\alpha}(a_t). \end{aligned}$$

First fix $s \in (S_{\Delta\backslash\Theta})\backslash(S^0_{\Delta\backslash\Theta})$. Then there is $\alpha_s \in \Delta\backslash\Theta$ so that $s'(0:0)_{\alpha_s} \neq 0$. But $s'(h:0)_{\alpha_s} = \mathrm{Re}\, s'(h:\nu)_{\alpha_s} \ge 0$ for all $(h,\nu) \in \mathrm{cl}(\mathcal{D}) \times \mathfrak{a}^*$ by (4.7). Thus $s'(0:0)_{\alpha_s} = c_s > 0$ and there is a neighborhood $U_s(0)$ of 0 in $\mathrm{cl}(\mathcal{D})$ so that $\mathrm{Re}\, s'(h:\nu)_{\alpha_s} = s'(h:0)_{\alpha_s} > c_s/2$ for all $(h,\nu) \in U_s(0) \times \mathfrak{a}^*$. Now for all $(h,\nu) \in U_s(0) \times \mathfrak{a}^*$,

$$\begin{aligned} \left|\underline{\alpha}(a_t)^{s'(h:\nu)}\right| &= \left|\underline{\alpha}(a)^{s'(h:\nu)} \prod_{\alpha\in\Delta\backslash\Theta} e^{-t\epsilon_\alpha s'(h:\nu)_\alpha}\right| \\ &\le e^{-t\epsilon_{\alpha_s} s'(h:0)_{\alpha_s}} \le e^{-t\epsilon_{\alpha_s} c_{\alpha_s}/2}. \end{aligned}$$

Thus for any $0 < \epsilon < \epsilon_{\alpha_s} c_{\alpha_s}/2$ and $(h,\nu,a) \in U_s(0) \times \mathfrak{a}^* \times \mathrm{cl}(A_0^+), (h,\nu) \notin L$, we have

$$\lim_{t\to+\infty} e^{\epsilon t}\underline{\alpha}(a_t)^{s'(h:\nu)} \log^m \underline{\alpha}(a_t) F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}(a_t)) = 0.$$

Now fix $s \in S^0_{\Delta\backslash\Theta}, m \in M(s)$. Then for any $(h,\nu) \notin L$,

$$G(h:\nu:z) = F^{\Delta\backslash\Theta}_{s,m}(h:\nu:z) - F^{\Delta\backslash\Theta}_{s,m}(h:\nu:(z_\alpha)_{\alpha\in\Theta}, (0)_{\alpha\in\Delta\backslash\Theta})$$

is a holomorphic function of $z \in C(\Theta)$ which is zero if $z_\alpha = 0$ for all $\alpha \in \Delta\backslash\Theta$. Thus for each $\alpha \in \Delta\backslash\Theta$ there is a holomorphic function $G_\alpha(h:\nu:z)$ of $z \in C(\Theta)$ so that

$$G(h:\nu:z) = \sum_{\alpha\in\Delta\backslash\Theta} z_\alpha G_\alpha(h:\nu:z).$$

Now

$$F_{s,m}^{\Delta\backslash\Theta}(h:\nu:\underline{\alpha}(a_t)) - F_{s,m}^{\Delta\backslash\Theta}(h:\nu:\underline{\alpha}^{\Theta}(a_t)) = G(h:\nu:\underline{\alpha}(a_t))$$
$$= \sum_{\alpha\in\Delta\backslash\Theta} \alpha(a)e^{-t\epsilon_\alpha} G_\alpha(h:\nu:\underline{\alpha}(a_t))$$

where

$$\lim_{t\to+\infty} G_\alpha(h:\nu:\underline{\alpha}(a_t)) = G_\alpha(h:\nu:\underline{\alpha}^{\Theta}(a)).$$

Thus if $0<\epsilon<\min_{\alpha\in\Delta\backslash\Theta}\epsilon_\alpha$ and $(h,\nu,a)\in \mathrm{cl}(\mathcal{D})\times\mathfrak{a}^*\times\mathrm{cl}(A_0^+), (h,\nu)\notin L$, then

$$\lim_{t\to+\infty} e^{\epsilon t}\left(F_{s,m}^{\Delta\backslash\Theta}(h:\nu:\underline{\alpha}(a_t)) - F_{s,m}^{\Delta\backslash\Theta}(h:\nu:\underline{\alpha}^{\Theta}(a_t))\right)$$
$$\cdot\, \underline{\alpha}(a_t)^{s'(h:\nu)}\log^m \underline{\alpha}(a_t) = 0.$$

Thus

$$\lim_{t\to+\infty} e^{\epsilon t}\left(e^{\rho_{\mathfrak{a}_0}}(a_t)F(h:\nu:a_t) - F_\Theta(h:\nu:a_t)\right) = 0$$

if $0<\epsilon<\min\{\{\epsilon_\alpha : \alpha\in\Delta\backslash\Theta\}\cup\{\epsilon_{\alpha_s}c_s/2 : s\in(S_{\Delta\backslash\Theta})\backslash(S^0_{\Delta\backslash\Theta})\}\}$ and $h\in U(0)=\cap U_s(0), s\in S_{\Delta\backslash\Theta}\backslash S^0_{\Delta\backslash\Theta}$. □

Our next task is to compare the constant term of F with respect to Θ defined above to the constant terms defined in §3. Suppose that in addition to satisfying the conditions of (4.1), the restriction of F to $\mathrm{cl}(\mathcal{D}_{\mathbf{C}})\times\mathfrak{a}^*\times G$ is an element of $J(\mathrm{cl}(\mathcal{D}_{\mathbf{C}}):G)$. For example, F could be a holomorphic family of Eisenstein integrals. Let P_0 be the minimal parabolic subgroup corresponding to $\Phi^+_{\mathfrak{a}_0}$ and let $P_0\subseteq Q$ be the standard parabolic subgroup with $\mathfrak{a}_Q=\{H\in\mathfrak{a}_0 : \log\alpha(H)=0 \text{ for all } \alpha\in\Theta\}$. Then $\mathfrak{a}_Q^+=\{H\in\mathfrak{a}_Q : \log\alpha(H)<0 \text{ for all } \alpha\in\Delta\backslash\Theta\}$. Define the constant terms $\psi_{F,s_i}, 1\le i\le w$, of F with respect to Q as in (3.12b). Recall that $\psi_{F,s_i}=0$ unless $i\in I^0(0)$ and that the constant terms ψ_{F,s_i} are only defined in some neighborhood U_C of 0 in $\mathrm{cl}(\mathcal{D}_{\mathbf{C}})$. Define

$$\widetilde{F}_Q = \sum_{i\in I^0(0)} \det s_i\, \psi_{F,s_i}.$$

Theorem 4.11. *For all* $(h,\nu,a)\in U_{\mathbf{C}}\times\mathfrak{a}^*\times\mathrm{cl}(A_0^+)$,

$$\widetilde{F}_Q(h:\nu:a) = \pi_G(h:\nu)F_\Theta(h:\nu:a).$$

Proof. Choose H_0 as in (4.10). Then $H_0\in\mathfrak{a}_Q^+$ so that combining (4.10) and (3.14) we see that there are an $\epsilon>0$ and a neighborhood $U(0)$ of 0 in $\mathrm{cl}(\mathcal{D})$ such that for all $(h,\nu,a)\in U(0)\times\mathfrak{a}^*\times\mathrm{cl}(A_0^+)$ such that $(h,\nu)\notin L$,

$$\lim_{t\to+\infty} e^{\epsilon t}\left(\widetilde{F}_Q(h:\nu:a_t) - \pi_G(h:\nu)F_\Theta(h:\nu:a_t)\right) = 0$$

where $a_t = a\exp(tH_0)$.

From the proof of (4.10) we know that

$$F_\Theta(h:\nu:a_t) = \sum_{s\in S^0_{\Delta\backslash\Theta}} \sum_{m\in M(s)} F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}^\Theta(a))\underline{\alpha}^{s'(h:\nu)}(a)$$

$$\cdot \exp\left(-t\sum_{\alpha\in\Delta\backslash\Theta}\epsilon_\alpha s'(h:\nu)_\alpha\right)\prod_{\alpha\in\Delta\backslash\Theta}(\log\alpha(a)-t\epsilon_\alpha)^{m_\alpha}$$

where $\epsilon_\alpha = -\alpha(H_0) > 0$ for all $\alpha \in \Delta\backslash\Theta$ and

$$\mathrm{Re}\ s'(h:\nu)_\alpha = \mathrm{Re}\, s(h:\nu)_\alpha + \langle\rho_{\mathfrak{a}_0},\mu_\alpha\rangle \ \geq 0, \mathrm{Re}\, s'(0:\nu)_\alpha = 0$$

for all $(h,\nu)\in U(0)\times\mathfrak{a}^*, s\in S^0_{\Delta\backslash\Theta}, \alpha\in\Delta\backslash\Theta$.

Similarly, using (3.10),

$$\widetilde{F}_Q(h:\nu:a_t) = \sum_{i\in I^0(0)} \det s_i\ \psi_{F,s_i}(h:\nu:a)e^{ts_i\Lambda_{h,\nu}(H_0)}$$

where $\mathrm{Re}\, s_i\Lambda_{h,\nu}(H_0)\leq 0$ and $\mathrm{Re}\, s_i\Lambda_{0,\nu}(H_0) = 0$ for $(h,\nu)\in U(0)\times\mathfrak{a}^*$.

Thus we can assume that $U(0)$ is small enough that for all $(h,\nu)\in U(0)\times\mathfrak{a}^*, s\in S^0_{\Delta\backslash\Theta}, i\in I^0(0)$,

$$0\leq \mathrm{Re}\sum_{\alpha\in\Delta\backslash\Theta}\epsilon_\alpha s'(h:\nu)_\alpha < \epsilon$$

and $0\leq -\mathrm{Re}\, s_i\Lambda_{h,\nu}(H_0) < \epsilon$. Thus for fixed (h,ν,a) as above there are finitely many distinct complex numbers $w_1,\ldots,w_k$ with $0\leq \mathrm{Re}\ w_i<\epsilon$ and polynomials $p_i(t), 1\leq i\leq k$, so that

$$\widetilde{F}_Q(h:\nu:a_t) - \pi_G(h:\nu)F_\Theta(h:\nu:a_t) = \sum_{i=1}^k p_i(t)e^{-tw_i}$$

for all $t\geq 0$. It now follows from (3.16) that $\widetilde{F}_Q(h:\nu:a) = \pi_G(h:\nu)F_\Theta(h:\nu:a)$ for all $(h,\nu,a)\in U(0)\times\mathfrak{a}^*\times \mathrm{cl}(A_0^+)$. But both sides are meromorphic functions of $h\in U_{\mathbf{C}}$ so the equality extends to $(h,\nu,a)\in U_{\mathbf{C}}\times\mathfrak{a}^*\times\mathrm{cl}(A_0^+)$. □

Corollary 4.12. *There is a bijection between the set of all $i\in I^0(0)$ such that $\psi_{F,s_i}\neq 0$ and $S^0_{\Delta\backslash\Theta}$ such that if i corresponds to s,*

$$e^{s_i\Lambda_{h,\nu}}(a) = e^{\rho_{\mathfrak{a}_0}}(a)\underline{\alpha}(a)^{s(h:\nu)}$$

for all $(h,\nu,a)\in\mathfrak{v}_{\mathbf{C}}\times\mathfrak{a}^*\times A_Q$ and

$$\det s_i\ \psi_{F,s_i}(h:\nu:a) = \pi_G(h:\nu)e^{\rho_{\mathfrak{a}_0}}(a)F^{\Delta\backslash\Theta}_{s,0}(h:\nu:\underline{\alpha}^\Theta(a))\,\underline{\alpha}(a)^{s(h:\nu)}$$

for all $(h,\nu,a) \in U_{\mathbf{C}} \times \mathfrak{a}^* \times \mathrm{cl}(A_0^+)$. *Further, for all* $s \in S^0_{\Delta\backslash\Theta}, m \in M(s), m \neq 0$,

$$F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}^{\Theta}(a)) = 0$$

for all $(h,\nu,a) \in \mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}} \times \mathrm{cl}(A_0^+)$.

Proof. Let $a_1 \in \mathrm{cl}(A_0^+)$ such that $\log\alpha(a_1) = 0$ for all $\alpha \in \Delta\backslash\Theta$. Then for all $(h,\nu,a_2) \in U_{\mathbf{C}} \times \mathfrak{a}^* \times A_Q^+$, using (4.11) and the notation in the proof of (4.10),

$$\begin{aligned}\widetilde{F}_Q(h:\nu:a_1a_2) &= \sum_{i\in I^0(0)} \det s_i\ \psi_{F,s_i}(h:\nu:a_1)e^{s_i\Lambda_{h,\nu}}(a_2)\\ &= \pi_G(h:\nu)\sum_{s\in S^0_{\Delta\backslash\Theta}}\ \sum_{m\in M(s)} F^{\Delta\backslash\Theta}_{s,m}(h:\nu:\underline{\alpha}^{\Theta}(a_1))\\ &\quad\cdot \underline{\alpha}(a_2)^{s'(h:\nu)}\log^m \underline{\alpha}(a_2).\end{aligned}$$

Let L' be the set of all $(h,\nu) \in \mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*$ such that there are $i \neq j$ in $I^0(0)$ with $e^{s_i\Lambda_{h,\nu}}(a_2) = e^{s_j\Lambda_{h,\nu}}(a_2)$ for all $a_2 \in A_Q$. Then $(\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*)\backslash(L \cup L')$ is a dense open subset of $\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*$ and the set of functions $a_2 \to e^{s_i\Lambda_{h,\nu}}(a_2), i \in I^0(0)$, and $a_2 \to \underline{\alpha}(a_2)^{s'(h:\nu)}\log^m \underline{\alpha}(a_2), s \in S^0_{\Delta\backslash\Theta}, m \in M(s)$, are linearly independent on A_2 for $(h,\nu) \notin L \cup L'$ except that we can have pairs $i \in I^0(0)$ and $s \in S^0_{\Delta\backslash\Theta}$ such that $e^{s_i\Lambda_{h,\nu}}(a) = \underline{\alpha}(a)^{s'(h:\nu)}$ for all $(h,\nu,a) \in \mathfrak{v}_{\mathbf{C}} \times \mathfrak{a}^* \times A_Q$. □

Corollary 4.13. *For all* $i \in I^0(0)$,

$$(h,\nu) \to \psi_{F,s_i}(h:\nu:a)$$

has a meromorphic extension to $(h,\nu) \in \mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}}$ *for all* $a \in \mathrm{cl}(A_0^+)$. *For all* $s \in S^0_{\Delta\backslash\Theta}, a \in \mathrm{cl}(A_0^+)$,

$$(h,\nu) \to \pi_G(h:\nu)F^{\Delta\backslash\Theta}_{s,0}(h:\nu:\underline{\alpha}^{\Theta}(a))$$

is smooth for $(h,\nu) \in U_{\mathbf{C}} \times \mathfrak{a}^*$.

Proof. This follows from combining (4.12) with (4.7) and (3.13). □

Fix $s_0 \in S^0_{\Delta\backslash\Theta}$ and write $(S_{\Delta\backslash\Theta})\backslash\{s_0\} = S' \cup S''$ where $S' = \left(S^0_{\Delta\backslash\Theta}\right)\backslash\{s_0\}$ and $S'' = (S_{\Delta\backslash\Theta})\Big\backslash\left(S^0_{\Delta\backslash\Theta}\right)$. For each $s \in S'$, define

(4.14 a)

$$L_{s_0,s} = \{(h,\nu) \in \mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}} : s_0(h:\nu)_\alpha - s(h:\nu)_\alpha = 0 \text{ for all } \alpha \in \Delta\backslash\Theta\}$$

and set

(4.14 b) $$L_{s_0} = \cup_{s\in S'} L_{s_0,s}.$$

Lemma 4.15. *There is a neighborhood U of $(0,0)$ in $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$ so that*

$$(h,\nu) \to F_{s_0,0}^{\Delta\backslash\Theta}\,(h:\nu:\underline{\alpha}^{\Theta}(a))$$

is holomorphic in $U\backslash(U \cap L_{s_0})$ for all $a \in \mathrm{cl}(A_0^+)$.

Proof. For each $s \in S'$, define

$$L'_{s_0,s} = \{(h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* : s_0(h:\nu)_\alpha - s(h:\nu)_\alpha \in \mathbf{Z} \quad \text{for all } \alpha \in \Delta\backslash\Theta\}$$

and set $L'_{s_0} = \cup_{s\in S'} L'_{s_0,s}$. We will show that there is a neighborhood U of $(0,0)$ in $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$ so that

$$(h,\nu) \to F_{s_0,0}^{\Delta\backslash\Theta}\,(h:\nu:\underline{\alpha}^{\Theta}(a))$$

is holomorphic in $U\backslash(U \cap L'_{s_0})$ for all $a \in \mathrm{cl}(A_0^+)$. But for all $s \in S'$, $s_0(0:0)_\alpha = s(0:0)_\alpha = 0$. Thus by shrinking U if necessary, we can guarantee that $U \cap L_{s_0} = U \cap L'_{s_0}$.

The proof uses monodromy transformations as in [**CM**, A.1.7] and [**HW4**, 8.6]. For each $\alpha_0 \in \Delta\backslash\Theta$, define $e(\alpha_0) \in \mathbf{C}^{\Delta\backslash\Theta}$ by

$$e(\alpha_0)_\alpha = \begin{cases} 1, & \text{if } \alpha = \alpha_0; \\ 0, & \text{otherwise.} \end{cases}$$

Now for $\alpha \in \Delta\backslash\Theta$, let T_α^* be the monodromy transformation satisfying $T_\alpha^*(\log z) = \log z - 2\pi i e(\alpha)$. Thus for any $s \in \mathbf{C}^{\Delta\backslash\Theta}, m \in \mathbf{Z}_+^{\Delta\backslash\Theta}$,

$$T_\alpha^* z^s \log^m(z) = e^{-2\pi i s_\alpha} z^s \sum_{k=0}^{m_\alpha} \binom{m_\alpha}{k} (-2\pi i)^{m_\alpha - k} \log^{m-ke(\alpha)} z.$$

Fix $(h_0,\nu_0) \notin L'_{s_0}$ and $s \in S'$. Then there is $\alpha = \alpha_s \in \Delta\backslash\Theta$ so that $s(h_0:\nu_0)_\alpha - s_0(h_0:\nu_0)_\alpha \notin \mathbf{Z}$. Pick $n_s > \max_{m\in M(s)} m_\alpha$. Then for all $(h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$,

$$\left(T_\alpha^* - e^{-2\pi i s(h:\nu)_\alpha}\right)^{n_s} \sum_{m\in M(s)} z^{s(h:\nu)} \log^m(z) F_{s,m}^{\Delta\backslash\Theta}(h:\nu:z) = 0$$

for all $z \in \mathbf{C}^{\Delta\backslash\Theta}$. Now,

$$\begin{aligned} &\left(T_\alpha^* - e^{-2\pi i s(h:\nu)_\alpha}\right)^{n_s} z^{s_0(h:\nu)} F_{s_0,0}^{\Delta\backslash\Theta}(h:\nu:z) \\ &= \left(e^{-2\pi i s_0(h:\nu)_\alpha} - e^{-2\pi i s(h:\nu)_\alpha}\right)^{n_s} z^{s_0(h:\nu)} F_{s_0,0}^{\Delta\backslash\Theta}(h:\nu:z). \end{aligned}$$

Write

$$T(s_0) = \prod_{s\in S'} (T_{\alpha_s}^* - e^{-2\pi i s(h:\nu)_{\alpha_s}})^{n_s}.$$

Then if $M'(s_0) = \{m \in M(s_0)$ such that $m \neq 0\}$, we have

$$T(s_0)F(h:\nu:z) = \prod_{s\in S'}\left(e^{-2\pi i s_0(h:\nu)_{\alpha_s}} - e^{-2\pi i s(h:\nu)_{\alpha_s}}\right)^{n_s} z^{s_0(h:\nu)} F^{\Delta\backslash\Theta}_{s_0,0}(h:\nu:z)$$
$$+ \sum_{m\in M'(s_0)} T(s_0) z^{s_0(h:\nu)} \log^m(z) F^{\Delta\backslash\Theta}_{s_0,m}(h:\nu:z)$$
$$+ \sum_{s\in S'', m\in M(s)} T(s_0) z^{s(h:\nu)} \log^m(z) F^{\Delta\backslash\Theta}_{s,m}(h:\nu:z).$$

Write

$$a(s_0:h:\nu) = \prod_{s\in S'}\left(e^{-2\pi i s_0(h:\nu)_{\alpha_s}} - e^{-2\pi i s(h:\nu)_{\alpha_s}}\right)^{n_s}.$$

It is a holomophic function on $\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}}$ and there is a neighborhood $U(h_0,\nu_0)$ of $(h_0,\nu_0) \in \mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}}$ so that $a(s_0:h:\nu) \neq 0$ for all $(h,\nu) \in U(h_0,\nu_0)$.

Fix $H_0 \in \mathfrak{a}_0$ satisfying the conditions of Theorem 4.10. For $a \in \mathrm{cl}(A_0^+)$, $t > 0$, write $a_t = a\exp(tH_0)$. Now for $s \in S''$,

$$\underline{\alpha}(a_t)^{s(h:\nu)-s_0(h:\nu)} = \underline{\alpha}(a)^{s(h:\nu)-s_0(h:\nu)} \exp\left(-t\sum_{\alpha\in\Delta\backslash\Theta} \epsilon_\alpha(s(h:\nu)_\alpha - s_0(h:\nu)_\alpha)\right).$$

Now there is $\alpha_s \in \Delta\backslash\Theta$ so that

$$s(0:0)_{\alpha_s} - s_0(0:0)_{\alpha_s} = s(0:0)_{\alpha_s} + \langle \rho_{\mathfrak{a}_0}, \mu_{\alpha_s}\rangle > 0.$$

Further, for all $\alpha \in \Delta\backslash\Theta$,

$$s(0:0)_\alpha - s_0(0:0)_\alpha = s(0:0)_\alpha + \langle \rho_{\mathfrak{a}_0}, \mu_\alpha\rangle \geq 0.$$

Thus

$$\sum_{\alpha\in\Delta\backslash\Theta} \epsilon_\alpha(s(0:0)_\alpha - s_0(0:0)_\alpha) > 0$$

so that there is a neighborhood U_s of $(0,0)$ in $\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}}$ so that

$$\mathrm{Re}\sum_{\alpha\in\Delta\backslash\Theta} \epsilon_\alpha(s(h:\nu)_\alpha - s_0(h:\nu)_\alpha) > 0$$

for all $(h,\nu) \in U_s$. Let $U = \cap_{s\in S''} U_s$ and let $g(h:\nu)$ be a holomorphic function on $\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}}$ so that $(h,\nu,z) \to g(h:\nu)F^{\Delta\backslash\Theta}_{s,m}(h:\nu:z)$ is holomorphic on $\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}} \times C(\Theta)$ for all $s \in S_{\Delta\backslash\Theta}, m \in M(s)$. Such a function exists by Theorem 4.7.

Let $D \in D(\mathfrak{v}^*_{\mathbf{C}} \times \mathfrak{a}^*_{\mathbf{C}})$. Then for all $(h,\nu) \in U$,

$$\lim_{t\to+\infty} D\left(a(s_0:h:\nu)g(h:\nu)F^{\Delta\backslash\Theta}_{s_0,0}(h:\nu:\underline{\alpha}(a_t))\right)$$

$$= D\left(a(s_0 : h : \nu) g(h : \nu) F_{s_0,0}^{\Delta\backslash\Theta}(h : \nu : \underline{\alpha}^{\Theta}(a))\right).$$

Further, for any $s \in S_{\Delta\backslash\Theta}, m \in M(s)$, there are finitely many $m' \in \mathbf{Z}_+^{\Delta\backslash\Theta}$ and holomorphic functions $c_{s,m,m'}(h : \nu)$ on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$ so that

$$\begin{aligned} &T(s_0) z^{s(h:\nu)} \log^m(z) F_{s,m}^{\Delta\backslash\Theta}(h : \nu : z) \\ &= \sum_{m'} c_{s,m,m'}(h : \nu) z^{s(h:\nu)} \log^{m'}(z) F_{s,m}^{\Delta\backslash\Theta}(h : \nu : z). \end{aligned}$$

Now as in [**HW4**, 8.8],

$$D\left(g(h : \nu) z^{-s_0(h:\nu)} T(s_0) z^{s(h:\nu)} \log^m(z) F_{s,m}^{\Delta\backslash\Theta}(h : \nu : z)\right)$$

is a finite sum of terms of the form

$$d(h : \nu) z^{s(h:\nu) - s_0(h:\nu)} \log^{m''}(z) D''\left(g(h : \nu) F_{s,m}^{\Delta\backslash\Theta}(h : \nu : z)\right)$$

where d is a holomophic function on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*, D'' \in D(\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*)$ and $m'' \in \mathbf{Z}_+^{\Delta\backslash\Theta}$. Now $D''(g(h : \nu) F_{s,m}^{\Delta\backslash\Theta}(h : \nu : z))$ is holomorphic on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* \times C(\Theta)$ so that using the argument of Theorem 4.10, if $s \in S'', m \in M(s)$, for all $(h, \nu) \in U$, since

$$\operatorname{Re} \sum_{\alpha \in \Delta\backslash\Theta} \epsilon_\alpha (s(h : \nu)_\alpha - s_0(h : \nu)_\alpha) > 0,$$

we have

$$\begin{aligned} \lim_{t \to +\infty} D\Big(g(h : \nu) \underline{\alpha}(a_t)^{-s_0(h:\nu)} T(s_0) \underline{\alpha}(a_t)^{s(h:\nu)} \\ \cdot \log^m \underline{\alpha}(a_t) F_{s,m}^{\Delta\backslash\Theta}(h : \nu : \underline{\alpha}(a_t))\Big) = 0. \end{aligned}$$

Further, if $m \in M'(s_0)$, by (4.12) $F_{s_0,m}^{\Delta\backslash\Theta}(h : \nu : \underline{\alpha}^{\Theta}(a)) = 0$ for all (h, ν, a). Thus

$$D\left(g(h : \nu) \underline{\alpha}(a_t)^{-s_0(h:\nu)} T(s_0) \underline{\alpha}(a_t)^{s_0(h:\nu)} \log^m \underline{\alpha}(a_t) F_{s_0,m}^{\Delta\backslash\Theta}(h : \nu : \underline{\alpha}(a_t))\right)$$

can be written as a finite sum of terms of the form

$$\begin{aligned} &d(h : \nu) \log^{m''} \underline{\alpha}(a_t) D'' \\ &\qquad \cdot \left(g(h : \nu) \left[F_{s_0,m}^{\Delta\backslash\Theta}(h : \nu : \underline{\alpha}(a_t)) - F_{s_0,m}^{\Delta\backslash\Theta}(h : \nu : \underline{\alpha}^{\Theta}(a))\right]\right) \end{aligned}$$

so again as in (4.10),

$$\lim_{t \to +\infty} D\Big(g(h : \nu) \underline{\alpha}(a_t)^{-s_0(h:\nu)} T(s_0) \underline{\alpha}(a_t)^{s_0(h:\nu)}$$

$$\cdot \log^m \underline{\alpha}(a_t) F^{\Delta\backslash\Theta}_{s_0,m}(h:\nu:\underline{\alpha}(a_t))\Big) = 0.$$

Thus

$$\lim_{t\to+\infty} D\left(g(h:\nu)\underline{\alpha}(a_t)^{-s_0(h:\nu)}T(s_0)F(h:\nu:a_t)\right)$$
$$= D\left(a(s_0:h:\nu)g(h:\nu)F^{\Delta\backslash\Theta}_{s_0,0}(h:\nu:\underline{\alpha}^{\Theta}(a))\right)$$

for all $(h,\nu) \in U$.

If $g(h_0:\nu_0) \neq 0$, then of course $F^{\Delta\backslash\Theta}_{s,m}(h:\nu:z)$ is holomorphic at (h_0,ν_0) for all $z \in C(\Theta)$. Thus we may as well assume that (h_0,ν_0) is on hyperplanes $L_1,\ldots,L_n$ such that $L_i \subseteq L$. Then there are linear functionals μ_i and integers $r_i \geq 0$ so that $L_i = \{(h,\nu) : \mu_i(h-h_0,\nu-\nu_0) = 0\}$ and $g(h:\nu) = \prod_{i=1}^n \mu_i^{r_i}(h:\nu)g_1(h:\nu)$ where $g_1(h:\nu) \neq 0$ for (h,ν) in a neighborhood U' of (h_0,ν_0). Let D_i be the directional derivative in the direction μ_i. Then a holomorphic function $\phi(h:\nu)$ on U' is divisible by $g(h:\nu)$ if for all $1 \leq i \leq n$ we have $D_i^k\phi(h:\nu) = 0$ for all $0 \leq k < r_i$ and $(h,\nu) \in L_i \cap U'$.

But $\underline{\alpha}(a_t)^{-s_0(h:\nu)}T(s_0)F(h:\nu:a_t)$ is holomorphic in U', so that for any $D = D_i^k, 0 \leq k < r_i$ and $(h,\nu) \in L_i \cap U'$,

$$D\left(g(h:\nu)\underline{\alpha}(a_t)^{-s_0(h:\nu)}T(s_0)F(h:\nu:a_t)\right) = 0$$

for all $t > 0$ so that

$$\lim_{t\to+\infty} D\left(g(h:\nu)\underline{\alpha}(a_t)^{-s_0(h:\nu)}T(s_0)F(h:\nu:a_t)\right) = 0.$$

Thus for all such D and (h,ν),

$$D\left(a(s_0:h:\nu)g(h:\nu)F^{\Delta\backslash\Theta}_{s_0,0}(h:\nu:\underline{\alpha}^{\Theta}(a))\right) = 0.$$

But $a(s_0:h:\nu)g(h:\nu)F^{\Delta\backslash\Theta}_{s_0,0}(h:\nu:\underline{\alpha}^{\Theta}(a))$ is holomorphic in U. Thus

$$a(s_0:h:\nu)g(h:\nu)F^{\Delta\backslash\Theta}_{s_0,0}(h:\nu:\underline{\alpha}^{\Theta}(a))$$

is divisible by $g(h:\nu)$ so that

$$a(s_0:h:\nu)F^{\Delta\backslash\Theta}_{s_0,0}(h:\nu:\underline{\alpha}^{\Theta}(a))$$

is holomorphic in $U \cap U'$. Finally, $a(s_0:h:\nu) \neq 0$ in a neighborhood of (h_0,ν_0) so that $F^{\Delta\backslash\Theta}_{s_0,0}(h:\nu:\underline{\alpha}^{\Theta}(a))$ is holomorphic at (h_0,ν_0). □

We now need to vary the chamber $\mathcal{D}$. Recall that as in (2.11) the set of all chambers $\mathcal{D}$ with $0 \in \mathrm{cl}(\mathcal{D})$ can be parameterized by Σ_0. As before we let $s_i, 1 \leq i \leq w$, be coset representatives for $W_Q \backslash W_G$. Let

$$I^0 = \{1 \leq i \leq w : s_i \Lambda_{h,0}(H) = 0 \quad \text{for all } h \in \mathfrak{v}_{\mathbf{C}}^*, H \in \mathfrak{a}_Q\}$$

and for each $\epsilon \in \Sigma_0$, let

$$I^0(0 : \epsilon) = \{1 \leq i \leq w : \text{ for all } H \in \mathfrak{a}_Q^+, s_i \Lambda_{h,0}(H) \leq 0 \\ \text{for all } h \in \mathrm{cl}(\mathcal{D}(\epsilon)) \text{ and } s_i \Lambda_{0,0}(H) = 0\}.$$

Now $I^0 \subseteq I^0(0 : \epsilon)$ for all $\epsilon \in \Sigma_0$. For $1 \leq i \neq j \leq w$, define

$$L_{i,j} = \{(h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* : s_i \Lambda_{h,\nu}(H) = s_j \Lambda_{h,\nu}(H) \text{ for all } H \in \mathfrak{a}_Q\}$$

and for $i \in I^0, \epsilon \in \Sigma_0$, set

$$L_i(\epsilon) = \cup_{j \in I^0(0:\epsilon), j \neq i} L_{i,j}.$$

Recall that $\pi_G(h : \nu) = \prod_{\beta \in \Phi^+} \pi_\beta(h : \nu)$ where $\pi_\beta(h : \nu) = \langle \lambda(h) + i\nu, \beta \rangle$. For each $\beta \in \Phi$, write $\mathcal{H}_\beta = \{(h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* : \pi_\beta(h : \nu) = 0\}$. For $i \in I^0, \epsilon \in \Sigma_0, \beta \in \Phi$, write $\mathcal{H}'_\beta(i : \epsilon) = \{(h, \nu) \in \mathcal{H}_\beta : (h, \nu) \notin L_i(\epsilon)\}$.

Lemma 4.16. *Suppose that $i \in I^0$ such that $s_i^{-1} \mathfrak{a}_Q \subseteq \mathfrak{a}$. Let $\gamma \in \Phi^+$ such that $\pi_\gamma(0 : 0) = 0$, and*

$$h \mapsto \pi_\gamma(h : 0) \quad \text{and} \quad \nu \mapsto \pi_\gamma(0 : \nu)$$

are non-trivial linear functionals on $\mathfrak{v}_{\mathbf{C}}^*$ and $\mathfrak{a}_{\mathbf{C}}^*$ respectively. Then there is $\epsilon \in \Sigma_0$ such that $\mathcal{H}'_\gamma(i : \epsilon)$ is a dense open subset of $\mathcal{H}_\gamma$.

Proof. Assume that $\mathcal{H}'_\gamma(i : \epsilon)$ is not a dense open subset of $\mathcal{H}_\gamma$ for any $\epsilon \in \Sigma_0$. Since $\pi_\gamma(0 : 0) = 0$ and $\pi_\gamma(h : 0)$ is not identically zero, there are $\epsilon^\pm \in \Sigma_0$ so that $\pi_\gamma(h : 0) > 0$ for some $h \in \mathcal{D}(\epsilon^+)$ and $\pi_\gamma(h : 0) < 0$ for some $h \in \mathcal{D}(\epsilon^-)$. Write

$$\mathcal{H} = \mathcal{H}_\gamma, \mathcal{H}^\pm = \mathcal{H}'_\gamma(i : \epsilon^\pm), J^\pm = \{j \in I^0(0 : \epsilon^\pm) : j \neq i\}.$$

Then since $\pi_\gamma(0 : 0) = 0$, $\mathcal{H}$ is the co-dimension one subspace of $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$ which is the kernel of the linear functional $\pi_\gamma(h : \nu)$. Further, for each $j \in J^\pm$, $L_{i,j} = \cap_{\alpha \in \Delta \backslash \Theta} L_{i,j}(\alpha)$ where each $L_{i,j}(\alpha)$ is the kernel of the linear functional $g_{j,\alpha}(h : \nu) = \langle s_i \Lambda_{h,\nu} - s_j \Lambda_{h,\nu}, \mu_\alpha \rangle$. Thus there are $j_\pm \in J^\pm, \alpha_\pm \in \Delta \backslash \Theta$ such that $\mathcal{H} \subseteq L_{i,j_\pm}(\alpha)$ for all $\alpha \in \Delta \backslash \Theta$ and $\mathcal{H} = L_{i,j_\pm}(\alpha_\pm)$. Thus there are complex numbers $c_\pm \neq 0$ so that

$$g_\pm(h : \nu) = g_{j_\pm, \alpha_\pm}(h : \nu) = c_\pm \pi_\gamma(h : \nu)$$

for all $(h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$.

Since $i \in I^0$ we have $\langle s_i \Lambda_{h,0}, \mu_\alpha \rangle = 0$ for all $h \in \mathfrak{v}_{\mathbf{C}}^*$. Since $j_\pm \in I^0(0 : \epsilon^\pm)$ and the μ_α are dual to the negative simple roots, we have $\langle s_{j_\pm} \Lambda_{h,0}, \mu_\alpha \rangle \geq 0$ for all $h \in \mathcal{D}(\epsilon^\pm), \alpha \in \Delta \backslash \Theta$. Thus $g_\pm(h : 0) \leq 0$ for all $h \in \mathcal{D}(\epsilon^\pm)$. Further, if one or both of $g_\pm(h : 0) = 0$ for all $h \in \mathfrak{v}_{\mathbf{C}}^*$, then $\pi_\gamma(h : 0) = 0$ for all $h \in \mathfrak{v}_{\mathbf{C}}^*$. This contradicts one of the hypotheses of the lemma. Thus $g_\pm(h : 0) < 0$ for all $h \in \mathcal{D}(\epsilon^\pm)$. Thus by our choice of $\epsilon^\pm$ we must have $c_+ < 0$ and $c_- > 0$ so that $c = c_+ c_-^{-1} < 0$.

But now for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$,

$$g_+(0 : \nu) = c_+ \pi_\gamma(0 : \nu) = c g_-(0 : \nu).$$

Thus we will have a contradiction if we can show that there is $\nu_0 \in \mathfrak{a}_{\mathbf{C}}^*$ such that both of $g_\pm(0 : \nu_0) > 0$.

Let $\nu_0 = (-i s_i^{-1} \mu_{\alpha_+})^{y^{-1}}$. Since we assume that $s_i^{-1} \mathfrak{a}_Q \subseteq \mathfrak{a}$ we have $\nu_0 \in \mathfrak{a}_{\mathbf{C}}^*$. Now

$$g_+(0 : \nu_0) = \langle \mu_{\alpha_+}, \mu_{\alpha_+} \rangle - \langle s_{j_+} s_i^{-1} \mu_{\alpha_+}, \mu_{\alpha_+} \rangle.$$

Thus $g_+(0 : \nu_0) \geq 0$ and if $g_+(0 : \nu_0) = 0$, then $s_{j_+} s_i^{-1} \mu_{\alpha_+} = \mu_{\alpha_+}$. But this would imply that $s_i^{-1} \mu_{\alpha_+} = s_{j_+}^{-1} \mu_{\alpha_+}$ so that $g_+(h : \nu) = 0$ for all $(h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$. Thus $g_+(0 : \nu_0) > 0$. Now suppose that $\alpha_+ = \alpha_-$. Then by the same argument as above, $g_-(0 : \nu_0) > 0$ and we are done.

Now we can take $\alpha_+ = \alpha_-$ unless $L_{i,j_-}(\alpha_+) = \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$. In this case

$$\langle s_i \Lambda_{h,\nu}, \mu_{\alpha_+} \rangle = \langle s_{j_-} \Lambda_{h,\nu}, \mu_{\alpha_+} \rangle$$

for all $(h, \nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^*$, so in particular, at $(h, \nu) = (0, \nu_0)$, we have

$$\langle \mu_{\alpha_+}, \mu_{\alpha_+} \rangle = \langle s_{j_-} s_i^{-1} \mu_{\alpha_+}, \mu_{\alpha_+} \rangle.$$

Thus $s_{j_-} s_i^{-1} \mu_{\alpha_+} = \mu_{\alpha_+}$ so that $0 = g_-(0 : \nu_0) = c g_+(0 : \nu_0)$. This contradicts the fact that $g_+(0 : \nu_0) > 0$. Thus we can assume that $\alpha_+ = \alpha_-$. □

Lemma 4.17. *Let $i \in I^0$ and $\gamma \in \Phi^+$ such that $\pi_\gamma(0 : 0) = 0$, and*

$$h \mapsto \pi_\gamma(h : 0) \quad \text{and} \quad \nu \mapsto \pi_\gamma(0 : \nu)$$

are non-trivial linear functionals on $\mathfrak{v}_{\mathbf{C}}^*$ and $\mathfrak{a}_{\mathbf{C}}^*$ respectively. Suppose that $\mathcal{D} = \mathcal{D}(\epsilon)$ for $\epsilon \in \Sigma_0$ such that $\mathcal{H}'_\gamma(i : \epsilon)$ is a dense open subset of $\mathcal{H}_\gamma$. Then for any $F \in J(\mathrm{cl}(\mathcal{D}_{\mathbf{C}}) : G)$ such that F has an extension to $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* \times G$ satisfying (4.1) *we have*

$$\psi_{F,s_i}(h_0 : \nu_0 : a) = 0$$

for all $(h_0, \nu_0, a) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}}^* \times \mathrm{cl}(A_0^+)$ *such that* $\pi_\gamma(h_0 : \nu_0) = 0$.

Proof. If $\psi_{F,s_i} = 0$ there is nothing to prove. Thus we assume that ψ_{F,s_i} is not identically zero. Thus by [**HW5**, 7.28] we have $s_i^{-1}\mathfrak{a}_Q \subseteq \mathfrak{a}$. Further, by (4.12) there is $s_0 \in S^0_{\Delta\backslash\Theta}$ such that

$$\det s_i\ \psi_{F,s_i}(h : \nu : a) = \pi_G(h : \nu) F^{\Delta\backslash\Theta}_{s_0,0}(h : \nu : \underline{\alpha}^\Theta(a))\, \underline{\alpha}(a)^{s_0'(h:\nu)}.$$

Now by (4.15), there is a neighborhood U of $(0,0)$ so that $F^{\Delta\backslash\Theta}_{s_0,0}(h : \nu : \underline{\alpha}^\Theta(a))$ is holomorphic for $(h,\nu) \in U\backslash(U \cap L_{s_0})$. Thus $\pi_G(h : \nu)$ $F^{\Delta\backslash\Theta}_{s_0,0}(h : \nu : \underline{\alpha}^\Theta(a)) = 0$ for all $(h,\nu) \in (\mathcal{H}_\gamma \cap U)\backslash(\mathcal{H}_\gamma \cap U \cap L_{s_0})$. But $L_{s_0} \subseteq L_i(\epsilon)$ so that $\mathcal{H}'_\gamma(i : \epsilon) \subseteq \mathcal{H}_\gamma\backslash(\mathcal{H}_\gamma \cap L_{s_0})$. Thus $\pi_G(h : \nu) F^{\Delta\backslash\Theta}_{s_0,0}(h : \nu : \underline{\alpha}^\Theta(a)) = 0$ for all $(h,\nu) \in \mathcal{H}_\gamma$. □

5. Poles of the constant term.

Fix $H = TA$ a θ-stable Cartan subgroup and $(\lambda, \chi) \in X(T), \tau_1, \tau_2 \in \widehat{K}(\chi)$. Let $U(0)$ be a neighborhood of 0 in $i\mathfrak{v}^*$ satisfying the conditions of [**H3**, 4.6] and (3.18). We assume that the Plancherel function $m(H : h : \nu)$ corresponding to $\pi(H : \lambda : \chi : h : \nu)$ is jointly smooth as a function of $(h,\nu) \in (U(0) \cap \mathrm{cl}(\mathcal{D})) \times \mathfrak{a}^*$ for every connected component $\mathcal{D}$ of $\{h \in i\mathfrak{v}^* : \langle\lambda(h), \alpha\rangle \neq 0, \alpha \in \Phi_M^+\}$. As in (2.10) we define F_0 and $H_F = T_F A_F, (\lambda_F, \chi_F) \in X(T_F)$ for every $F \subseteq F_0$.

Suppose for each $F \subseteq F_0$ we have a function

$$\Phi(F) : i\mathfrak{v}^* \times \mathfrak{a}_F^* \times G \to W = W(\tau_1 : \tau_2)$$

satisfying the conditions of (2.16) so that

$$\text{(5.1)}\quad \Phi(x) = \sum_{F \subseteq F_0} \int_{i\mathfrak{v}^*} \int_{\mathfrak{a}_F^*} \Phi(F : h : \nu_F : x) m(H_F : \lambda_F : \chi_F : h : \nu_F) d\nu_F dh$$

is an elementary mixed wave packet.

For each $F \subseteq F_0, \epsilon \in \Sigma_0$, let $\Phi(F : \epsilon)$ denote the restriction of $\Phi(F)$ to $U_F(\epsilon) \times \mathfrak{a}_F^* \times G$. Then each $\Phi(F : \epsilon) \in J_F^0(U : \epsilon : G)$. Thus for each standard parabolic subgroup Q of G and $s \in W_G$ representing a coset of $W_Q\backslash W_G$ we can define constant terms $\widetilde{\Phi}_{Q,s}(F : \epsilon) = \psi_{\Phi(F:\epsilon),s}$ as in (3.12). By (3.18) $\widetilde{\Phi}_{Q,s}(F : \epsilon) \in C^\infty(U_F(\epsilon) \times \mathfrak{a}_F^* \times L_Q^*)$.

Recall that for any $F \subseteq F_0, \epsilon \in \Sigma_0$, we can write as in [**H3**, 5.3]

$$\int_{\mathfrak{a}_F^*} \Phi(F : \epsilon : h : \nu : x) m(H_F : \lambda_F : \chi_F : h : \nu) d\nu$$

$$= (\pi i)^{-|F|} \int_{\mathfrak{a}_F^*} \frac{g(F:\epsilon:h:\nu:x)}{p_F(h:\nu)} d\nu$$

where $g(F:\epsilon) \in J_F^0(U:\epsilon:G)$ is defined as in (2.22). In order to carry out the estimates needed to prove that the elementary mixed wave packet defined by (5.1) is a Schwartz function we will need to know that $g(F:\epsilon) \in J_F^1(U:\epsilon:G)$.

Theorem 5.2. *For any $F \subseteq F_0, \epsilon \in \Sigma_0$,*

$$(h,\nu,x) \to g_{Q,s}(F:\epsilon:h:\nu:x) = \pi(F:h:\nu)^{-1}\psi_{g(F:\epsilon),s}(h:\nu:x)$$

extends to a C^∞ function on $U_F(\epsilon) \times \mathfrak{a}_F^* \times G$. That is, $g(F:\epsilon) \in J_F^1(U:\epsilon:G)$.

The remainder of this section is devoted to the proof of (5.2). Recall from (2.22) that for fixed F, ϵ there is a constant c so that

(5.3a)
$$g(F:\epsilon:h:\nu:x) = c\pi(F:h:\nu)\Phi(F:\epsilon:h:\nu:x)q(F:h:\nu) \cdot \prod_{\alpha\in\Phi'_{F,R}} m_\alpha(F:h:\nu) \sum_{\psi\in\mathcal{T}_F} \epsilon(\psi)t(F:\psi:h:\nu).$$

Thus

(5.3 b)
$$g_{Q,s}(F:\epsilon:h:\nu:x) = \widetilde{\Phi}_{Q,s}(F:\epsilon:h:\nu:x)g'(F:h:\nu)$$

where
(5.3 c)
$$g'(F:h:\nu) = cq(F:h:\nu) \prod_{\alpha\in\Phi'_{F,R}} m_\alpha(F:h:\nu) \sum_{\psi\in\mathcal{T}_F} \epsilon(\psi)t(F:\psi:h:\nu).$$

Now it is proven in [**H3**, 5.8; 5.9], that

$$(h,\nu) \to \pi(F:h:\nu)g'(F:h:\nu)$$

is jointly smooth on $U(0) \times \mathfrak{a}_F^*$. Recall that

$$\pi(F:h:\nu) = \prod_{\alpha\in\Phi_F^+} \pi_\alpha(F:h:\nu)$$

where

$$\pi_\alpha(F:h:\nu) = \langle\alpha, \lambda_F(h) + i\nu\rangle, (h,\nu) \in i\mathfrak{b}^* \times \mathfrak{a}_F^*.$$

In order to prove Theorem 5.2 we will show that there is a subset $\Phi_{F,0}$ of Φ_F^+ so that if we define

$$\pi_0(F : h : \nu) = \prod_{\alpha \in \Phi_{F,0}} \pi_\alpha(F : h : \nu),$$

then

$$(h, \nu) \to \pi_0(F : h : \nu) g'(F : h : \nu)$$

and

$$(h, \nu, x) \to \pi_0(F : h : \nu)^{-1} \widetilde{\Phi}_{Q,s}(F : \epsilon : h : \nu : x)$$

are both jointly smooth.

The subset $\Phi_{F,0}$ needed consists of three types of roots. We will use the notation of [**H3**, §4; §5]. Let $\Phi_{F,R}^+$ denote the real roots in Φ_F^+. For every $\alpha \in \Phi_{F,R}^+$, we have the Plancherel factor

$$m_\alpha^*(F : h : \nu) = \prod_{\beta \in \Phi_\alpha^+} \pi_\beta(F : h : \nu) \frac{\sinh \pi\nu_\alpha}{\cosh \pi\nu_\alpha - \epsilon_\alpha(F : h)}$$

defined as in [**H3**, 4.5]. First, define Φ_F^1 to be the set of all $\alpha \in \Phi_{F,R}^+$ such that $\epsilon_\alpha(F : h) = 1$ for all h. Second, suppose $\alpha \in \Phi_{F,R}^+ \backslash \Phi_F^1$ such that $\epsilon_\alpha(F : 0) = 1$. Then $\epsilon_\alpha(F : h) = \cos \pi h_\alpha$ as in [**H1**, 10.4]. Suppose further that $m_\alpha^*(F : h : \nu)$ is jointly smooth at $(0, 0)$. Then as in [**H3**, 4.7] there are $\gamma, \gamma' \in \Phi_\alpha^+$ and non-zero constants c, c' so that

$$\pi_\gamma(F : h : \nu) = c(h_\alpha + i\nu_\alpha), \quad \pi_{\gamma'}(F : h : \nu) = c'(h_\alpha - i\nu_\alpha).$$

Define Φ_F^2 to be the set of all γ, γ' obtained in this way. (There could be more that one pair $\{\gamma, \gamma'\}$ which satisfy this condition for a given $\alpha \in \Phi_{F,R}^+$. The set Φ_F^2 should consist of just one pair for each α.) Finally, suppose in the notation of [**H3**, 4.12] that $\alpha \in F, \beta \in [\alpha], \Phi(\alpha) \neq \Phi(\beta)$. Define $\gamma, \bar{\gamma}$ as in [**H3**, 4.12]. Then as in [**H3**, 5.8], $\pi_\gamma(F : h : \nu)$ is a multiple of $\langle i\nu, \beta - \alpha \rangle$. Let Φ_F^3 be the set of all $\gamma \in \Phi_F$ of this form. (We choose only one of each pair $\{\gamma, \bar{\gamma}\}$ for Φ_F^3.) Now we define

(5.4a) $$\Phi_{F,0} = \Phi_F^1 \cup \Phi_F^2 \cup \Phi_F^3$$

and set

(5.4b) $$\pi_0(F : h : \nu) = \prod_{\alpha \in \Phi_{F,0}} \pi_\alpha(F : h : \nu).$$

Lemma 5.5. *Let $U(0)$ be a neighborhood of 0 in $i\mathfrak{v}^*$ defined as in* [**H3**, 4.6]. *Define $g'(F:h:\nu)$ as in* (5.3c). *Then*

$$(h,\nu) \to \pi_0(F:h:\nu)g'(F:h:\nu)$$

is jointly smooth on $U(0) \times \mathfrak{a}_F^*$.

Proof. It is proven in [**H3**, 5.8; 5.9], that

$$(h,\nu) \to \pi(F:h:\nu)q(F:h:\nu) \prod_{\alpha \in \Phi'_{F,R}} m_\alpha(F:h:\nu) \sum_{\psi \in \mathcal{T}_F} \epsilon(\psi)t(F:\psi:h:\nu)$$

is jointly smooth on $U(0) \times \mathfrak{a}_F^*$. In fact, from the proof of [**H3**, 5.8; 5.9] we see the following.

First, as in [**H3**, 5.9], if $\alpha \in \Phi'_{F,R}$, then

$$m_\alpha^*(F:h:\nu) = \prod_{\beta \in \Phi_\alpha^+} \pi_\beta(F:h:\nu)m_\alpha(F:h:\nu)$$

is jointly smooth in $U(0) \times \mathfrak{a}_F^*$. Now

$$m_\alpha(F:h:\nu) = \frac{\sinh \pi\nu_\alpha}{\cosh \pi\nu_\alpha - \epsilon_\alpha(F:h)}$$

is jointly smooth unless $\epsilon_\alpha(F:0) = 1$. Now if $\epsilon_\alpha(F:h) = 1$ for all h, then $\alpha \in \Phi_F^1$ and $\pi_\alpha(F:h:\nu)m_\alpha(F:h:\nu)$ is jointly smooth. If $\epsilon_\alpha(F:0) = 1$, but $\epsilon_\alpha(F:h)$ is not identically one, then there are $\gamma, \gamma' \in \Phi_F^2 \cap \Phi_\alpha^+$ so that

$$\pi_\gamma(F:h:\nu)\pi_{\gamma'}(F:h:\nu)m_\alpha(F:h:\nu)$$

is jointly smooth for $(h,\nu) \in U(0) \times \mathfrak{a}_F^*$.

Second, as in [**H3**, 5.8],

$$(h,\nu) \to \prod_{\alpha \in \Phi''_{F,R}(1) \backslash \Phi^+_{F,R}(0)} (\nu_\alpha + ih_\alpha) \prod_{\alpha \in F} \prod_{\beta \in [\alpha], \Phi(\beta) \neq \Phi(\alpha)} (\nu_\beta - \nu_\alpha)q(F:h:\nu)$$
$$\cdot \sum_{\psi \in \mathcal{T}_F} \epsilon(\psi)t(F:\psi:h:\nu)$$

is jointly smooth on $U(0) \times \mathfrak{a}_F^*$. Now for $\alpha \in \Phi''_{F,R}(1) \backslash \Phi^+_{F,R}(0)$, if $\epsilon_\alpha(F:h) = 1$ for all h, then $\alpha \in \Phi_F^1$ and $\nu_\alpha + ih_\alpha$ is a non-zero multiple of $\pi_\alpha(F:h:\nu)$. If $\epsilon_\alpha(F:h)$ is not identically 1, then as above there is $\gamma \in \Phi_F^2 \cap \Phi_\alpha^+$ so that $\pi_\gamma(F:h:\nu)$ is a non-zero multiple of $\nu_\alpha + ih_\alpha$. Further, for $\alpha \in F$ and $\beta \in [\alpha]$ with $\Phi(\beta) \neq \Phi(\alpha)$, there is $\gamma \in \Phi_F^3$ so that $\pi_\gamma(F:h:\nu)$ is a non-zero multiple of $\nu_\beta - \nu_\alpha$. □

Lemma 5.6. *Let $F \subseteq F_0, \epsilon \in \Sigma_0$. Then*

$$(h, \nu, x) \to \pi_0(F : h : \nu)^{-1}\widetilde{\Phi}_{Q,s}(F : \epsilon : h : \nu : x)$$

extends to a C^∞ function on $U_F(\epsilon) \times \mathfrak{a}_F^* \times L_Q^*$.

Proof. By definition, there are finitely many

$$\Psi_i \in \mathcal{S}\left(M_F^\dagger : \lambda_F : \chi_F : \mathcal{D}_F(\epsilon) : W\right), \alpha_i \in \mathcal{C}(\mathcal{D}_F(\epsilon) \times \mathfrak{a}_F^*)_0$$

so that

$$\Phi(F : \epsilon : h : \nu : x) = \sum_i \alpha_i(h : \nu)E(P_F : \Psi_i : h : \nu : x).$$

Thus it suffices to prove that each

$$\pi_0(F : h : \nu)^{-1}\widetilde{E}_{Q,s}(P_F : \Psi_i : h : \nu : x)$$

is jointly smooth. This follows from Theorem 5.7 below. □

Theorem 5.7. *Let $F \subseteq F_0, \epsilon \in \Sigma_0, \Psi \in \mathcal{S}\left(M_F^\dagger : \lambda_F : \chi_F : \mathcal{D}_F(\epsilon) : W\right)$. Let $\alpha \in \Phi_{F,0}$. Then*

$$(h, \nu, x) \to \pi_\alpha(F : h : \nu)^{-1}\widetilde{E}_{Q,s}(P_F : \Psi : h : \nu : x)$$

is jointly smooth on $U_F(\epsilon) \times \mathfrak{a}_F^* \times L_Q^*$.

The remainder of this section is devoted to the proof of Theorem 5.7. Recall $\Phi_{F,0} = \Phi_F^1 \cup \Phi_F^2 \cup \Phi_F^3$. Now for roots $\alpha \in \Phi_F^1 \cup \Phi_F^3$, $\pi_\alpha(F : h : \nu) = \langle i\nu, \alpha\rangle$ is independent of h, and showing that $\pi_\alpha(F : h : \nu)^{-1}\widetilde{E}_{Q,s}(P_F : \Psi : h : \nu : x)$ extends to a smooth function will use Harish-Chandra's theory of the c-function when $\widetilde{E}_{Q,s}$ is an ordinary constant term. For roots $\alpha \in \Phi_F^2, \pi_\alpha(F : h : \nu)$ depends on both h and ν and showing that $\pi_\alpha(F : h : \nu)^{-1}\widetilde{E}_{Q,s}(P_F : \Psi : h : \nu : x)$ extends to a smooth function will use results from [**H2**] and §4 in the case that $\widetilde{E}_{Q,s}$ is an ordinary constant term. In both cases, we will use matching conditions to extend the results from ordinary constant terms to all constant terms. Thus the first step is to show that any holomorphic family of Eisenstein integrals is contained in a matching family of holomorphic families of Eisenstein integrals. We will do this first in the case that $P = M = G$ so that $E(P : \Psi) = \Psi$.

Assume for now that G is a connected reductive group with rank $G =$ rank K and that $H = T$ is a relatively compact Cartan subgroup of G. Fix $(\lambda, \chi) \in X(T)$. Define $F_0 = \{\alpha \in \Phi_G^+ : \langle\lambda, \alpha\rangle = 0\}$. Then as in (2.10) we define $H_F, (\lambda_F, \chi_F) \in X(T_F), F \subseteq F_0, \Sigma_0$, and the corresponding family

$\{\Theta(H_F : h : \nu_F)\}_{F\subseteq F_0}$ of matching characters. For $\epsilon \in \Sigma_0$, let $\widetilde{\Theta}(H_F : \epsilon)$ denote the restriction of $\sigma_F(\epsilon)\Theta(H_F)$ to $\mathcal{D}_F(\epsilon) \times \mathfrak{a}_F^* \times G$. It is clear from the character formulas [**HW1**, 2.10, **HW4**, 2.6] that for each $x \in G'$, $(h,\nu) \to \widetilde{\Theta}(H_F : \epsilon : h : \nu : x)$ extends to a holomorphic function on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$.

Suppose for each $F \subseteq F_0, \epsilon \subseteq \Sigma_0$, we have a holomorphic function $f(F : \epsilon : h : \nu)$ of $(h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$. Then we say the collection $\{f(F : \epsilon)\}$ satisfies the same matching conditions as the characters if, in the notation of (2.14), we have for any $1 \le i \le m, \epsilon \in \Sigma_i, E \subseteq F_0, k \ge 0$,

(5.8)
$$(\partial/\partial h_i)^k[f(E : \epsilon^+(i) : h_0 : \nu_E) - f(E : \epsilon^-(i) : h_0 : \nu_E)]$$
$$= \sum_{E\subset F\subseteq E(i)} c_{|F\setminus E|} D_{F\setminus E}^k [f(F : \epsilon^+(i) : h_0 : (\nu_E, 0)) + f(F : \epsilon^-(i) : h_0 : (\nu_E, 0))]$$

for all $\nu_E \in \mathfrak{a}_E^*, h_0 \in \mathcal{H}_i \cap \mathrm{cl}(\mathcal{D}_E(\epsilon))$. Here $D_{F\setminus E} = \partial/\partial h_i - i\sum_{\alpha\in F\setminus E} \partial/\partial\mu_\alpha$.

For each $F \subseteq F_0, \epsilon \in \Sigma_0, (h,\nu) \in \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^*$, let

$$(\pi(F : \epsilon : h : \nu), \mathcal{H}(F : \epsilon : h : \nu)) \tag{5.9 a}$$

denote the representation with character $\Theta(H_F : \epsilon : h : \nu)$ and the space on which it acts. For $\tau \in \widehat{K}(\chi)$, let

$$\mathcal{H}(F : \epsilon : \tau : h : \nu) \tag{5.9 b}$$

denote the τ_h-isotypic subspace of $\mathcal{H}(F : \epsilon : h : \nu)$. For $\tau_1, \tau_2 \in \widehat{K}(\chi)$, let

$$V(F : \epsilon : \tau_1 : \tau_2 : h : \nu) \tag{5.9 c}$$

be the linear span in $C^\infty(G)$ of the matrix coefficients

$$\langle \pi(F : \epsilon : h : \nu)(x)w_2, w_1\rangle, w_j \in \mathcal{H}(F : \epsilon : \tau_j : h : \nu),$$

and for any open subset U of $\mathfrak{v}_{\mathbf{C}}^*$ let

$$\mathcal{F}'(F : \epsilon : U : \tau_1 : \tau_2) \tag{5.9 d}$$

be the set of all functions $f \in C^\infty(U \times \mathfrak{a}_{F,\mathbf{C}}^* \times G)$ such that $(h,\nu) \to f(h : \nu : x)$ is holomorphic on $U \times \mathfrak{a}_{F,\mathbf{C}}^*$ for all $x \in G$ and $f(h : \nu) \in V(F : \epsilon : \tau_1 : \tau_2 : h : \nu)$ when $(h,\nu) \in \mathrm{cl}(\mathcal{D}_F(\epsilon)) \cap U \times \mathfrak{a}_F^*$. We call elements f of $\mathcal{F}'(F : \epsilon : U : \tau_1 : \tau_2)$ holomorphic families of matrix coefficients.

Lemma 5.10. *Let U be a neighborhood of 0 in $\mathfrak{v}_{\mathbf{C}}^*, \epsilon_0 \in \Sigma_0$, and let $\phi \in \mathcal{F}'(\emptyset : \epsilon_0 : U : \tau_1 : \tau_2)$. Then there are a neighborhood U' of 0 in U and for each $F \subseteq F_0, \epsilon \in \Sigma_0$, a function $\phi(F : \epsilon) \in \mathcal{F}'(F : \epsilon : U' : \tau_1 : \tau_2)$ so* *that $\phi(\emptyset : \epsilon_0) = \phi$ and the collection $\{\phi(F : \epsilon)\}$ satisfies the same matching conditions as the characters for $h \in U'$.*

Proof. We will carry out the construction of [**HW4**, §3; §5] simultaneously for the matching family of characters $\Theta(H_F : \epsilon), F \subseteq F_0, \epsilon \in \Sigma_0$.

Fix $\tau \in \widehat{K}(\chi)$ and for $(h,k) \in \mathfrak{v}_{\mathbf{C}}^* \times K$ define

$$\delta(h : k) = e^{-h}(k)\,\mathrm{degree}(\tau)\,\mathrm{trace}\,\tau^*(k)$$

where τ^* is the contragredient of τ. For each $f \in C_c^\infty(G), x \in G$, we can define

$$\delta(h) *_K f(x) = \int_K \delta(h : k) f(k^{-1}x)dk.$$

Now for each $F \subseteq F_0, \epsilon \in \Sigma_0$ we define

$$\widetilde{\Theta}(H_F : \epsilon : \tau : h : \nu : f) = \int_{G/Z_G} \widetilde{\Theta}(H_F : \epsilon : h : \nu : x)(\delta(h) *_K f)(x)d(xZ_G), f \in C_c^\infty(G).$$

As in [**HW4**, 3.7] we see that $\widetilde{\Theta}(H_F : \epsilon : \tau : f)$ is holomorphic on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$ and that for each $(h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$,

$$f \to \widetilde{\Theta}(H_F : \epsilon : \tau : h : \nu : f)$$

defines a distribution on G. As in [**HW4**, 3.7], we can differentiate under the integrals, so that for every $f \in C_c^\infty(G), \left\{\widetilde{\Theta}(H_F : \epsilon : \tau : f)\right\}$ satisfies the same matching conditions as the characters.

Now as in [**HW4**, 3.11] there are real analytic functions $\widetilde{T}(H_F : \epsilon : \tau)$ on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^* \times G$ so that for every $f \in C_c^\infty(G), (h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$,

$$\widetilde{\Theta}(H_F : \epsilon : \tau : h : \nu : f) = \int_G \widetilde{T}(H_F : \epsilon : \tau : h : \nu : x)f(x)dx.$$

As in [**HW4**, 3.12] we can differentiate under the integral and see that for each $x \in G$ the functions $\widetilde{T}(H_F : \epsilon : \tau : x)$ are holomorphic on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$ and that the collection

$$\left\{\widetilde{T}(H_F : \epsilon : \tau : x)\right\}_{F \subseteq F_0, \epsilon \in \Sigma_0}$$

satisfies the same matching conditions as the characters.

Now fix $\epsilon_0 \in \Sigma_0$ and let τ_0 be the lowest K-type of the relative discrete series (or limit of relative discrete series) representation of G with character $\Theta(H_\emptyset : \epsilon_0 : 0)$. Let $\mu \in i\mathfrak{t}^*$ be the highest weight of τ_0. For each $F \subseteq F_0, \epsilon \in \Sigma_0, (h,\nu,x) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^* \times G$ we define

$$\psi(H_F : \epsilon : h : \nu : x) = \int_{T/Z_G} \widetilde{T}(H_F : \epsilon : \tau_0 : h : \nu : tx)e^{-\mu-h}(t)d(tZ).$$

Again, differentiating under the integrals, we see that for each $x \in G$ the functions $\psi(H_F : \epsilon : x)$ are holomorphic on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$ and that the collection

$$\{\psi(H_F : \epsilon : x)\}_{F \subseteq F_0, \epsilon \in \Sigma_0}$$

satisfies the same matching conditions as the characters.

Finally, for $D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}}), \tau_1, \tau_2 \in \widehat{K}(\chi)$, we define

$$\begin{aligned}&\psi(H_F : \epsilon : D_1 : D_2 : \tau_1 : \tau_2 : h : \nu : x)\\&= \delta(\tau_1^* : h) *_{K/Z} \psi(H_F : \epsilon : h : \nu : D_1^* ; x; D_2) *_{K/Z} \delta(\tau_2 : h).\end{aligned}$$

As in [**HW4**, 5.12] we see that for each $x \in G$ the functions $\psi(H_F : \epsilon : D_1 : D_2 : \tau_1 : \tau_2 : x)$ are holomorphic on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$ and that the collection

$$\{\psi(H_F : \epsilon : D_1 : D_2 : \tau_1 : \tau_2 : x)\}_{F \subseteq F_0, \epsilon \in \Sigma_0}$$

satisfies the same matching conditions as the characters. Further, as in [**HW4**, 5.10; 5.12] we see that for each $F \subseteq F_0, \epsilon \in \Sigma_0, D_1, D_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$, $\tau_1, \tau_2 \in \widehat{K}(\chi)$,

$$\psi(H_F : \epsilon : D_1 : D_2 : \tau_1 : \tau_2) \in \mathcal{F}'(F : \epsilon : \mathfrak{v}_{\mathbf{C}}^* : \tau_1 : \tau_2).$$

Now, as in [**HW4**, 5.14], there are a neighborhood J of 0 in $\mathrm{cl}(\mathcal{D}_{\emptyset}(\epsilon_0))$ and $D_i, D_j' \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$, so that $\left\{\psi\left(T : \epsilon_0 : D_i : D_j' : \tau_1 : \tau_2 : h\right)\right\}$ is a basis of $V(\emptyset : \epsilon_0 : \tau_1 : \tau_2 : h)$ for all $h \in J$. In fact the argument shows that there is a neighborhood W of 0 in $\mathfrak{v}_{\mathbf{C}}^*$ so that the collection

$$\left\{\psi\Big(T : \epsilon_0 : D_i : D_j' : \tau_1 : \tau_2 : h\Big)\right\}$$

is linearly independent for all $h \in W$. Let $U' = U \cap W$. As in [**HW4**, 5.17], when we expand $\phi \in \mathcal{F}'(\emptyset : \epsilon_0 : \tau_1 : \tau_2)$ in terms of this basis as

$$\phi(h) = \sum_{i,j} \beta_{i,j}(h) \psi\left(T : \epsilon_0 : D_i : D_j' : \tau_1 : \tau_2 : h\right),$$

the coefficient functions $\beta_{i,j}(h)$ are holomorphic on U'. Now for any $F \subseteq F_0, \epsilon \in \Sigma_0, (h, \nu) \in U' \times \mathfrak{a}_{F,\mathbf{C}}^*$, set

$$\phi(F : \epsilon : h : \nu) = \sum_{i,j} \beta_{i,j}(h) \psi\left(H_F : \epsilon : D_i : D_j' : \tau_1 : \tau_2 : h : \nu\right).$$

□

Now we return to the general case that G is an arbitrary connected reductive Lie group and $H = TA$ is any θ-stable Cartan subgroup. Let $(\lambda, \chi) \in X(T), \tau_1, \tau_2 \in \widehat{K}(\chi), W = W(\tau_1 : \tau_2)$. Let $\mathcal{D}$ be a connected component of $\{h \in \mathfrak{v}^* : \langle \lambda(h), \alpha \rangle \neq 0$ for all $\alpha \in \Phi_M\}$ such that $0 \in \mathrm{cl}(\mathcal{D})$. Define $F_0, H_F, (\lambda_F, \chi_F), F \subseteq F_0, \Sigma_0$, as in (2.10) and fix $\epsilon_0 \in \Sigma_0$ so that $\mathcal{D}_\emptyset(\epsilon_0) = \mathcal{D}$. Recall that for $(h, \nu) \in \mathcal{D}_F(\epsilon) \times \mathfrak{a}_F^*, \Theta(H_F : \epsilon : h : \nu)$ is the character of the representation

(5.11 a) $$\pi(F : \epsilon : h : \nu) = \mathrm{Ind}_{Z_{M_F}(M_F^0)M_F^0 A_F N_F}^G \left(\chi_F(h) \otimes \pi^0_{\lambda_F(h)} \otimes e^{i\nu} \otimes 1\right).$$

Let $P_F^\dagger = P_F \cap M^\dagger = M_F^\dagger A_{F,M} N_{F,M}$ where $A_{F,M} = A_F \cap M^0, N_{F,M} = N_F \cap M^0$. As in [**H1**, 10.16], using induction by stages, for $\nu \in \mathfrak{a}_F^*$ with $\nu_1 = \nu|_{\mathfrak{a}_{F,M}}, \nu_2 = \nu|_{\mathfrak{a}}$, then

(5.11 b) $$\pi(F : \epsilon : h : \nu) = \mathrm{Ind}_{M^\dagger AN}^G \left(\pi\left(M^\dagger : F : \epsilon : h : \nu_1\right) \otimes e^{i\nu_2} \otimes 1\right)$$

where

(5.11 c)
$$\pi\left(M^\dagger : F : \epsilon : h : \nu_1\right) = \mathrm{Ind}_{Z_{M_F}(M_F^0)M_F^0 A_{F,M} N_{F,M}}^{M^\dagger} \left(\chi_F(h) \otimes \pi^0_{\lambda_F(h)} \otimes e^{i\nu_1} \otimes 1\right).$$

Now the characters $\Theta\left(M^\dagger : F : \epsilon : h : \nu_1\right)$ of the representations $\pi(M^\dagger : F : \epsilon : h : \nu_1)$ satisfy the same matching conditions as the characters $\Theta(F : \epsilon : h : \nu)$ if we extend $\Theta\left(M^\dagger : F : \epsilon : h : \nu_1\right)$ to $\nu \in \mathfrak{a}_{F,\mathbf{C}}^*$ by $\Theta\left(M^\dagger : F : \epsilon : h : \nu\right) = \Theta\left(M^\dagger : F : \epsilon : h : \nu_1\right), \nu_1 = \nu|_{\mathfrak{a}_{F,M}}$ as above. Thus we will not distinguish between the matching conditions satisfied by the characters corresponding to $M^\dagger$ and G.

For each $F \subseteq F_0, \epsilon \in \Sigma_0$, and neighborhood U of 0 in $\mathfrak{v}_{\mathbf{C}}^*$ we can define

(5.12 a) $$\mathcal{S}\left(M^\dagger : F : \epsilon : U : W\right)$$

to be the set of all $\Psi \in C^\infty\left(U \times \mathfrak{a}_{F,M,\mathbf{C}}^* \times M^\dagger : W\right)$ such that

(5.12 b) $$\Psi(h : \nu : k_1 x k_2) = \tau_{1,h}(k_1)\Psi(h : \nu : x)\tau_{2,h}(k_2)$$

for all $(h, \nu) \in U \times \mathfrak{a}_{F,M,\mathbf{C}}^*, x \in M^\dagger, k_1, k_2 \in K_M^\dagger$,

(5.12 c) $$(h, \nu) \to \Psi(h : \nu : x) \text{ is holomorphic on } U \times \mathfrak{a}_{F,M,\mathbf{C}}^*$$

for all $x \in M^\dagger$, and

(5.12 d) $$x \to \langle \Psi(h : \nu : x), w^* \rangle$$

is a finite sum of matrix coefficients of $\pi\left(M^\dagger : F : \epsilon : h : \nu\right)$ for all $(h, \nu) \in U \cap \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_{F,M}^*$. We will not require any growth condition on these spaces since we will only need to use these holomorphic families to study Eisenstein integrals in a neighborhood of some fixed point.

Lemma 5.13. *Let U be a neighborhood of 0 in $\mathfrak{v}_{\mathbf{C}}^*$ and let $\Psi \in \mathcal{S}(M^\dagger : \emptyset : \epsilon_0 : U : W)$. Then there are a neighborhood U' of 0 in U and for each $F \subseteq F_0, \epsilon \in \Sigma_0$, a function $\Psi(F : \epsilon) \in \mathcal{S}\left(M^\dagger : F : \epsilon : U' : W\right)$ so that $\Psi(\emptyset : \epsilon_0) = \Psi|_{U' \times M^\dagger}$ and the collection $\{\Psi(F : \epsilon)\}$ satisfies the same matching conditions as the characters for $h \in U'$.*

Proof. Write $K_M^0 = K \cap M^0, K_M^\dagger = K \cap M^\dagger$. For $F \subseteq F_0, \epsilon \in \Sigma_0$, let $\pi\left(M^0 : F : \epsilon : h : \nu_1\right)$ be the representation of M^0 so that

$$\pi\left(M^\dagger : F : \epsilon : h : \nu_1\right) = \chi(h) \otimes \pi\left(M^0 : F : \epsilon : h : \nu_1\right).$$

For U a neighborhood of 0 in $\mathfrak{v}_{\mathbf{C}}^*$, write $\mathcal{F}'\left(M^0 : F : \epsilon : U : \tau_1 : \tau_2\right)$ for the holomorphic families of matrix coefficients of the representations $\pi(M^0 : F : \epsilon)$ defined as above with K_M^0-types $\tau_i|_{K_M^0}$. Similarly we could define holomorphic families $\mathcal{F}'\left(M^\dagger : F : \epsilon : U : \tau_1 : \tau_2\right)$ of matrix coefficients of the representations $\pi\left(M^\dagger : F : \epsilon : h : \nu_1\right)$ of $M^\dagger$. (See [**H1**, §5].) Now every $\phi \in \mathcal{F}\left(M^\dagger : \emptyset : \epsilon_0 : U : \tau_1 : \tau_2\right)$ is a finite sum of terms of the form

$$\phi(h : zm) = \psi(h : z)\phi^0(h : m), z \in Z_M(M^0), m \in M^0$$

where $\psi(0)$ is a matrix coefficient of $\chi(0), \psi(h) = \psi(0) \otimes e^h, h \in \mathfrak{v}_{\mathbf{C}}^*$, and $\phi^0 \in \mathcal{F}(M^0 : \emptyset : \epsilon_0 : U : \tau_1 : \tau_2)$. Now if ϕ^0 is embedded in a matching family $\{\phi^0(F : \epsilon)\}$ of elements of $\mathcal{F}(M^0 : F : \epsilon : U' : \tau_1 : \tau_2)$ as in (5.10), $\phi = \psi \cdot \phi^0$ can be embedded in the matching family $\{\phi(F : \epsilon)\}$ defined by

$$\phi(F : \epsilon : h : zm) = \psi(h : z)\phi^0(F : \epsilon : h : m), z \in Z_M\left(M^0\right), m \in M^0.$$

Thus every $\phi \in \mathcal{F}\left(M^\dagger : \emptyset : \epsilon_0 : U : \tau_1 : \tau_2\right)$ can be embedded in a matching family.

Now corresponding to any $\phi \in \mathcal{F}\left(M^\dagger : F : \epsilon : U' : \tau_1 : \tau_2\right)$, the procedure in [**HW5**, §5] gives a canonical way of constructing a spherical function $F(\phi) \in \mathcal{S}\left(M^\dagger : F : \epsilon : U' : W\right)$. If a family $\{\phi(F : \epsilon)\}$ satifies matching conditions, so will the corresponding functions $F(\phi(F : \epsilon))$. Finally, for any $S \in \mathrm{End}_{K_M^\dagger}(W), \phi \in \mathcal{F}\left(M^\dagger : F : \epsilon : U' : \tau_1 : \tau_2\right)$, we can define

$$(SF(\phi))(h : \nu : x) = S(F(\phi)(h : \nu : x)), (h, \nu, x) \in U' \times \mathfrak{a}_{F,M,\mathbf{C}}^* \times M^\dagger.$$

Then $SF(\phi) \in \mathcal{S}(M^\dagger : F : \epsilon : U' : W)$, and again, if a family $\{\phi(F : \epsilon)\}$ satifies matching conditions, so will the corresponding family $\{SF(\phi(F : \epsilon))\}$.

Now using the argument of [**H1**, §5], given $\Psi \in \mathcal{S}(M^\dagger : \emptyset : \epsilon_0 : U : W)$ there are a neighborhood U'' of 0 in U' and elements $\phi_i \in \mathcal{F}(M^\dagger : \emptyset : \epsilon_0 : U'' : \tau_1 : \tau_2), S_i \in \mathrm{End}_{K_M^\dagger}(W)$, so that

$$\Psi(h : x) = \sum_i S_i F(\phi_i)(h : x), (h, x) \in U'' \times M^\dagger.$$

Now embedding each ϕ_i in a matching family $\phi_i(F : \epsilon)$, we can embed Ψ in the matching family given by

$$\Psi(F : \epsilon : h : \nu : x) = \sum_i S_i F(\phi_i(F : \epsilon))(h : \nu : x).$$

□

For each $F \subseteq F_0, \epsilon \in \Sigma_0$, and neighborhood U of 0 in $\mathfrak{v}_{\mathbf{C}}^*$ we can define

$$\mathcal{S}(G : F : \epsilon : U : W) \tag{5.14 a}$$

to be the set of all $\Phi \in C^\infty(U \times \mathfrak{a}_{F,\mathbf{C}}^* \times G : W)$ such that

$$\Phi(h : \nu : k_1 x k_2) = \tau_{1,h}(k_1)\Phi(h : \nu : x)\tau_{2,h}(k_2) \tag{5.14 b}$$

for all $(h, \nu) \in U \times \mathfrak{a}_{F,\mathbf{C}}^*, x \in G, k_1, k_2 \in K$,

$$(h, \nu) \to \Phi(h : \nu : x) \quad \text{is holomorphic on } U \times \mathfrak{a}_{F,\mathbf{C}}^* \tag{5.14 c}$$

for all $x \in G$, and

$$x \to \langle \Phi(h : \nu : x), w^* \rangle \tag{5.14 d}$$

is a finite sum of matrix coefficients of $\pi(F : \epsilon : h : \nu)$ for all $(h, \nu) \in U \cap \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^*$. For each $\Psi \in \mathcal{S}\,(M^\dagger : F : \epsilon : U : W)$, we extend Ψ from $U \times \mathfrak{a}_{F,M,\mathbf{C}}^* \times M^\dagger$ to $U \times \mathfrak{a}_{F,M,\mathbf{C}}^* \times G$ by

(5.15 a)

$$\Psi(h : \nu : kman) = \tau_{1,h}(k)\Psi(h : \nu : m), k \in K, m \in M^\dagger, a \in A, n \in N.$$

Then we define

(5.15 b)

$$E(P : \Psi : h : \nu : x) = \int_{K/Z_G} \Psi(h : \nu_1 : xk)\tau_{2,h}(k^{-1})e^{(i\nu_2 - \rho_P)H_P(xk)} d(kZ_G),$$

where for $\nu \in \mathfrak{a}_{F,\mathbf{C}}^*, \nu_1 = \nu|_{\mathfrak{a}_{F,M}}$ and $\nu_2 = \nu|_{\mathfrak{a}}$.

Lemma 5.16. *Let $F \subseteq F_0, \epsilon \in \Sigma_0$. For each $\Psi \in \mathcal{S}(M^\dagger : F : \epsilon : U : W), E(P : \Psi) \in \mathcal{S}(G : F : \epsilon : U : W)$. Further, there are a neighborhood U'*

of 0 *in* U *and finitely many* $\Psi_i \in \mathcal{S}\left(M_F^\dagger : \lambda_F : \chi_F : \mathcal{D}_F(\epsilon) : W\right)$ *so for every* $\Phi \in \mathcal{S}(G : F : \epsilon : U' : W), (h, \nu, x) \in U' \cap \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^* \times G$,

$$\Phi(h : \nu : x) = \sum_i \alpha_i(h : \nu) E(P_F : \Psi_i : h : \nu : x)$$

where $\alpha_i(h : \nu) \in \mathbf{C}$ *for all* $(h, \nu) \in U' \cap \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^*$.

Proof. As in [**H1**, §4] we see that if σ is a unitary representation of $M^\dagger$ and Ψ is any $\left(\tau_{1,h}|_{K_M^\dagger}, \tau_{2,h}|_{K_M^\dagger}\right)$-spherical function of matrix coefficients of σ, then $E(P : \Psi : h : \nu_2)$ is a $(\tau_{1,h}, \tau_{2,h})$-spherical function of matrix coefficients of $\mathrm{Ind}_{M^\dagger AN}^G(\sigma \otimes e^{i\nu_2} \otimes 1)$ for all $\nu_2 \in \mathfrak{a}^*$. The first part of the lemma follows since we know that

$$\pi(F : \epsilon : h : \nu) = \mathrm{Ind}_{M^\dagger AN}^G(\pi\left(M^\dagger : F : \epsilon : h : \nu_1\right) \otimes e^{i\nu_2} \otimes 1).$$

As in [**H1**, §5] we know that there are a neighborhood U' of 0 in U and finitely many $\Psi_i \in \mathcal{S}\left(M_F^\dagger : \lambda_F : \chi_F : \mathcal{D}_F(\epsilon) : W\right)$ so that for all $h \in U' \cap \mathrm{cl}(\mathcal{D}_F(\epsilon)), \{\Psi_i(h)\}$ is a basis for the space of $(\tau_{1,h}, \tau_{2,h})$-spherical functions of matrix coefficients of the relative discrete series representation $\chi_F(h) \otimes \pi_{\lambda_F}^0(h)$ of $M_F^\dagger$. Now the Eisenstein integrals $E(P_F : \Psi_i : h : \nu : x)$ can be defined as in (5.15) relative to the parabolic subgroup P_F, and for each $(h, \nu) \in U' \cap \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^*$, they will span the space of $(\tau_{1,h}, \tau_{2,h})$-spherical functions of matrix coefficients of the representations $\pi(F : \epsilon : h : \nu)$. Thus, given $\Phi \in \mathcal{S}(G : F : \epsilon : U' : W)$ there are complex numbers $\alpha_i(h : \nu)$ so that for all $(h, \nu_1, x) \in U' \cap \mathrm{cl}(\mathcal{D}_F(\epsilon)) \times \mathfrak{a}_F^* \times G$,

$$\Phi(h : \nu : x) = \sum_i \alpha_i(h : \nu) E(P_F : \Psi_i : h : \nu : x).$$

□

Lemma 5.17. *Let* U *be a neighborhood of* 0 *in* $\mathfrak{v}_{\mathbf{C}}^*$ *and let* $\Psi \in \mathcal{S}(M^\dagger : \emptyset : \epsilon_0 : U : W)$. *Then there are a neighborhood* U' *of* 0 *in* U *and for each* $F \subseteq F_0, \epsilon \in \Sigma_0$, *a function* $\Phi(F : \epsilon) \in \mathcal{S}(G : F : \epsilon : U' : W)$ *so that* $\Phi(\emptyset : \epsilon_0) = E(P : \Psi)|_{U' \times G}$ *and the collection* $\{\Phi(F : \epsilon)\}$ *satisfies the same matching conditions as the characters for* $h \in U'$.

Proof. Using (5.13), there is U' so that for each $F \subseteq F_0, \epsilon \in \Sigma_0$, we have $\Psi(F : \epsilon) \in \mathcal{S}\left(M^\dagger : F : \epsilon : U' : W\right)$ so that $\Psi(\emptyset : \epsilon_0) = \Psi|_{U' \times M^\dagger}$ and the collection $\{\Psi(F : \epsilon)\}$ satisfies the same matching conditions as the characters for $h \in U'$. Now by differentiating under the integrals we see that the collection $\{E(P : \Psi(F : \epsilon))\}$ also satisfies the same matching conditions as the characters. Now using (5.16), each $E(P : \Psi(F : \epsilon)) \in \mathcal{S}(G : F : \epsilon : U' : W)$. □

Suppose that U is a neighborhood of 0 in $\mathfrak{v}_{\mathbf{C}}^*$ and for each $F \subseteq F_0, \epsilon \in \Sigma_0$, we have $\Phi(F:\epsilon) \in \mathcal{S}(G:F:\epsilon:U:W)$ such that $\{\Phi(F:\epsilon)\}$ satisfy the same matching conditions as the characters. Then given a standard parabolic subgroup Q of G and $s \in W_G$ representing a coset of $W_Q \backslash W_G$ we have constant terms $\widetilde{\Phi}_{Q,s}(F:\epsilon)$ which by (3.21) also satisfy the matching conditions. Further, by (4.13), each constant term $\widetilde{\Phi}_{Q,s}(F:\epsilon)$ is a meromorphic function on $U \times \mathfrak{a}_{F,\mathbf{C}}^*$ which is holomorphic in a complex neighborhood of $U_F(\epsilon) \times \mathfrak{a}_F^*$ where $U_F(\epsilon) = U \cap \mathrm{cl}(\mathcal{D}_F(\epsilon))$. For each $h \in \mathfrak{v}_{\mathbf{C}}^*, E \subseteq F \subseteq F_0$, since $\mathfrak{a}_F = \mathfrak{a}_E \oplus \sum_{\alpha \in F\backslash E} \mathbf{R} H_{c_F\alpha}$, we can define $\nu_{F\backslash E}(h) \in \mathfrak{a}_{F,\mathbf{C}}^*$ by

$$\nu_{F\backslash E}(h)|_{\mathfrak{a}_E} = 0, \quad \nu_{F\backslash E}(h)(H_{c_F\alpha}) = -i\langle h_M(h), \alpha\rangle, \alpha \in F\backslash E.$$

For $h \in \mathfrak{v}_{\mathbf{C}}^*, \nu_E \in \mathfrak{a}_{E,\mathbf{C}}^*$, we also write $(\nu_E, \nu_{F\backslash E}(h)) = (\nu_E, 0) + \nu_{F\backslash E}(h)$.

Lemma 5.18. *Let $1 \le i \le m, \epsilon \in \Sigma_i, E \subseteq F_0$. Then for all $(h, \nu_E, x) \in U \times \mathfrak{a}_{E,\mathbf{C}}^* \times L_Q^*$,*

$$\begin{aligned}
&\widetilde{\Phi}_{Q,s}(E:\epsilon^+(i):h:\nu_E:x) - \widetilde{\Phi}_{Q,s}(E:\epsilon^-(i):h:\nu_E:x) \\
&= \sum_{E \subset F \subseteq E(i)} c_{|F\backslash E|}\Big[\widetilde{\Phi}_{Q,s}(F:\epsilon^+(i):h:(\nu_E,\nu_{F\backslash E}(h)):x) \\
&\quad + \widetilde{\Phi}_{Q,s}(F:\epsilon^-(i):h:(\nu_E,\nu_{F\backslash E}(h)):x)\Big].
\end{aligned}$$

Proof. Note that if $1 \le i \le m, E \subseteq F \subseteq E(i)$, and $h_0 \in \mathcal{H}_i$, then $\nu_{F\backslash E}(h_0) = 0$. Also, if f is a function on $\mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$ and g is the function on $\mathfrak{v}_{\mathbf{C}}^*$ defined by $g(h) = f(h:\nu(h))$, then $\partial/\partial h_i g(h_0) = (\partial/\partial h_i - i\sum_{\alpha \in F\backslash E} \partial/\partial\mu_\alpha) f(h_0:0)$ for all $h_0 \in \mathcal{H}_i$.

Now fix $(\nu_E, x) \in \mathfrak{a}_E^* \times G$ and for any $E \subseteq F \subseteq F_0, \epsilon \in \Sigma_0, h \in U$, define

$$f(F:\epsilon:h) = \widetilde{\Phi}_{Q,s}(F:\epsilon:h:(\nu_E,\nu_{F\backslash E}(h)):x).$$

By the above, for all $k \ge 0, h_0 \in \mathcal{H}_i \cap U$ we have

$$(\partial/\partial h_i)^k f(F:\epsilon:h_0) = D^k_{F\backslash E}\widetilde{\Phi}(F:\epsilon:h_0:(\nu_E,0):x).$$

Fix $1 \le i \le m$ and $\epsilon \in \Sigma_i$, and write

$$\begin{aligned}
g(h) &= f(E:\epsilon^+(i):h) - f(E:\epsilon^-(i):h) \\
&\quad - \sum_{E \subset F \subseteq E(i)} c_{|F\backslash E|}[f(F:\epsilon^+(i):h) + f(F:\epsilon^-(i):h)].
\end{aligned}$$

Then the matching conditions can be rephrased as saying that $(\partial/\partial h_i)^k g(h_0) = 0$ for all $k \ge 0, h_0 \in \mathcal{H}_i \cap U$. But since g is a meromorphic function of $h \in U$ we can conclude that $g(h) = 0$ for all $h \in U$. ☐

We are now ready to return to the proof of Theorem 5.7. We will first look at the case that $\alpha \in \Phi_F^1 \cup \Phi_F^3$. For every $F \subseteq F_0$ we can identify $\Phi_F^+ = \Phi^+(\mathfrak{g}_C, \mathfrak{h}_{F,\mathbf{C}})$ with $\Phi^+ = \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_C)$ via the Cayley transform c_F as in (2.10). Thus for every $\alpha \in \Phi^+, F \subseteq F_0$, we can write

$$\pi_\alpha(F : h : \nu) = \langle \lambda_F(h) + i\nu, \alpha \rangle, (h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*.$$

Lemma 5.19. *Let $E \subseteq F_0$. Suppose $\alpha \in \Phi_E^1$. Then for all $E \subseteq F \subseteq F_0$, $\alpha \in \Phi_F^1$, and for all $(h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$,*

$$\pi_\alpha(F : h : \nu) = \pi_\alpha(E : h : \nu_E)$$

where ν_E is the restriction of ν to $\mathfrak{a}_E$.

Proof. We always have $\Phi_{E,R}^+ \subseteq \Phi_{F,R}^+$. Now by [**H3**, 5.5], $\epsilon_\alpha(F : h) = \epsilon_\alpha(E : h)$ for all h except possibly in the case that α is a long root in a simple factor of G which is isomorphic to the universal covering group of $Sp(n, \mathbf{R})$ for some n. But in this case $\epsilon_\alpha(E : h)$ is not independent of h, so $\alpha \notin \Phi_E^1$. Finally, since $\alpha \in \Phi_{E,R}^+$, α is orthogonal to $F_0 \backslash E$ so that $\pi_\alpha(F : h : \nu) = \pi_\alpha(E : h : \nu_E)$ for all $(h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$. □

Lemma 5.20. *Let $E \subseteq F_0$ and suppose $\gamma \in \Phi_E^3$. Then for all $E \subseteq F \subseteq F_0, \gamma \in \Phi_{F,CPX}^+$ and $\pi_\gamma(F : h : \nu) = \pi_\gamma(E : h : \nu_E)$ for all $(h,\nu) \in \mathfrak{v}_{\mathbf{C}}^* \times \mathfrak{a}_{F,\mathbf{C}}^*$, where ν_E is the restriction of ν to $\mathfrak{a}_E$.*

Proof. We may as well assume that G is simple. We know from [**H3**, 4.12] that $\gamma - \gamma'$ is orthogonal to $\lambda_E(h)$ for all h. Thus $s_\gamma s_{\gamma'}(F_0 \backslash E) \subset \pm(F_0 \backslash E)$. Suppose that $F_0 \backslash E = \delta$ consists of only one root. If $s_\gamma s_{\gamma'}\delta = -\delta$, then $\gamma - \gamma'$ is a multiple of δ. But this cannot be so since $\gamma - \gamma'$ is orthogonal to $\lambda_E(h)$ for all h while δ is orthogonal to $\lambda(0)$, but not to $\lambda(h)$ for some $h \neq 0$. Thus $s_\gamma s_{\gamma'}\delta = +\delta$ so that $\gamma - \gamma'$ is orthogonal to δ. Thus in this case γ and γ' are both orthogonal to $F_0 \backslash E$. In $|F_0 \backslash E| \geq 2$, then G has real rank at least four, and as in [**H3**, 5.12] we see by looking at the three possible cases that we have γ, γ' orthogonal to $F_0 \backslash E$. Thus in any case we have γ, γ' orthogonal to $F_0 \backslash E$ so that for all $E \subseteq F \subseteq F_0, \gamma \in \Phi_{F,CPX}^+$ and $\pi_\gamma(F : h : \nu) = \pi_\gamma(E : h : \nu_E)$. □

Lemma 5.21. *Let $E \subseteq F_0, \epsilon \in \Sigma_0, \Psi \in \mathcal{S}(M_E^\dagger : \lambda_E : \chi_E : \mathcal{D}_E(\epsilon) : W)$. Let $\alpha \in \Phi_E^1 \cup \Phi_E^3$. Then*

$$(h,\nu,x) \to \pi_\alpha(E : h : \nu)^{-1} \widetilde{E}_{Q,s}(P_E : \Psi : h : \nu : x)$$

is jointly smooth on $U_E(\epsilon) \times \mathfrak{a}_E^* \times L_Q^*$.

Proof. Fix $E_0 \subseteq F_0$. For every $E_0 \subseteq F \subseteq F_0$ we can identify $\Phi^+(\mathfrak{g}_C, \mathfrak{h}_{F,\mathbf{C}})$ with $\Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ via the Cayley transform c_F. We will prove by induction on $F_0\backslash F$ that if $\nu_0 \in \mathfrak{a}^*_{F,\mathbf{C}}$ such that $\pi_\alpha(F : h : \nu_0) = 0$ where $\alpha \in \Phi^1_{E_0} \cup \Phi^3_{E_0}$, and if $\Psi_F \in \mathcal{S}\left(M^\dagger_F : \lambda_F : \chi_F : \mathcal{D}_F(\epsilon) : W\right), \epsilon \in \Sigma_0$, then

$$\widetilde{E}_{Q,s}(P_F : \Psi_F : h : \nu_0 : x) = 0$$

for all $(h,x) \in U_F(\epsilon) \times G$. In particular, the case of $F = E_0$ will show that $\widetilde{E}_{Q,s}(P_{E_0} : \Psi : h : \nu_0 : x) = 0$. Now, since $\widetilde{E}_{Q,s}(P_{E_0} : \Psi)$ is jointly smooth on $U_{E_0}(\epsilon) \times \mathfrak{a}^*_{E_0} \times L^*_Q$ and $\pi_\alpha(E_0 : h : \nu) = \pi_\alpha(E_0 : 0 : \nu)$ is a real linear form on $\mathfrak{a}^*_{E_0}$, if follows that

$$(h,\nu,x) \to \pi_\alpha(E_0 : h : \nu)^{-1}\widetilde{E}_{Q,s}(P_{E_0} : \Psi : h : \nu : x)$$

is smooth.

Suppose first that $F = F_0, \epsilon \in \Sigma_0$. Then $0 \in \mathcal{D}_{F_0}(\epsilon)$ is regular so that $s = s_i$ for some $i \in I^0$. Thus

$$\widetilde{E}_{Q,s}(P_{F_0} : \Psi_{F_0} : h : \nu : x) = \pi(F_0 : h : \nu)E_{Q,s}(P_{F_0} : \Psi_{F_0} : h : \nu : x)$$

is an ordinary constant term. Then if $Q = P_{F_0}$ we know from an easy extension of Harish-Chandra's result (see [**H2**, 5.4]) that there is a constant $c(A_{F_0}) > 0$ so that

$$\begin{aligned}&\left\|\widetilde{E}_{Q,s}(P_{F_0} : \Psi_{F_0} : h : \nu)\right\|^2_{L^2(M_{F_0}/Z)}\\ &= c(A_{F_0})^2\left|\pi(F_0 : h : \nu)^2 m(H_{F_0} : h : \nu)^{-1}\right| \|\Psi_{F_0}(h)\|^2_{L^2(M_{F_0}/Z)}\end{aligned}$$

for all $(h,\nu) \in \mathrm{cl}(\mathcal{D}_{F_0}(\epsilon)) \times \mathfrak{a}^*_{F_0}$. As in [**H3**, 4.5], there is a constant $c(F_0 : \epsilon)$ so that

$$\begin{aligned}&\pi(F_0 : h : \nu)^2 m(H_{F_0} : h : \nu)^{-1}\\ &= c(F_0 : \epsilon)^{-1} \prod_{\beta \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{F_0,\mathbf{C}})\backslash \Phi^+_R(\mathfrak{g},\mathfrak{h}_{F_0})} \pi_\beta(F_0 : h : \nu)\\ &\quad\cdot \prod_{\alpha \in \Phi^+_R(\mathfrak{g},\mathfrak{h}_{F_0})} \frac{\pi_\alpha(F_0 : h : \nu)(\cosh \pi\nu_\alpha - \epsilon_\alpha(F_0 : h))}{\sinh \pi\nu_\alpha}.\end{aligned}$$

Now $(h,\nu) \to \pi(F_0 : h : \nu)^2 m(H_{F_0} : h : \nu)^{-1}$ is a smooth function on $\mathrm{cl}(\mathcal{D}_{F_0}(\epsilon)) \times \mathfrak{a}^*_{F_0}$ since for $\alpha \in \Phi^+_R(\mathfrak{g},\mathfrak{h}_{F_0}), \pi_\alpha(F_0 : h : \nu) = i\nu_\alpha/2$.

Fix $h \in \mathrm{cl}(\mathcal{D}_{F_0}(\epsilon))$. Suppose $\alpha \in \Phi^1_{E_0}$. Then $\alpha \in \Phi^1_{F_0}$ by (5.19). Thus

$$\pi(F_0 : h : \nu_0)^2 m(H_{F_0} : h : \nu_0)^{-1} = 0$$

for any $\nu_0 \in \mathfrak{a}^*_{F_0}$ such that $\pi_\alpha(F_0 : h : \nu_0) = 0$. Thus $\widetilde{E}_{Q,s}(P_{F_0} : \Psi_{F_0} : h : \nu_0) = 0$ for any $\nu_0 \in \mathfrak{a}^*_{F_0}$ such that $\pi_\alpha(F_0 : h : \nu_0) = 0$.

Similarly, if $\alpha \in \Phi^3_{E_0}$, then by (5.20) $\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{F_0,\mathbf{C}}) \backslash \Phi^+_R(\mathfrak{g}, \mathfrak{h}_{F_0})$. Thus we again have $\pi(F_0 : h : \nu_0)^2 m(H_{F_0} : h : \nu_0)^{-1} = 0$ for any $\nu_0 \in \mathfrak{a}^*_{F_0}$ such that $\pi_\alpha(F_0 : h : \nu_0) = 0$. Thus $\widetilde{E}_{Q,s}(P_{F_0} : \Psi_{F_0} : h : \nu_0) = 0$ for any $\nu_0 \in \mathfrak{a}^*_{F_0}$ such that $\pi_\alpha(F_0 : h : \nu_0) = 0$. The case of general Q follows as in [**HW5**, 9.16]. Now, since $\widetilde{E}_{Q,s}(P_{F_0} : \Psi_{F_0})$ is a meromorphic function, it follows that $\widetilde{E}_{Q,s}(P_{F_0} : \Psi_{F_0} : h : \nu_0) = 0$ for any $h \in \mathfrak{v}^*_{\mathbf{C}}$ and $\nu_0 \in \mathfrak{a}^*_{F_0,\mathbf{C}}$ such that $\pi_\alpha(F_0 : h : \nu_0) = 0$.

Now suppose $E_0 \subseteq E \subset F_0$ and assume the result is proven for all F such that $E \subset F \subseteq F_0$. Let $\epsilon' \in \Sigma_0$, $\Psi \in \mathcal{S}(M_E^\dagger : \lambda_E : \chi_E : \mathcal{D}_E(\epsilon') : W)$. If for some $\epsilon_0 \in \Sigma_0, s = s_i$ for some $i \in I^0$, then $i \in I^0$ with respect to any $\epsilon \in \Sigma_0$, and the result follows exactly as above since $\widetilde{E}_{Q,s}$ is an ordinary constant term. Now suppose that there is $\epsilon_0 \in \Sigma_0$ so that $s = s_i, i \in I^0(0) \cap I^-$. That is, for all $H \in \mathfrak{a}^+_Q$, $\lambda_i(h : H) < 0$ for all $h \in \mathcal{D}_E(\epsilon_0)$ and $\lambda_i(0 : H) = 0$. Since λ_i is a linear functional, there must be $\epsilon_0' \in \Sigma_0$ so that $\lambda_i(h : H) > 0$ for some $h \in \mathcal{D}_E(\epsilon_0'), H \in \mathfrak{a}^+_Q$.

Apply (5.17) to the case when $H = H_E, \lambda = \lambda_E, \mathcal{D} = \mathcal{D}_E(\epsilon')$. Then there are a neighborhood U of 0 in $\mathfrak{v}^*_{\mathbf{C}}$ and for each $E \subseteq F \subseteq F_0, \epsilon \in \Sigma_0$, a function $\Phi(F : \epsilon) \in \mathcal{S}(G : F : \epsilon : U : W)$ so that $\Phi(E : \epsilon') = E(P_E : \Psi)|_{U \times G}$ and the collection $\{\Phi(F : \epsilon) : E \subseteq F \subseteq F_0\}$ satisfies the same matching conditions as the characters for $h \in U$. Hence the constant terms $\left\{\widetilde{\Phi}_{Q,s}(F : \epsilon)\right\}$ satisfy the matching conditions (5.18). That is, fix $x \in L^*_Q$ and for $(h, \nu_E) \in U \times \mathfrak{a}^*_{E,\mathbf{C}}$, write $f(F : \epsilon : h : \nu_E) = \widetilde{\Phi}_{Q,s}(F : \epsilon : h : (\nu_E, \nu_{F\backslash E}(h)) : x)$. By (5.18) we know that for all $1 \le i \le m, \epsilon \in \Sigma_i, (h, \nu_E) \in U \times \mathfrak{a}^*_{E,\mathbf{C}}$,

$$\begin{aligned} &f(E : \epsilon^+(i) : h : \nu_E) - f(E : \epsilon^-(i) : h : \nu_E) \\ &= \sum_{E \subset F \subseteq E(i)} c_{|F\backslash E|}[f(F : \epsilon^+(i) : h : \nu_E) + f(F : \epsilon^-(i) : h : \nu_E)]. \end{aligned}$$

Write $\mathfrak{a}_F = \mathfrak{a}_E \oplus \mathfrak{h}_{F\backslash E}$ where $\mathfrak{h}_{F\backslash E} = \sum_{\alpha \in F\backslash E} \mathbf{R} H_{c_F \alpha}$. Then every $\nu_F \in \mathfrak{a}^*_{F,\mathbf{C}}$ can be written as $\nu_F = (\nu_E, \nu')$ where $\nu_E = \nu_F|_{\mathfrak{a}_E}$ and $\nu' = \nu|_{\mathfrak{h}_{F\backslash E}}$. Now suppose $\nu_{E,0} \in \mathfrak{a}^*_{E,\mathbf{C}}$ satisfies $\pi_\alpha(E : h : \nu_{E,0}) = 0$ where $\alpha \in \Phi^1_{E_0} \cup \Phi^3_{E_0}$. Then by (5.19), (5.20), every $\nu_{F,0} = (\nu_{E,0}, \nu'), \nu' \in \mathfrak{h}^*_{F\backslash E,\mathbf{C}}$, satisfies $\pi_\alpha(F : h : \nu_{F,0}) = 0$ for all $h \in \mathfrak{v}^*_{\mathbf{C}}$. Thus for all $h \in U$ we have $\pi_\alpha(F : h : (\nu_{E,0}, \nu_{F\backslash E}(h))) = 0$. Thus by the induction hypothesis and (5.16),

$$f(F : \epsilon^\pm(i) : h : \nu_{E,0}) = 0$$

for all $h \in U$. Now using the matching conditions we have for all $h \in U$,

$$f(E : \epsilon^+(i) : h : \nu_{E,0}) = f(E : \epsilon^-(i) : h : \nu_{E,0}).$$

But now this is the case for every $1 \le i \le m$ and $\epsilon \in \Sigma_i$. Thus $f(E : \epsilon : h : \nu_{E,0}), h \in U$, is independent of $\epsilon \in \Sigma_0$. But as above there is an $\epsilon_0' \in \Sigma_0$ so that $s = s_i, i \in I^+$ with respect to ϵ_0'. Thus $f(E : \epsilon_0' : h : \nu_E) = 0$ for all $(h, \nu_E) \in U \times \mathfrak{a}_{E,\mathbf{C}}^*$. Now for all $\epsilon \in \Sigma_0, h \in U$, $f(E : \epsilon : h : \nu_{E,0}) = f(E : \epsilon_0' : h : \nu_{E,0}) = 0$. □

Lemma 5.22. *Let $E \subseteq F_0, \epsilon \in \Sigma_0, \Psi \in \mathcal{S}\left(M_E^\dagger : \lambda_E : \chi_E : \mathcal{D}_E(\epsilon) : W\right)$. Then*

$$(h, \nu, x) \to \pi_\gamma(E : h : \nu)^{-1} \widetilde{E}_{Q,s}(P_E : \Psi : h : \nu : x)$$

extends to a smooth function on $U_E(\epsilon) \times \mathfrak{a}_E^* \times L_Q^*$ for any $\gamma \in \Phi_E^2$.

Proof. The proof is by induction on $|F_0 \backslash E|$. Suppose that $E = F_0$. Then $U(0) \subset \mathcal{D}_{F_0}(\epsilon)$ consists entirely of regular elements so that

$$\widetilde{E}_{Q,s}(P_E : \Psi : h : \nu : x) = \pi(E : h : \nu) E_{Q,s}(P_E : \Psi : h : \nu : x)$$

is an ordinary constant term. But in this case the lemma follows from [**H2**, 5.1], since for $\gamma \in \Phi_{F_0}^2, \pi_\gamma(F_0 : h : \nu)$ is not independent of h.

Now suppose that $E \subset F_0$. Let $\gamma \in \Phi_E^2 \cap \Phi_\alpha^+, \alpha \in \Phi_{E,R}^+$, so that $\pi_\gamma(E : h : \nu) = c(h_\alpha \pm i\nu_\alpha)$. We can assume by induction that the theorem is true for all F such that $E \subset F \subseteq F_0$. Let $\epsilon_0 \in \Sigma_0, \Psi \in \mathcal{S}\left(M_E^\dagger : \lambda_E : \chi_E : \mathcal{D}_E(\epsilon_0) : W\right)$. In order to show that π_γ divides $\widetilde{E}_{Q,s}(P_E : \Psi : h : \nu : x)$ it suffices to show that for every $(h_0, \nu_0) \in \mathfrak{b}_\mathbf{C}^* \times \mathfrak{a}_{E,\mathbf{C}}^*$ such that $\pi_\gamma(E : h_0 : \nu_0) = 0$ we have

$$\widetilde{E}_{Q,s}(P_E : \Psi : h_0 : \nu_0 : x) = 0.$$

As in (5.21), for each $E \subseteq F \subseteq F_0, \epsilon \in \Sigma_0$, we can find a function $\Phi(F : \epsilon) \in \mathcal{S}(G : F : \epsilon : U : W)$ so that $\Phi(E : \epsilon_0) = E(P_E : \Psi)|_{U \times G}$ and the collection $\{\widetilde{\Phi}_{Q,s}(F : \epsilon) : E \subseteq F \subseteq F_0\}$ satisfies the matching conditions of (5.18). Fix $x \in L_Q^*$ and for $(h, \nu_E) \in U \times \mathfrak{a}_{E,\mathbf{C}}^*$, write

$$f(F : \epsilon : h : \nu_E) = \widetilde{\Phi}_{Q,s}(F : \epsilon : h : (\nu_E, \nu_{F\backslash E}(h)) : x).$$

Then by the matching conditions we know that for all $1 \leq i \leq m, \epsilon \in \Sigma_i$, $(h, \nu_E) \in U \times \mathfrak{a}_{E,\mathbf{C}}^*$,

$$\begin{aligned} &f(E : \epsilon^+(i) : h : \nu_E) - f(E : \epsilon^-(i) : h : \nu_E) \\ &= \sum_{E \subset F \subseteq E(i)} c_{|F\backslash E|}[f(F : \epsilon^+(i) : h : \nu_E) + f(F : \epsilon^-(i) : h : \nu_E)]. \end{aligned}$$

Fix $E \subset F \subseteq E(i)$. If γ is orthogonal to $F \backslash E$, then $\gamma \in \Phi_F^2$ and

$$\pi_\gamma(F : h : (\nu_E, \nu_{F\backslash E}(h))) = \pi_\gamma(E : h : \nu_E)$$

for all (h, ν_E). Thus by the induction hypothesis and (5.16),

$$f(F : \epsilon^{\pm}(i) : h_0 : \nu_{E,0}) = 0$$

for all $(h_0, \nu_{E,0})$ such that $\pi_\gamma(E : h_0 : \nu_{E,0}) = 0$.

If γ is not orthogonal to $F \backslash E$ for any $\gamma \in \Phi^+_{E,\alpha}$ with $\pi_\gamma(E : h : \nu) = c(h_\alpha \pm i\nu_\alpha)$, then there is no $\gamma \in \Phi^+_{F,\alpha}$ such that $\pi_\gamma(F : h : \nu) = c(h_\alpha \pm i\nu_\alpha)$. Thus if $\epsilon_\alpha(F : h) = \epsilon_\alpha(E : h) = \cos \pi h_\alpha$, then $m^*_\alpha(F : h : \nu)$ is not jointly smooth. Thus there is $\beta \in F \backslash E$ so that $\alpha \in [\beta], \Phi(\alpha) \neq \Phi(\beta)$, and we have $\gamma \in \Phi^3_F$. If $\epsilon_\alpha(F : h) \neq \epsilon_\alpha(E : h)$, then by [**H3**, 5.5] there is $\beta \in F \backslash E$ so that α, β are both long roots in a simple factor of G isomorphic to the universal covering group of $Sp(n, \mathbf{R})$ for some n. In this case $\gamma = (\alpha \pm \beta)/2 \in \Phi^1_F$. In either case we have $\pi_\gamma(E : h : \nu_E) = \pi_\gamma(F : h : (\nu_E, \nu_{F \backslash E}(h)))$ for all (h, ν_E). Thus in either case, using (5.21), we know that

$$f(F : \epsilon^{\pm}(i) : h_0 : \nu_{E,0}) = 0$$

for all $(h_0, \nu_{E,0})$ such that $\pi_\gamma(E : h_0 : \nu_{E,0}) = 0$.

Thus as in (5.21) we have now proven that

$$f(E : \epsilon : h_0 : \nu_{E,0}) = f(E : \epsilon' : h_0 : \nu_{E,0})$$

for all $\epsilon, \epsilon' \in \Sigma_0, (h_0, \nu_{E,0})$ such that $\pi_\gamma(E : h_0 : \nu_{E,0}) = 0$. Again, if $s = s_i, i \in I^0(0) \cap I^-$ with respect to some $\epsilon_0 \in \Sigma_0$, then $i \in I^+$ with respect to some other $\epsilon'_0 \in \Sigma_0$, so that $f(E : \epsilon'_0 : h : \nu_E) = 0$ for all (h, ν_E). If $i \in I^0$, then using (4.16) and (4.17) there is $\epsilon'_0 \in \Sigma_0$ so that $f(E : \epsilon'_0 : h_0 : \nu_{E,0}) = 0$ for all $(h_0, \nu_{E,0})$ such that $\pi_\gamma(E : h_0 : \nu_{E,0}) = 0$. Thus in either case we know that for all $\epsilon \in \Sigma_0$ we have $f(E : \epsilon : h_0 : \nu_{E,0}) = 0$ for all $(h_0, \nu_{E,0})$ such that $\pi_\gamma(E : h_0 : \nu_{E,0}) = 0$. □

6. Elementary mixed wave packets.

Let $H = TA$ be a θ-stable Cartan subgroup of G and let $(\lambda, \chi) \in X(T)$, $\tau_1, \tau_2 \in \widehat{K}(\chi), W = W(\tau_1 : \tau_2)$. As in (2.10) we define F_0 and $H_F = T_F A_F, (\lambda_F, \chi_F) \in X(T_F)$ for every $F \subseteq F_0$. Let U be a neighborhood of 0 in $i\mathfrak{v}^*$ satisfying the conditions of [**H3**, 4.6] and (3.18). We assume that the Plancherel function $m(H : h : \nu)$ corresponding to $\pi(H : \lambda : \chi : h : \nu)$ is jointly smooth as a function of $(h, \nu) \in (U \cap \mathrm{cl}(\mathcal{D})) \times \mathfrak{a}^*$ for every connected component $\mathcal{D}$ of $\{h \in i\mathfrak{v}^* : \langle \lambda(h), \alpha \rangle \neq 0, \alpha \in \Phi^+_M\}$.

Let P_0 be a minimal parabolic subgroup of G and fix $P_0 \subseteq P, s \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{0,\mathbf{C}})$ such that $s^{-1}\mathfrak{a}_P \subseteq \mathfrak{a}_{F_0}$. Assume that for each $F \subseteq F_0$, we have

$$f(F) : i\mathfrak{v}^* \times \mathfrak{a}^*_F \times L^*_P \to W$$

satisfying the following conditions. For each $\epsilon \in \Sigma_0$, let $f(F:\epsilon)$ denote the restriction of $f(F)$ to $U_F(\epsilon) \times \mathfrak{a}_F^* \times L_P^*$. Then we assume that each $f(F:\epsilon) \in J_F^1(U:\epsilon:L_P^*:s)$ and that $\mathbf{f} = \{f(F:\epsilon)\}$ satisfies the matching conditions of (2.14) for each $x \in L_P^*$. We will say that $\mathbf{f}$ is a matching collection of functions in $J^1(U:L_P^*:s)$. For each F, ϵ we have seminorms $T_{D,r}^0$ defined on $J_F^0(U:\epsilon:L_P^*:s)$ as in (3.2b). Define

(6.1 a) $$T_{D,r}^0(\mathbf{f}) = \sum_{F\subseteq F_0,\epsilon\in\Sigma_0} T_{D,r}^0(f(F:\epsilon)).$$

For each $(h,x) \in i\mathfrak{v}^* \times L_P^*$, define

(6.1 b) $$\Phi(\mathbf{f}:h:x) = \sum_{F\subseteq F_0} (\pi i)^{-|F|} \int_{\mathfrak{a}_F^*} \frac{f(F:h:\nu:x)}{p_F(h:\nu)} d\nu.$$

Theorem 6.2. *Let $P_0 \subseteq P, s \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{0,\mathbf{C}})$ such that $s^{-1}\mathfrak{a}_P \subseteq \mathfrak{a}_{F_0}$. Given $r \geq 0, D \in D(i\mathfrak{v}^*), l_1, l_2 \in \mathcal{U}(\mathfrak{l}_{P,\mathbf{C}})$, there is a finite subset $E \subset \widetilde{\mathcal{L}_P}$ so that given any $r' \geq 0$ there is $C > 0$ such that*

$$\sup_{h\in U, a\in \mathrm{cl}(A_0^+)} \Xi_P^{-1}(a)(1+\sigma(a))^r \|\Phi(\mathbf{f}:h;D:l_1;a;l_2)\| \leq C \sum_{D'\in E} T_{D',r'}^0(\mathbf{f})$$

for all matching collections $\mathbf{f}$ *of functions in* $J^1(U:L_P^*:s)$.

Theorem 6.3. *Let Φ be an elementary mixed wave packet as in* (2.16). *Then*

$$\Phi \in \mathcal{C}(G:W).$$

Proof. It follows from [**H3**, 7.2] that $(h,x) \mapsto \Phi(h:x)$ is jointly smooth on $i\mathfrak{v}^* \times G$. Thus using [**H3**, 2.8] it suffices to prove that for all $r \geq 0, D \in D(i\mathfrak{v}^*), g_1, g_2 \in \mathcal{U}(\mathfrak{g}_{\mathbf{C}})$,

$$\sup_{h\in i\mathfrak{v}^*, a\in \mathrm{cl}(A_0^+)} \Xi^{-1}(a)(1+\sigma(a))^r(1+|h|)^r \|\Phi(h;D:g_1;a;g_2)\| < \infty.$$

In fact, since $\Phi(h:x)$ has compact support $\omega \subset U$, the term $(1+|h|)^r$ is not needed and it suffices to take the sup over $h \in U$. Thus, using (2.22) and (5.2), the result follows from the special case $P = G, s = 1$, of Theorem 6.2. □

The remainder of this section will be devoted to the proof of Theorem 6.2. Since the proof is long, it will be divided up into a series of lemmas. A number of calculus lemmas which will be needed in the course of the proof will be deferred to the next section.

Lemma 6.4. *It is enough to prove* (6.2) *in the case that* $l_1, l_2 \in \mathcal{U}(\mathfrak{m}_{P,\mathbf{C}})$.

Proof. It is clear from [**H3**, §7] that we can differentiate under the integrals so that for $l_1, l_2 \in \mathcal{U}(\mathfrak{l}_{P,\mathbf{C}})$,

$$\Phi(\mathbf{f} : h : l_1 ; a; l_2) = \sum_{F \subseteq F_0} (\pi i)^{-|F|} \int_{\mathfrak{a}_F^*} \frac{f(F : h : \nu : l_1 ; a; l_2)}{p_F(h : \nu)} d\nu.$$

Write $\mathcal{U}(\mathfrak{l}_{P,\mathbf{C}}) = \mathcal{U}(\mathfrak{m}_{P,\mathbf{C}})S(\mathfrak{a}_{P,\mathbf{C}})$, where $S(\mathfrak{a}_{P,\mathbf{C}}) \subseteq \mathcal{Z}_P$. Then if $l_i = m_i u_i, m_i \in \mathcal{U}(\mathfrak{m}_{P,\mathbf{C}}), u_i \in S(\mathfrak{a}_{P,\mathbf{C}}), i = 1, 2$, and if $a \in \mathrm{cl}(A_0^+)$, then for each $F \subseteq F_0$,

$$f(F : h : \nu : m_1 u_1 ; a; m_2 u_2) = u_1 u_2 (\Lambda_{h,\nu,s}^F) f(F : h : \nu : m_1 ; a; m_2).$$

Using (3.22) and the fact that $u_1 u_2 \left(\Lambda_{h,\nu,s}^F\right)$ is a polynomial in (h, ν), we see that for any matching family $\mathbf{f}$ in $J^1(U : L_P^* : s)$, $\mathbf{f}' = \left\{u_1 u_2 \left(\Lambda_{h,\nu,s}^F\right) f(F : \epsilon)\right\}$ is also a matching family in $J^1(U : L_P^* : s)$. Further, given $D' \in \widetilde{\mathcal{L}}_P$ there is $D'' \in \widetilde{\mathcal{L}}_P$ so that $T_{D',r}^0(\mathbf{f}') \le T_{D'',r}^0(\mathbf{f})$. Thus we may as well assume that $u_1 = u_2 = 1$. □

Let Θ be the set of simple roots in $\Delta(P_0, A_0)$. For each $\alpha \in \Theta$ pick $H^\alpha \in \mathfrak{a}_0$ so that $\alpha(H^\alpha) = 1, \beta(H^\alpha) = 0, \beta \in \Theta, \beta \ne \alpha$. Then each $H \in \mathfrak{a}_0$ can be written uniquely as $H = \sum_{\alpha \in \Theta} \alpha(H) H^\alpha$. Now $H \in \mathrm{cl}(\mathfrak{a}_0^+)$ just in case $\alpha(H) \ge 0$ for all $\alpha \in \Theta$ and $H \in \mathfrak{a}_P$ just in case $\alpha(H) = 0, \alpha \in \Theta_P$. Write $\mathfrak{a}^P = \sum_{\alpha \in \Theta_P} \mathbf{R} H^\alpha \subseteq \mathfrak{a}_0$. For each $H \in \mathfrak{a}_0$, write $H_P = \sum_{\alpha \in \Theta \backslash \Theta_P} \alpha(H) H^\alpha \in \mathfrak{a}_P$. Write $H^P = H - H_P = \sum_{\alpha \in \Theta_P} \alpha(H) H^\alpha \in \mathfrak{a}^P$. Note that $H \in \mathrm{cl}(\mathfrak{a}_0^+)$ if and only if both of $H_P, H^P \in \mathrm{cl}(\mathfrak{a}_0^+)$ and $H \in \mathfrak{a}_0^+$ if and only if $H_P \in \mathfrak{a}_P^+ = \{H_P \in \mathfrak{a}_P : \alpha(H) > 0 \text{ for all } \alpha \in \Theta \backslash \Theta_P\}$ and $H^P \in \mathfrak{a}_+^P = \{H^P \in \mathfrak{a}^P : \alpha(H) > 0 \text{ for all } \alpha \in \Theta_P\}$. Now for each $a \in A_0^+$ we will write $a = a_1 a_2$ where if $a = \exp(H), a_1 = \exp(H^P) \in A_+^P \subseteq \mathrm{cl}(A_0^+), a_2 = \exp(H_P) \in A_P^+ \subseteq \mathrm{cl}(A_0^+)$.

We are assuming that $s^{-1}\mathfrak{a}_P \subseteq \mathfrak{a}_{F_0} = \mathfrak{a}_\emptyset + \sum_{\alpha \in F_0} \mathbf{R} H_{c_{F_0}\alpha}^*$ so that for any $a_2 \in A_P$ we can write $s^{-1}a_2 = a_\emptyset \exp\left(\sum_{\alpha \in F_0} t_\alpha H_{c_{F_0}\alpha}^*\right)$ for some $a_\emptyset \in A_\emptyset$ where $t_\alpha = t_\alpha(s^{-1}a_2) = s c_{F_0} \alpha(\log a_2)/2$. We will write $F_0 = F_0' \cup F_0''$ where $F_0'' = \{\alpha \in F_0 : t_\alpha(s^{-1}a_2) = 0 \text{ for all } a_2 \in A_P\}$ and $F_0' = F_0 \backslash F_0''$. Then we have $s^{-1}\mathfrak{a}_P \subseteq \mathfrak{a}_{F_0'}$. For each $F \subseteq F_0$ we will write $F' = F \cap F_0'$ and $F'' = F \cap F_0''$.

Since $U = \cup_{\epsilon \in \Sigma_0} U(\epsilon)$ where $U(\epsilon) = U \cap \mathrm{cl}(\mathcal{D}(\epsilon))$ and Σ_0 is a finite set, it suffices in Theorem 6.2 to estimate the sup over $h \in U(\epsilon)$ for each $\epsilon \in \Sigma_0$. Further, since $\Phi(\mathbf{f} : h : x)$ is jointly smooth on $U(\epsilon) \times \mathrm{cl}(A_0^+)$, it suffices to estimate the sup over $U^0(\epsilon) \times A_0^+$ where $U^0(\epsilon) = U \cap \mathcal{D}(\epsilon)$. Fix $\epsilon \in \Sigma_0$. We may as well assume that the ordering on Φ_M^+ was chosen so that

$\alpha(h_M(h)) > 0$ for all $\alpha \in F_0, h \in \mathcal{D}(\epsilon)$. That is, $\epsilon_\alpha = 1$ for all $\alpha \in F_0$. Define $F_0'(s) = F_0' \cap s^{-1}\Phi^+$.

Lemma 6.5. *For all $F \subseteq F_0$ we have $f(F:\epsilon) = 0$ unless $F_0'(s) \subseteq F$.*

Proof. Fix $F \subseteq F_0$. Then for all $(h, \nu, a = a_1 a_2) \in U^0(\epsilon) \times \mathfrak{a}_F^* \times A_0^+$ we have

$$f(F:\epsilon:h:\nu:a_1a_2) = \exp\left(\Lambda^F_{h,\nu,s}(\log a_2)\right) f(F:\epsilon:h:\nu:a_1).$$

Now for all $a_2 \in A_P$ we have

$$\begin{aligned}\exp\left(\Lambda^F_{h,\nu,s}(\log a_2)\right) &= \exp((\lambda_F(h) + i\nu)^{c_F^{-1}y}(s^{-1}\log a_2)) \\ &= e^{i\nu}(a_\emptyset) \prod_{\alpha\in F'} e^{i\nu_\alpha t_\alpha} \prod_{\alpha \in F_0'\backslash F'} e^{h_\alpha t_\alpha}\end{aligned}$$

where h_α and ν_α are defined as in (2.21b).

Now because of the growth condition, and because we assume the root ordering was chosen so that $h_\alpha > 0$ for all $\alpha \in F_0, h \in \mathcal{D}(\epsilon)$, $f(F:\epsilon) = 0$ unless $t_\alpha = t_\alpha(s^{-1}a_2) \leq 0$ for all $a_2 \in A_P^+, \alpha \in F_0'\backslash F'$. But $t_\alpha(s^{-1}a_2) \geq 0$ for all $a_2 \in A_P^+$ if $\alpha \in s^{-1}\Phi^+$ and $t_\alpha(s^{-1}a_2) \leq 0$ for all $a_2 \in A_P^+$ if $\alpha \in s^{-1}\Phi^-$. Further, for $\alpha \in F_0' \cap s^{-1}\Phi^+$, there is $a_2 \in A_P^+$ such that $t_\alpha(s^{-1}a_2) > 0$. Thus $f(F:\epsilon) = 0$ unless $F_0'\backslash F' \subseteq F_0'\backslash F_0'(s)$. □

Lemma 6.6. *Fix $F \subseteq F_0$ such that $F_0'(s) \subseteq F$. Then for all $(h,a) \in U^0(\epsilon) \times A_0^+$ we have*

$$\begin{aligned}&(\pi i)^{-|F|} \int_{\mathfrak{a}_F^*} \frac{f(F:h:\nu:a_1a_2)}{p_F(h:\nu)} d\nu \\ &= \frac{(-1)^{|F_0'\backslash F'|}}{(\pi i)^{|F_0'\cup F''|}} P.V. \int_{\mathfrak{a}^*_{F_0'\cup F''}} \frac{e^{i\nu}(s^{-1}a_2) f(F:h:\nu:a_1)}{p_{F_0'\cup F''}(h:\nu)} d\nu.\end{aligned}$$

Proof. Write

$$s^{-1}a_2 = a_F \exp\left(\sum_{\alpha \in F_0'\backslash F'} t_\alpha H^*_{c_{F_0}\alpha}\right)$$

where

$$a_F = a_\emptyset \exp\left(\sum_{\alpha\in F'} t_\alpha H^*_{c_{F_0}\alpha}\right)$$

as before. Then using the normalizations of Haar measures in [**H3**, 7.8], we have

$$\frac{(-1)^{|F_0'\backslash F'|}}{(\pi i)^{|F_0'\cup F''|}}\,\text{P.V.}\int_{\mathfrak{a}^*_{F_0'\cup F''}} \frac{e^{i\nu}(s^{-1}a_2)f(F:h:\nu:a_1)}{p_{F_0'\cup F''}(h:\nu)}d\nu$$
$$= \prod_{\alpha\in F_0'\backslash F'}(-1/2\pi i)\,\text{P.V.}\int_{\mathbf{R}}\frac{e^{i\nu_\alpha t_\alpha}}{\nu_\alpha+ih_\alpha}d\nu_\alpha(\pi i)^{-|F|}\int_{\mathfrak{a}^*_F}\frac{e^{i\nu}(a_F)f(F:h:\nu:a_1)}{p_F(h:\nu)}d\nu.$$

Now since $F_0'(s)\subseteq F'$, we have $h_\alpha t_\alpha<0$ and $t_\alpha<0$ for all $h\in\mathcal{D}(\epsilon), a\in A_0^+, \alpha\in F_0'\backslash F'$. Thus using (7.1), for each $\alpha\in F_0'\backslash F'$, we have

$$(-1/2\pi i)\,\text{P.V.}\int_{\mathbf{R}}\frac{e^{i\nu_\alpha t_\alpha}}{\nu_\alpha+ih_\alpha}d\nu_\alpha = e^{h_\alpha t_\alpha}.$$

Thus

$$\prod_{\alpha\in F_0'\backslash F'}(-1/2\pi i)\,\text{P.V.}\int_{\mathbf{R}}\frac{e^{i\nu_\alpha t_\alpha}}{\nu_\alpha+ih_\alpha}d\nu_\alpha(\pi i)^{-|F|}\int_{\mathfrak{a}^*_F}\frac{e^{i\nu}(a_F)f(F:h:\nu:a_1)}{p_F(h:\nu)}d\nu$$
$$=(\pi i)^{-|F|}\int_{\mathfrak{a}^*_F}\frac{e^{i\nu}(a_F)\prod_{\alpha\in F_0'\backslash F'}e^{h_\alpha t_\alpha}f(F:h:\nu:a_1)}{p_F(h:\nu)}d\nu$$
$$=(\pi i)^{-|F|}\int_{\mathfrak{a}^*_F}\frac{f(F:h:\nu:a_1a_2)}{p_F(h:\nu)}d\nu$$

since

$$e^{i\nu}(a_F)\prod_{\alpha\in F_0'\backslash F'}e^{h_\alpha t_\alpha}f(F:h:\nu:a_1)=e^{\Lambda^F_{h,\nu,s}}(a_2)f(F:h:\nu_F:a_1)$$
$$=f(F:h:\nu_F:a_1a_2).$$

□

For any $F''\subseteq F_0''$ write $\mathfrak{h}_{F''}=\sum_{\alpha\in F''}\mathbf{R}H_{c_F\alpha}$. Combining the above we have

Lemma 6.7. *For all $a=a_1a_2\in A_0^+, h\in U^0(\epsilon)$ we have*

$$\Phi(\mathbf{f}:h:a)=(\pi i)^{-|F_0'|}\,\text{P.V.}\int_{\mathfrak{a}^*_{F_0'}}\frac{e^{i\nu'}(s^{-1}a_2)}{p_{F_0'}(h:\nu')}d\nu'$$
$$\cdot\sum_{F_0'(s)\subseteq F\subseteq F_0}(-1)^{|F_0'\backslash F'|}(\pi i)^{-|F''|}\int_{\mathfrak{h}^*_{F''}}\frac{f(F:h:\nu'+\nu'':a_1)}{p_{F''}(h:\nu'')}d\nu''.$$

Fix $\alpha \in F_0' \backslash F_0'(s)$ and define $I \subseteq \{1, 2, ..., m\}$ as in (2.23) corresponding to ϵ and $\mathcal{H}_\alpha$.

Lemma 6.8. *Fix $E \subseteq F_0$ such that $F_0'(s) \subseteq E$ and $E' \neq E(I)'$. Then for all $a_1 \in A_P^+, \nu_E \in \mathfrak{a}_E^*, h_0 \in \mathcal{H}_I \cap U(\epsilon), k \geq 0$,*

$$\sum_{E \subseteq F \subseteq E(I)} (-1)^{|F \backslash E|} \left(D_{E(I)\backslash E}^k f\right)(F : \epsilon : h_0 : (\nu_E, 0) : a_1) = 0.$$

Here $D_{E(I)\backslash E} = \partial/\partial h_I - i \sum_{\alpha \in E(I)\backslash E} \partial/\partial \mu_\alpha$.

Proof. Fix E, a_1, ν_E, h_0, k as in the lemma. For $E \subseteq F \subseteq E(I)$ we write

$$a^\pm(F) = \left(D_{E(I)\backslash E}^k f\right)(F : \epsilon^\pm(I) : h_0 : (\nu_E, 0) : a_1).$$

Thus by (2.24), (2.25) we know that

$$a^-(E) = \sum_{E \subseteq F \subseteq E(I)} (-1)^{|F \backslash E|} a^+(F).$$

Since $F_0'(s) \subseteq E'$ we have $t_\alpha h_\alpha < 0$ for all $\alpha \in F_0' \backslash E', h \in \mathcal{D}(\epsilon), a \in A_0^+$. Thus $t_\alpha h_\alpha > 0$ for all $\alpha \in E(I)' \backslash E', h \in \mathcal{D}(\epsilon^-(I)), a \in A_0^+$. Thus, as in (6.5), $a^-(E) = 0$ since by assumption $E(I)' \backslash E' \neq \emptyset$. $\square$

Lemma 6.9. *Fix $\alpha \in F_0' \backslash F_0'(s)$ and define I as before. Let $E \subset F_0$ so that $F_0'(s) \subseteq E \subseteq F_0' \backslash \{\alpha\}$. Write $E(\alpha) = E \cup \{\alpha\}$. Then for any $a_1 \in A_+^P, k \geq 0$, we have*

$$(\partial/\partial h_I - i\partial/\partial \mu_\alpha)^k \sum_{E \subseteq F \subseteq E(\alpha)} (-1)^{|F \backslash E|} f(F : \epsilon : h_0 : \nu_0 : a_1) = 0$$

for all $(h_0, \nu_0) \in U(\epsilon) \times \mathfrak{a}_{F_0}^*$ such that $(h_0)_\alpha = (\nu_0)_\alpha = 0$.

Proof. Write $s(E) = |E(I) \backslash E| \geq 1$. The proof is by induction on $s(E)$. Suppose $s(E) = 1$. Thus $E(I) = E(\alpha)$ and $D_{E(I)\backslash E} = D_\alpha = \partial/\partial h_I - i\partial/\partial \mu_\alpha$. The result follows from (6.8) in the case that $\nu_E = (\nu_0)_E$ is the restriction of ν_0 to $\mathfrak{a}_E^*$ since $(\nu_0)_\alpha = 0$ implies that $((\nu_0)_E, 0)$ is the restriction of ν_0 to $\mathfrak{a}_{E(I)}^*$.

Now suppose $s(E) \geq 2$. Again, by (6.8) we can write

$$0 = \sum_{E \subseteq F \subseteq E(I)} (-1)^{|F \backslash E|} \left(D_{E(I)\backslash E}^k f\right)(F : \epsilon : h_0 : (\nu_E, 0) : a_1).$$

Now as above, if $\nu_E = (\nu_0)_E$ and $F \in \{E, E(\alpha)\}$, then

$$\left(D_{E(I)\backslash E}^k f\right)(F : \epsilon : h_0 : (\nu_E, 0) : a_1) = D_\alpha^k f(F : h_0 : \nu_0 : a_1).$$

Thus it suffices to prove that for any $E \subset \widetilde{E} \subseteq E(I) \backslash \{\alpha\}$ we have

$$\sum_{\widetilde{E} \subseteq F \subseteq \widetilde{E}(\alpha)} (-1)^{|F \backslash \widetilde{E}|} \left(D^k_{E(I)\backslash E} f\right)(F : \epsilon : h_0 : (\nu_E, 0) : a_1) = 0.$$

But for such an $\widetilde{E}$ we have $s\left(\widetilde{E}\right) < s(E)$ so that by the induction hypothesis we have

$$D^k_\alpha \sum_{\widetilde{E} \subseteq F \subseteq \widetilde{E}(\alpha)} (-1)^{|F \backslash \widetilde{E}|} f(F : \epsilon : h_0 : \nu_0 : a_1) = 0$$

for all ν_0 such that $(\nu_0)_\alpha = 0, k \geq 0$. Now $D^k_{E(I)\backslash E} = \sum_{j=0}^k \binom{k}{j} D^j_\alpha D_0^{k-j}$ where $D_0 = -i \sum_{\beta \in E(I) \backslash E(\alpha)} \partial / \partial \mu_\beta$. Now we can differentiate the above equation with respect to D_0 and evaluate at $((\nu_0)_E, 0)$ to obtain

$$\sum_{\widetilde{E} \subseteq F \subseteq \widetilde{E}(\alpha)} (-1)^{|F \backslash \widetilde{E}|} \left(D^k_{E(I)\backslash E} f\right)(F : \epsilon : h_0 : (\nu_E, 0) : a_1) = 0.$$

□

Lemma 6.10. *Fix $F'' \subseteq F_0''$. The mapping $(h, \nu', a_1) \mapsto$*

$$p_{F_0' \backslash F_0'(s)}(h : \nu')^{-1} \sum_{F_0'(s) \subseteq F' \subseteq F_0'} (-1)^{|F_0' \backslash F'|} \int_{\mathfrak{h}^*_{F''}} \frac{f(F' \cup F'' : h : \nu' + \nu'' : a_1)}{p_{F''}(h : \nu'')} d\nu''$$

is smooth on $U(\epsilon) \times \mathfrak{a}^*_{F_0'} \times A^P_+$.

Proof. Each of the functions $(h, \nu', \nu'', a_1) \to f(F' \cup F'' : h : \nu' + \nu'' : a_1)$ is jointly smooth on $U(\epsilon) \times \mathfrak{a}^*_{F_0'} \times \mathfrak{h}^*_{F''} \times A^P_+$ by definition. Now using [**H3**, 7.5] we see that

$$(h, \nu', a_1) \to \int_{\mathfrak{h}^*_{F''}} \frac{f(F' \cup F'' : h : \nu' + \nu'' : a_1)}{p_{F''}(h : \nu'')} d\nu''$$

is jointly smooth for all $F_0'(s) \subseteq F' \subseteq F_0'$. Fix $\alpha \in F_0' \backslash F_0'(s)$ and define I, D_α as above. It suffices to prove that for each $k \geq 0$, we have

$$\lim_{(h,\nu') \to (h_0, \nu_0')} D^k_\alpha \sum_{F_0'(s) \subseteq F' \subseteq F_0'} (-1)^{|F_0' \backslash F'|} \int_{\mathfrak{h}^*_{F''}} \frac{f(F' \cup F'' : h : \nu' + \nu'' : a_1)}{p_{F''}(h : \nu'')} d\nu'' = 0$$

for all $h_0 \in U(\epsilon) \cap \mathcal{H}_I$ semiregular and $\nu_0' \in \mathfrak{a}^*_{F_0'}$ such that $(\nu_0')_\alpha = 0$. Of course the limit is taken through $h \in U^0(\epsilon)$.

But using an easy extension of [**H3**, 7.10], if we write $F_I'' = F'' \cap F_0^I$ there are constants $C(E''), E'' \subseteq F_I''$, so that

$$\lim_{(h,\nu') \to (h_0, \nu_0')} D^k_\alpha \sum_{F_0'(s) \subseteq F' \subseteq F_0'} (-1)^{|F_0' \backslash F'|} \int_{\mathfrak{h}^*_{F''}} \frac{f(F' \cup F'' : h : \nu' + \nu'' : a_1)}{p_{F''}(h : \nu'')} d\nu''$$

$$= \int_{\mathfrak{h}_{F''\backslash F_I''}} d\nu_0'' \sum_{E''\subseteq F_I''} C(E'') \prod_{\beta\in E''} P.V. \int_{\mathbf{R}} \frac{d\nu_\beta}{\nu_\beta}$$

$$\cdot \sum_{F_0'(s)\subseteq F'\subseteq F_0'} (-1)^{|F_0'\backslash F'|} D^k_{F_I''(\alpha)} \frac{f(F'\cup F'' : h_0 : \nu_0' + (\nu_0'', (\nu_\beta)_{\beta\in E''}, 0) : a_1)}{p_{F''\backslash F_I''}(h_0 : \nu_0'')}.$$

Here $D_{F_I''(\alpha)} = \partial/\partial h_I - i\partial/\partial\mu_\alpha - \sum_{\beta\in F_I''} \partial/\partial\mu_\beta$.

Since we have assumed that h_0 is semiregular, $p_{F''\backslash F_I''}(h_0 : \nu_0'') \neq 0$ for all ν_0''. Of course it is also independent of $F_0'(s) \subseteq F' \subseteq F_0'$. Thus it is enough to prove that for all $k \geq 0$ we have

$$\sum_{F_0'(s)\subseteq F'\subseteq F_0'} (-1)^{|F_0'\backslash F'|} D^k_{F_I''(\alpha)} f(F'\cup F'' : h_0 : \nu_0' + (\nu_0'', (\nu_\beta)_{\beta\in E''}, 0) : a_1) = 0.$$

But $\{F = F'\cup F'' : F_0'(s) \subseteq F' \subseteq F_0'\}$ can be written as the disjoint union of sets $\{F, F(\alpha)\}$ where F runs over all subsets of F_0 such that $F_0'(s)\cup F'' \subseteq F \subseteq F_0'\backslash\{\alpha\} \cup F''$. Thus it is enough to prove that for any such F and for any $E'' \subseteq F_I''$ we have

$$\sum_{F\subseteq\widetilde{F}\subseteq F(\alpha)} (-1)^{|\widetilde{F}\backslash F|} D^k_{F_I''(\alpha)} f(F : h_0 : \nu_0' + (\nu_0'', (\nu_\beta)_{\beta\in E''}, 0) : a_1) = 0.$$

But for all such F, using (6.9) we know that

$$\sum_{F\subseteq\widetilde{F}\subseteq F(\alpha)} (-1)^{|\widetilde{F}\backslash F|} D^k_\alpha f(F : h_0 : \nu_0 : a_1) = 0$$

for all $\nu_0 \in \mathfrak{a}^*_{F_0}$ such that $(\nu_0)_\alpha = 0$. But if we write $\nu = (\nu', \nu'')$, then $\nu_\alpha = \nu_\alpha'$ is independent of ν''. Thus we can differentiate with respect to the $\partial/\partial\mu_\beta, \beta \in F_I''$ and evaluate at $\nu'' = (\nu_0'', (\nu_\beta)_{\beta\in E''}, 0)$. □

We now return to the proof of Theorem 6.2. Let $\{\alpha_1, ..., \alpha_d\} = \Theta_P$ be the simple roots of (L_P, A_0). The proof of (6.2) will be by induction on d, the number of simple roots.

Assume that $d = 0$. Then A_0 is central in L_P so that $P = P_0$. Since we assume that $s^{-1}\mathfrak{a}_P \subseteq \mathfrak{a}_{F_0} \subseteq \mathfrak{a}_0$, this occurs only when $\mathfrak{a}_{F_0} = \mathfrak{a}_0$ and $s \in W(\mathfrak{g}, \mathfrak{a}_0)$.

Lemma 6.11. *In the case that $d = 0$ it is enough to prove* (6.2) *in the case that $l_1 = l_2 = 1$.*

Proof. By (6.4) it suffices to prove the theorem in the case that $l_1, l_2 \in \mathcal{U}(\mathfrak{m}_{P_0,\mathbf{C}})$. But $M_0 = M_{P_0} \subset K$ and each $f(F)$ is K_{P_0}-spherical so that

$$f(F : h : \nu : l_1; a; l_2) = d\tau_{1,h}(l_1) f(F : h : \nu : a) d\tau_{2,h}(l_2).$$

Now since $d\tau_{i,h}(l_i), i = 1,2$, are polynomials in h and $f(F)$ is compactly supported as a function of h, for any $D \in D(i\mathfrak{v}^*)$, $\|\Phi(\mathbf{f} : h; D : l_1 ;a; l_2)\|$ can be bounded by a finite number of terms of the form $\|\Phi(\mathbf{f} : h; D' : a)\|, D' \in D(i\mathfrak{v}^*)$. Thus we may as well assume that $l_i = 1, i = 1,2$. □

Lemma 6.12. *Theorem* 6.2 *is true when* $d = 0$.

Proof. In the case that $P = P_0$ we have $\Xi_{P_0} = 1$ and $A_0^+ = A_{P_0}^+$ so that the decomposition $a = a_1 a_2$ is just given by $a = a_2, a_1 = 1$. Using Lemma 6.11 and the fact that $\Xi_{P_0} = 1$, it is enough to estimate

$$\sup_{h \in U^0(\epsilon), a \in A_0^+} (1 + \sigma(a))^r \|\Phi(\mathbf{f} : h; D : a)\|.$$

Use (6.7) to write

$$\Phi(\mathbf{f} : h : a) = \mathrm{P.V.} \int_{\mathfrak{a}^*_{F_0'}} \frac{e^{i\nu}(s^{-1}a) \sum_{F_0'(s) \subseteq F' \subseteq F_0'} g(F' : h : \nu)}{p_{F_0'}(h : \nu)} d\nu$$

where

$$g(F' : h : \nu) = (-1)^{|F_0' \backslash F'|} \sum_{F'' \subseteq F_0''} (\pi i)^{-|F_0' \cup F''|} \int_{\mathfrak{h}^*_{F''}} \frac{f(F' \cup F'' : h : \nu + \nu'' : 1)}{p_{F''}(h : \nu'')} d\nu''.$$

Fix $F = F' \cup F''$ as above. Then since $f(F : \epsilon) \in J_F^1(U : \epsilon : L_P^* : s)$ we know that $f(F : \epsilon : 1) \in \mathcal{C}(U(\epsilon) \times \mathfrak{a}_F^*)$. Now using [**H3**, 7.6] we see that $g(F') \in \mathcal{C}(U(\epsilon) \times \mathfrak{a}_{F'}^*)$ for each F'. By (6.10), the functions $g(F')$ satisfy the hypotheses for Theorem 7.7. Thus using (7.7) we see that given $D \in D(i\mathfrak{v}^*), r \geq 0$, there are $t \geq 0$ and a finite subset E_1 of $D(i\mathfrak{v}^* \times \mathfrak{a}_{F_0'}^*)$ so that

$$\sup_{h \in U^0(\epsilon), a \in A_0^+} (1 + \sigma(a))^r \|\Phi(\mathbf{f} : h; D : a)\| \leq \sum_{D_1 \in E_1} \sum_{F'} \sup_{(h, \nu_{F'}) \in U(\epsilon) \times \mathfrak{a}^*_{F'}} (1 + |\nu_{F'}|)^t \|g(F' : h : \nu_{F'}; D_1)\|.$$

But, again using [**H3**, 7.6], there is a finite subset E_2 of $D(i\mathfrak{v}^* \times \mathfrak{a}_{F_0}^*)$ so that for all $t \geq 0$,

$$\sum_{D_1 \in E_1} \sum_{F'} \sup_{(h, \nu_{F'}) \in U(\epsilon) \times \mathfrak{a}^*_{F'}} (1 + |\nu_{F'}|)^t \|g(F' : h : \nu_{F'}; D_1)\| \leq \sum_{D_2 \in E_2} \sum_{F \subseteq F_0} \sup_{(h, \nu_F) \in U(\epsilon) \times \mathfrak{a}^*_F} (1 + |\nu_F|)^{t + 2|F''|} \|f(F : \epsilon : h : \nu; D_2 : 1)\|.$$

Thus there is a finite subset $E \subset \widetilde{\mathcal{L}}_P$ so that

$$\sup_{h\in U^0(\epsilon),a\in A_0^+} (1+\sigma(a))^r \|\Phi(\mathbf{f}:h;D:a)\|$$
$$\leq \sum_{D'\in E}\sum_{F\subseteq F_0} \sup_{(h,\nu_F)\in U(\epsilon)\times \mathfrak{a}_F^*} \|D'f(F:\epsilon:h:\nu:1)\| \leq \sum_{D'\in E} T^0_{D',r'}(\mathbf{f})$$

for any $r' \geq 0$. □

Fix $d \geq 1$ and assume inductively that Theorem 6.2 is true when $d' < d$. For $1 \leq i \leq d$, let $\mathfrak{a}_i = \{H \in \mathfrak{a}_0 : \alpha_j(H) = 0 \text{ for } 1 \leq j \neq i \leq d\}$, and let $P_0 \subseteq Q_i = L_iN_i \subset P$ be the corresponding standard parabolic subgroup. For $H \in \mathfrak{a}_0$ let $\rho^i(H) = 1/2$ trace ad H on $\mathfrak{n}_i$, $\rho_P(H) = 1/2\sum m(\alpha)\alpha(H), \alpha \in \Delta^+(L_P, A_0), \rho_i(H) = \rho_P(H) - \rho^i(H)$. For $b \geq 0, 1 \leq i \leq d$, let $A^+(i:b) = \{a \in A_0^+ : \alpha_i(\log a) > b\rho_P(\log a)\}$ and $A_+^P(i:b) = \{a_1 \in A_+^P : \alpha_i(\log a) > b\rho_P(\log a)\}$. Fix b small enough that $A_0^+ \subseteq \cup_{i=1}^d A^+(i:b)$. Then if we write $a \in A_0^+$ as $a = a_1a_2$ where $a_1 \in A_+^P, a_2 \in A_P^+$, we see that $a \in A^+(i:b)$ if and only if $a_1 \in A_+^P(i:b)$ since $\rho_P(\log a) = \rho_P(\log a_1)$ and $\alpha_i(\log a) = \alpha_i(\log a_1)$.

Let $m_1, m_2 \in \mathcal{U}(\mathfrak{m}_{P,\mathbf{C}}), r \geq 0, D \in D(i\mathfrak{b}^*)$. Since $A_0^+ \subseteq \cup_{i=1}^d A^+(i:b)$, to complete the proof of (6.2) we must show that for each $1 \leq i \leq d$ there is $E_i \subseteq \widetilde{\mathcal{L}}_P$ so that given any $r' \geq 0$ there is $C \geq 0$ so that

(6.13)
$$\sup_{h\in U^0(\epsilon),a\in A^+(i:b)} \Xi_P^{-1}(a)(1+\sigma(a))^r\|\Phi(\mathbf{f}:h;D:m_1;a;m_2)\| \leq C\sum_{D'\in E_i} T^0_{D',r'}(\mathbf{f})$$

for every matching collection $\mathbf{f}$ of functions in $J^1(U:L_P^*:s)$. Fix an i and drop it from the notations so that $Q = LN = Q_i, A^+(b) = A^+(i:b)$.

Write

$$\mathcal{U}(\mathfrak{m}_{P,\mathbf{C}}) = \mathcal{U}(\mathfrak{k}_{P,\mathbf{C}})\mathcal{U}((\mathfrak{l}\cap\mathfrak{m}_P)_\mathbf{C})\mathcal{U}(\mathfrak{n}_\mathbf{C}) = \mathcal{U}(\theta(\mathfrak{n})_\mathbf{C})\mathcal{U}((\mathfrak{l}\cap\mathfrak{m}_P)_\mathbf{C})\mathcal{U}(\mathfrak{k}_{P,\mathbf{C}}).$$

Given $m_1, m_2 \in \mathcal{U}(\mathfrak{m}_{P,\mathbf{C}})$ there exist $b_1 \in \mathcal{U}(\mathfrak{k}_{P,\mathbf{C}})\mathcal{U}((\mathfrak{l}\cap\mathfrak{m}_P)_\mathbf{C}), b_2 \in \mathcal{U}((\mathfrak{l}\cap\mathfrak{m}_P)_\mathbf{C})\mathcal{U}(\mathfrak{k}_{P,\mathbf{C}})$ and $m_1' \in \mathcal{U}(\mathfrak{m}_{P,\mathbf{C}})\mathfrak{n}, m_2' \in \theta(\mathfrak{n})\mathcal{U}(\mathfrak{m}_{P,\mathbf{C}})$ such that $m_i = b_i + m_i', i = 1, 2$. Thus

$$\Phi(\mathbf{f}:h:m_1;a;m_2) = \Phi(\mathbf{f}:h:m_1';a;m_2) + \Phi(\mathbf{f}:h:b_1;a;m_2')$$
$$+\,\Phi(\mathbf{f}:h:b_1;a;b_2). \tag{6.14}$$

We will estimate each of these terms separately.

Lemma 6.15. *There is a finite subset $E \subset \widetilde{\mathcal{L}}_P$ so that given any $r' \geq 0$ there is $C \geq 0$ so that*

$$\sup_{h\in U^0(\epsilon),a\in A^+(b)} \Xi_P^{-1}(a)(1+\sigma(a))^r\|\Phi(\mathbf{f}:h;D:m_1';a;m_2)\| \leq C\sum_{D'\in E} T^0_{D',r'}(\mathbf{f}).$$

The same is true for $\Phi(\mathbf{f} : h; D : b_1 ; a; m_2')$.

Proof. Combining (6.7), (6.10), (7.7), and [**H3**, 7.6] as in (6.12), there are a constant C and a finite subset F_1 of $P(i\mathfrak{v}^* \times \mathfrak{a}_{F_0}^*)$ so that

$$\begin{aligned} &\sup_{h\in U^0(\epsilon), a_1\in A_+^P(b), a_2\in A_P^+} \Xi_P^{-1}(a_1)(1+\sigma(a_1))^r(1+\sigma(a_2))^r \\ &\qquad \cdot \|\Phi(\mathbf{f} : h; D : m_1' ; a_1a_2; m_2)\| \\ &\leq C \sum_{D_1\in F_1} \sup_{h\in U(\epsilon), a_1\in A_+^P(b), \nu\in\mathfrak{a}_{F_0}^*} \Xi_P(a_1)^{-1}(1+\sigma(a_1))^r \\ &\qquad \cdot \sum_{F\subseteq F_0} \|f(F : h : \nu; D_1 : m_1' ; a_1; m_2)\|. \end{aligned}$$

But now using (3.3) there are a finite subset $E \subseteq \widetilde{\mathcal{L}}_P$ and $r_0 \geq 0$ so that for all $r' \geq 0$ this is bounded by

$$T_{E,r'}^0(\mathbf{f}) \sup_{a_1\in A_+^P(b)} (1+\sigma(a_1))^{r+r'+r_0}\Xi_P^{-1}(a_1)\Xi_Q(a_1)e^{-\beta_Q(\log a_1)}d_Q^{-1}(a_1).$$

But there are constants $D, q \geq 0$ so that $\Xi_Q(a_1) \leq De^{-\rho_i(\log a_1)}(1+\sigma(a_1))^q$. Further, $d_Q^{-1}(a_1) = e^{-\rho^i(\log a_1)}$ and $e^{-\beta_Q(\log a_1)} = e^{-\alpha_i(\log a_1)} \leq e^{-b\rho_P(\log a_1)}$ since $a_1 \in A_+^P(b)$. Thus

$$\Xi_Q(a_1)d_Q^{-1}(a_1)e^{-\beta_Q(\log a_1)} \leq D\Xi_P(a_1)^{1+b}(1+\sigma(a_1))^q$$

so that

$$\begin{aligned} &\sup_{a_1\in A_+^P(b)} (1+\sigma(a_1))^{r+r'+r_0}\Xi_P^{-1}(a_1)\Xi_Q(a_1)e^{-\beta_Q(\log a_1)}d_Q^{-1}(a_1) \\ &\leq D \sup_{a_1\in A_+^P(b)} \Xi_P(a_1)^b(1+\sigma(a_1))^{r+r'+r_0+q}. \end{aligned}$$

But for all $a_1 = \exp(H_1) \in A_+^P, \Xi_P(a_1)^b \leq D'e^{-b\rho_P(H_1)}(1+\sigma(a_1))^{q'}$ as above. But there are $r_\alpha > 0, \alpha \in \Theta_P$, with $b\rho_P = \sum_{\alpha\in\Theta_P} r_\alpha\alpha$ so that $b\rho_P(a_1) = \sum_{\alpha\in\Theta_P} r_\alpha\alpha(H_1)$. Further, there is a constant C' so that $\sigma(a_1) \leq C'\left(\sum_{\alpha\in\Theta_P}\alpha(H_1)^2\right)^{1/2}$. Now for $a_1 \in A_+^P$ we have $\alpha(H_1) > 0$ for all $\alpha \in \Theta_P$ so that

$$\sup_{a_1\in A_+^P(b)} D\Xi_P(a_1)^b(1+\sigma(a_1))^{r+r'+r_0+q} = C_{r'} < \infty.$$

□

Lemma 6.16. *When $d \geq 1$ it is enough to prove* (6.2) *in the case that* $l_i = \beta_i' = d_Q^{-1} \circ \beta_i \circ d_Q$ *where* $\beta_i \in \mathcal{U}((\mathfrak{m}_P \cap \mathfrak{l})_{\mathbf{C}}), i = 1, 2.$

Proof. Because of (6.4) and (6.15) it is enough to consider terms of the form $\Phi(\mathbf{f} : h; D : b_1 ; a; b_2)$ where

$$b_1 \in \mathcal{U}(\mathfrak{k}_{P,\mathbf{C}})\mathcal{U}((\mathfrak{m}_P \cap \mathfrak{l})_{\mathbf{C}}),\ b_2 \in \mathcal{U}((\mathfrak{m}_P \cap \mathfrak{l})_{\mathbf{C}})\mathcal{U}(\mathfrak{k}_{P,\mathbf{C}}).$$

Write $b_1 = \kappa_1\beta_1', b_2 = \beta_2'\kappa_2$, where $\kappa_i \in \mathcal{U}(\mathfrak{k}_{P,\mathbf{C}}), \beta_i \in \mathcal{U}((\mathfrak{m}_P \cap \mathfrak{l})_{\mathbf{C}}), i = 1, 2, \beta_i' = d_Q^{-1} \circ \beta_i \circ d_Q$. Since each $f(F)$ is K_P-spherical, the result follows as in (6.11) for the $d = 0$ case. □

Fix $\beta_i \in \mathcal{U}((\mathfrak{m}_P \cap \mathfrak{l})_{\mathbf{C}}), i = 1, 2$, as above and for each $F \subseteq F_0, (h, \nu, a_1) \in U^0(\epsilon) \times \mathfrak{a}_F^* \times A_+^P$, write

$$\begin{aligned} d(F : \epsilon : h : \nu : a_1) &= f(F : \epsilon : h : \nu : \beta_1' ; a_1; \beta_2') \\ &\quad - \sum_{i=1}^{w} \det s_i\ d_Q^{-1}(a_1) f_{Q,s_i s}(F : \epsilon : h : \nu : \beta_1 ; a_1; \beta_2) \end{aligned}$$

where the constant terms $f_{Q,s_i s}(F : \epsilon)$ are defined as in (3.19).

Lemma 6.17. *There is a finite subset $E \subset \widetilde{\mathcal{L}}_P$ so that given any $r' \geq 0$ there is $C \geq 0$ so that*

$$\begin{aligned} &\sup_{h \in U^0(\epsilon), a_1 \in A_+^P(b), a_2 \in A_P^+} \Xi_P^{-1}(a_1)(1 + \sigma(a_1))^r (1 + \sigma(a_2))^r \\ &\cdot \Bigg\| D\ \text{P.V.} \int_{\mathfrak{a}_{F_0'}^*} \frac{e^{i\nu'}(s^{-1}a_2)}{p_{F_0'}(h : \nu')} \sum_{F_0'(s) \subseteq F \subseteq F_0} (-1)^{|F_0' \backslash F'|} (\pi i)^{-|F''|} \\ &\cdot \int_{\mathfrak{h}_{F''}^*} \frac{d(F : \epsilon : h : \nu' + \nu'' : a_1)}{p_{F''}(h : \nu'')} d\nu'' d\nu' \Bigg\| \\ &\leq C \sum_{D' \in E} T^0_{D',r'}(\mathbf{f}). \end{aligned}$$

Proof. By (3.21) all of the constant terms $\{f_{Q,s_i s}(F : \epsilon)\}$ satisfy the same matching conditions as the original family $\{f(F : \epsilon)\}$. Further, since $\mathbf{f} \in J^1(U : L_P^* : s)$, each $d(F : \epsilon)$ is jointly smooth on $U(\epsilon) \times \mathfrak{a}_F^*$. Thus we have the smoothness result of (6.10) with the terms $f(F : \epsilon)$ replaced by $d(F : \epsilon)$. Now as in (6.15) we can apply (7.7) to obtain a constant C and a finite subset F_1 of $P(i\mathfrak{v}^* \times \mathfrak{a}_{F_0}^*)$ so that

$$\sup_{h \in U^0(\epsilon), a_1 \in A_+^P(b), a_2 \in A_P^+} \Xi_P^{-1}(a_1)(1 + \sigma(a_1))^r (1 + \sigma(a_2))^r$$

$$\cdot \left\| D \text{ P.V.} \int_{\mathfrak{a}^*_{F_0'}} \frac{e^{i\nu'}(s^{-1}a_2)}{p_{F_0'}(h:\nu')} \sum_{F_0'(s)\subseteq F\subseteq F_0} (-1)^{|F_0'\setminus F'|}(\pi i)^{-|F''|} \right.$$
$$\left. \cdot \int_{\mathfrak{h}^*_{F''}} \frac{d(F:\epsilon:h:\nu'+\nu'':a_1)}{p_{F''}(h:\nu'')} d\nu'' d\nu' \right\|$$
$$\leq C \sum_{D_1\in F_1} \sup_{h\in U(\epsilon), a_1\in A_+^P(b), \nu\in\mathfrak{a}^*_{F_0}} \Xi_P^{-1}(a_1)(1+\sigma(a_1))^r$$
$$\cdot \sum_{F\subseteq F_0} \|d(F:\epsilon:h:\nu;D_1:a_1)\|.$$

Write $a_1 = a_1' \exp(TH)$ where $H = H^{\alpha_i} = H(P,Q), T = \alpha_i(\log a_1)$, and $a_1' = \exp(\sum_{\alpha\in\Theta_Q} \alpha(\log a_1)H^\alpha)$. Then $a_1' \in L_Q^*$. Then using (3.19) we have a finite subset $E \subset \widetilde{\mathcal{L}}_P, r_1 > 0$, and $\epsilon_0 > 0$ so that for all $r' \geq 0$ there is a $C > 0$ so that

$$\sum_{D_1\in F_1} \sup_{h\in U(\epsilon), a_1\in A_+^P(b), \nu\in\mathfrak{a}^*_{F_0}} \Xi_P^{-1}(a_1)(1+\sigma(a_1))^r \sum_{F\subseteq F_0} \|d(F:\epsilon:h:\nu;D_1:a_1)\|$$
$$\leq C \sum_{D'\in E} T^0_{D',r'}(\mathbf{f}) \sup_{a_1\in A_+^P(b)} e^{-\epsilon_0 T} d_Q(a_1)^{-1}\Xi_P^{-1}(a_1)\Xi_Q(a_1')(1+\sigma(a_1))^{r+r'+r_1}.$$

But as in (6.15), this last sup over a_1 is finite since $e^{-\epsilon_0 T} = e^{-\epsilon_0\alpha_i(\log a_1)} \leq e^{-\epsilon_0 b\rho_P(\log a_1)}$. □

Lemma 6.18. *Fix $1 \leq i \leq w$ such that $s_i^{-1}s^{-1}\mathfrak{a}_Q \subseteq \mathfrak{a}_{F_0}$. Then there is a finite subset $E \subset \widetilde{\mathcal{L}}_P$ so that given any $r' \geq 0$ there is $C > 0$ so that*

$$\sup_{h\in U^0(\epsilon), a_1\in A_+^P(b), a_2\in A_P^+} \Xi_P^{-1}(a_1)(1+\sigma(a_1))^r(1+\sigma(a_2))^r$$
$$\cdot \left\| D \text{ P.V.} \int_{\mathfrak{a}^*_{F_0'}} \frac{e^{i\nu'}(s^{-1}a_2)}{p_{F_0'}(h:\nu')} \sum_{F_0'(s)\subseteq F\subseteq F_0} (-1)^{|F_0'\setminus F'|}(\pi i)^{-|F''|} \right.$$
$$\left. \cdot \int_{\mathfrak{h}^*_{F''}} \frac{d_Q^{-1}(a_1) f_{Q,s_i s}(F:\epsilon:h:\nu'+\nu'':\beta_1;a_1;\beta_2)}{p_{F''}(h:\nu'')} d\nu'' d\nu' \right\|$$
$$\leq C \sum_{D'\in E} T^0_{D',r'}(\mathbf{f}).$$

Proof. First, as in (6.7) we can rewrite

$$\text{P.V.} \int_{\mathfrak{a}^*_{F_0'}} \frac{e^{i\nu'}(s^{-1}a_2)}{p_{F_0'}(h:\nu')} \sum_{F_0'(s)\subseteq F\subseteq F_0} (-1)^{|F_0'\setminus F'|}$$

$$\cdot (\pi i)^{-|F''|} \int_{\mathfrak{h}^*_{F''}} \frac{d_Q^{-1}(a_1) f_{Q,s_is}(F:\epsilon:h:\nu'+\nu'':\beta_1;a_1;\beta_2)}{p_{F''}(h:\nu'')} d\nu'' d\nu'$$

$$= (\pi i)^{|F_0'|} d_Q^{-1}(a_1) \sum_{F\subseteq F_0} (\pi i)^{-|F|} \int_{\mathfrak{a}^*_F} \frac{f_{Q,s_is}(F:h:\nu:\beta_1;a_1a_2;\beta_2)}{p_F(h:\nu)} d\nu.$$

Now $\Xi_P^{-1}(a_1)d_Q^{-1}(a_1) \leq D\Xi_Q^{-1}(a_1)(1+\sigma(a_1))^q$ and $A^+(b) \subseteq A_0^+$.
Thus

$$\sup_{h\in U^0(\epsilon), a_1\in A_+^P(b), a_2\in A_P^+} \Xi_P^{-1}(a_1)(1+\sigma(a_1))^r(1+\sigma(a_2))^r$$

$$\cdot \Bigg\| D \text{ P.V.} \int_{\mathfrak{a}^*_{F_0'}} \frac{e^{i\nu'}(s^{-1}a_2)}{p_{F_0'}(h:\nu')} \sum_{F_0'(s)\subseteq F\subseteq F_0} (-1)^{|F_0'\setminus F'|}(\pi i)^{-|F''|}$$

$$\cdot \int_{\mathfrak{h}^*_{F''}} \frac{d_Q^{-1}(a_1) f_{Q,s_is}(F:\epsilon:h:\nu'+\nu'':\beta_1;a_1;\beta_2)}{p_{F''}(h:\nu'')} d\nu'' d\nu' \Bigg\|$$

$$\leq C \sup_{h\in U^0(\epsilon), a\in A_0^+} \Xi_Q^{-1}(a)(1+\sigma(a))^{r+q}$$

$$\cdot \Bigg\| D \sum_{F\subseteq F_0} (\pi i)^{-|F|} \int_{\mathfrak{a}^*_F} \frac{f_{Q,s_is}(F:h:\nu:\beta_1;a;\beta_2)}{p_F(h:\nu)} d\nu \Bigg\|.$$

Now by the induction hypothesis and (3.19) there is $E' \subset \widetilde{\mathcal{L}}_Q$ such that for any $r' \geq 0$ there is C' so that the above is bounded by $C' \sum_{D'\in E'} T^0_{D',r'}(\mathbf{f}_{Q,s_is})$. Finally, using (3.19) there is a finite subset E of $\widetilde{\mathcal{L}}_P$ and $r_1 \geq 0$ so that for all $r' \geq 0$ there is $C > 0$ so that

$$C' \sum_{D'\in E'} T^0_{D',r'+r_1}(\mathbf{f}_{Q,s_is}) \leq C \sum_{D''\in E} T^0_{D'',r'}(\mathbf{f}).$$

□

Finally, to complete the proof of Theorem 6.2, we must deal with constant terms $f_{Q,s_is}, 1 \leq i \leq w$, for which $s_i^{-1}s^{-1}\mathfrak{a}_Q \not\subseteq \mathfrak{a}_{F_0}$. Fix such an i. Recall that Q is the standard parabolic subgroup of G corresponding to some simple root $\alpha \in \Theta_P$ and $\mathfrak{a}_Q = \mathfrak{a}_P + \mathbf{R}H^\alpha$. Now $s_i^{-1}s^{-1}\mathfrak{a}_Q = s^{-1}\mathfrak{a}_P + s_i^{-1}s^{-1}\mathbf{R}H^\alpha$. Since we have assumed that $s^{-1}\mathfrak{a}_P \subseteq \mathfrak{a}_{F_0}$, it must be the case that $s_i^{-1}s^{-1}\mathbf{R}H^\alpha \not\subseteq \mathfrak{a}_{F_0}$. Let G_0 be the simple factor of G containing the root α. For any subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, we will write $\mathfrak{b}_0 = \mathfrak{b} \cap \mathfrak{g}_0$. Fix $F \subseteq F_0$. Then we say that $i \in I_0^0$ if $\lambda_i(h:H) = 0$ for all $h \in i\mathfrak{v}^*, H \in \mathfrak{a}_{Q,0}$ and $i \in I_0^-$ if $\lambda_i(h:H) < 0$ for all $h \in U^0(\epsilon), H \in \mathfrak{a}^+_{Q,0}$.

Lemma 6.19. *Let $F \subseteq F_0$. Then $f_{Q,s_is}(F:\epsilon) = 0$ unless $i \in I_0^- \cap I^0(0)$.*

Proof. First, by definition $f_{Q,s_is}(F:\epsilon) = 0$ unless $i \in I^0(0) \subseteq I^0 \cup I^-$. Now if $i \in I^0$, then $i \in I_0^0$ since $\mathfrak{a}_{Q,0} \subseteq \mathfrak{a}_Q$. If $i \in I^-$, then $i \in I_0^0 \cup I_0^-$ since $\mathfrak{a}^+_{Q,0} \subseteq$

$\mathrm{cl}(\mathfrak{a}_Q^+)$. Suppose $i \in I_0^0$. Then the restriction of $f_{Q,s_is}(F : \epsilon)$ to the simple factor G_0 is the ordinary constant term of the restriction of $f(F : \epsilon)$ to G_0 as defined as in [**HW5**]. Thus by [**HW5**, 7.28] we know that $f_{Q,s_is}(F : \epsilon) = 0$ unless $s_i^{-1}s^{-1}\mathfrak{a}_{Q,0} \subseteq \mathfrak{a}_F \subseteq \mathfrak{a}_{F_0}$. But $s_i^{-1}s^{-1}\mathbf{R}H^\alpha \subseteq s_i^{-1}s^{-1}\mathfrak{a}_{Q,0}$ so this is not the case. □

Let $F_{0,0} \subseteq F_0$ be the roots in F_0 coming from the simple factor G_0. If $F_{0,0} = \emptyset$, then λ_F is regular in the simple factor G_0 for all $F \subseteq F_0$ so that $i \in I^0(0)$ implies that $i \in I_0^0$. Thus we may as well assume that $F_{0,0} \neq \emptyset$. Let $F_0^j, 1 \leq j \leq m$ be the equivalence class of F_0 containing $F_{0,0}$.

Lemma 6.20. *Let $F \subseteq F_0, h_0 \in U(\epsilon) \cap \mathcal{H}_j$. Then for any $D \in D(i\mathfrak{b}^* \times \mathfrak{a}_F^*)$ we have*

$$f_{Q,s_is}(F : \epsilon : h_0 : \nu; D : m) = 0$$

for all $(\nu, m) \in \mathfrak{a}_F^* \times L_Q^*$.

Proof. Define I as in (2.23) corresponding to $\mathcal{H}_j$ and ϵ. The proof is by induction on $n = |F(I)\backslash F|$. When $n = 0$, then $\lambda_F = \lambda_{F(I)}$ is regular in the G_0 factor, so that $i \notin I_0^-$. Thus by (6.19), $f_{Q,s_is}(F : \epsilon) = 0$ in this case. Now fix $n \geq 1$ and assume that the lemma is true for $F \subseteq F_0$ such that $|F(I)\backslash F| < n$.

Fix $E \subseteq F_0$ such that $|E(I)\backslash E| = n$. Because of (6.19) we may as well assume that $i \in I_0^-$ with respect to E and ϵ. Since $f_{Q,s_is}(E : \epsilon)$ is jointly smooth on $U(\epsilon)$, it is enough to show that for any choice of h_I as in (2.24), $(\partial/\partial h_I)^k f_{Q,s_is}(E : \epsilon : h_0 : \nu) = 0$ for all $k \geq 0, (h_0, \nu) \in (\mathcal{H}_I \cap U(\epsilon)) \times \mathfrak{a}_E^*$. But combining the matching conditions of (2.24) and (2.25) we can write

$$\begin{aligned}&(\partial/\partial h_I)^k f_{Q,s_is}(E : \epsilon : h_0 : \nu : m) - (\partial/\partial h_I)^k f_{Q,s_is}(E : \epsilon^-(I) : h_0 : \nu : m)\\ &= \sum_{E \subset F \subseteq E(I)} (-1)^{|F\backslash E|+1} D_{F\backslash E}^k f_{Q,s_is}(F : \epsilon : h_0 : (\nu, 0) : m).\end{aligned}$$

Fix $E \subset F \subseteq E(I)$. Then by the induction hypothesis we have that

$$D_{F\backslash E}^k f_{Q,s_is}(F : \epsilon : h_0 : (\nu, 0) : m) = 0.$$

Thus we have

$$(\partial/\partial h_I)^k f_{Q,s_is}(E : \epsilon : h_0 : \nu : m) - (\partial/\partial h_I)^k f_{Q,s_is}(E : \epsilon^-(I) : h_0 : \nu : m) = 0.$$

Fix $\beta_0 \in F_{0,0}$. Then since $\lambda_i(0 : H^\alpha) = 0$, but $\lambda_i(h : H^\alpha)$ is not identically zero, there is a constant $c_0 \neq 0$ so that $\lambda_i(h : H^\alpha) = s_i s h_{M_F}(h)^y(H^\alpha) = c_0 < h_M(h), \beta_0 >$ for all $h \in i\mathfrak{b}^*$. In particular we see that $\lambda_i(h : H^\alpha)$ changes sign as we cross the wall $\mathcal{H}_j$. Thus since $i \in I_0^-$ with respect to ϵ, we must have $i \in I^+$ with respect to $\epsilon^-(I)$. Thus $f_{Q,s_i s}(E : \epsilon^-(I)) = 0$ so that we also have $(\partial/\partial h_I)^k f_{Q,s_i s}(E : \epsilon : h_0 : \nu : m) = 0$. □

For $h \in U(\epsilon)$, let $d_j(h)$ be the Euclidean distance from h to $\mathcal{H}_j$. For $g \in J^0_F(U : \epsilon : L^*_Q : s_i s), D \in \widetilde{\mathcal{L}}_Q, r, t \geq 0$, define

$$T^0_{D,r,t}(g) = \sup_{(h,\nu,x) \in U(\epsilon) \times \mathfrak{a}^*_F \times L^*_Q} \|Dg(h : \nu : x)\| \Xi_Q(x)^{-1}(1+\sigma(x))^{-r}(1+d_j(h)^{-1})^t.$$

Note that $T^0_{D,r,0}(g) = T^0_{D,r}(g)$.

Lemma 6.21. *For all $D \in \widetilde{\mathcal{L}}_Q, r, t \geq 0$, there is a finite subset $S \subset \widetilde{\mathcal{L}}_Q$ so that $T^0_{D,r,t}(g) \leq \sum_{D' \in S} T^0_{D',r}(g)$ for every $g \in J^0_F(U : \epsilon : L^*_Q : s_i s)$ satisfying $g(h_0 : \nu; D : m) = 0$ for all $D \in D(i\mathfrak{b}^* \times \mathfrak{a}^*_F), (h_0, \nu, m) \in U(\epsilon) \times \mathfrak{a}^*_F \times L^*_Q$ such that $h_0 \in \mathcal{H}_j$.*

Proof. This is proved in the same way as [**H1**, 7.12]. □

Lemma 6.22. *Fix $1 \leq i \leq w$ so that $s_i^{-1}s^{-1}\mathfrak{a}_Q \not\subseteq \mathfrak{a}_{F_0}$. Then there is a finite subset $E \subset \widetilde{\mathcal{L}}_P$ so that given any $r' \geq 0$ there is $C > 0$ so that*

$$\sup_{h \in U^0(\epsilon), a_1 \in A^P_+(b), a_2 \in A^+_P} \Xi_P^{-1}(a_1)(1+\sigma(a_1))^r(1+\sigma(a_2))^r$$
$$\cdot \left\| D \text{ P.V.} \int_{\mathfrak{a}^*_{F'_0}} \frac{e^{i\nu'}(s^{-1}a_2)}{p_{F'_0}(h : \nu')} \sum_{F'_0(s) \subseteq F \subseteq F_0} (-1)^{|F'_0 \setminus F'|}(\pi i)^{-|F''|} \right.$$
$$\left. \cdot \int_{\mathfrak{h}^*_{F''}} \frac{d_Q^{-1}(a_1) f_{Q,s_i s}(F : \epsilon : h : \nu' + \nu'' : \beta_1 ; a_1 ; \beta_2)}{p_{F''}(h : \nu'')} d\nu'' d\nu' \right\|$$
$$\leq C \sum_{D' \in E} T^0_{D',r'}(\mathbf{f}).$$

Proof. By (3.21) all of the constant terms $\{f_{Q,s_i s}(F : \epsilon)\}$ satisfy the same matching conditions as the original family $\{f(F : \epsilon)\}$. Further, since $\mathbf{f} \in J^1(U : L^*_P : s)$, each $f_{Q,s_i s}(F : \epsilon)$ is jointly smooth on $U(\epsilon) \times \mathfrak{a}^*_F$. Thus we have the smoothness result of (6.10) with the terms $f(F : \epsilon)$ replaced by $f_{Q,s_i s}(F : \epsilon)$. Now as in (6.15) we can apply (7.7) to obtain a constant C and a finite subset E_1 of $D(i\mathfrak{b}^* \times \mathfrak{a}^*_{F_0})$ so that

$$\sup_{h \in U^0(\epsilon), a_1 \in A^P_+(b), a_2 \in A^+_P} \Xi_P^{-1}(a_1)(1+\sigma(a_1))^r(1+\sigma(a_2))^r$$
$$\cdot \left\| D \text{ P.V.} \int_{\mathfrak{a}^*_{F'_0}} \frac{e^{i\nu'}(s^{-1}a_2)}{p_{F'_0}(h : \nu')} \sum_{F'_0(s) \subseteq F \subseteq F_0} (-1)^{|F'_0 \setminus F'|}(\pi i)^{-|F''|} \right.$$

$$\cdot \int_{\mathfrak{h}^*_{F''}} \frac{d_Q^{-1}(a_1) f_{Q,s_i s}(F : \epsilon : h : \nu' + \nu'' : \beta_1 ; a_1 ; \beta_2)}{p_{F''}(h : \nu'')} d\nu'' d\nu' \Bigg\|$$
$$\leq C \sum_{D_1 \in E_1} \sup_{h \in U(\epsilon), a_1 \in A_+^P(b), \nu \in \mathfrak{a}^*_{F_0}} \Xi_P^{-1}(a_1)(1 + \sigma(a_1))^r d_Q^{-1}(a_1)$$
$$\cdot \sum_{F \subseteq F_0} \| f_{Q,s_i s}(F : \epsilon : h : \nu ; D_1 : \beta_1 ; a_1 ; \beta_2) \|.$$

Fix $F \subseteq F_0$. If $i \notin I_0^- \cap I^0(0)$, then by (6.19) we know that $f_{Q,s_i s}(F : \epsilon) = 0$. Thus we assume that $i \in I_0^- \cap I^0(0)$.

Write $a_1 = a_1' \exp(TH)$ as in (6.17). Then using (3.19), for each $D_1 \in E_1$ we have a finite subset $S \subset \widetilde{\mathcal{L}}_P, r_1 > 0$, and $\epsilon_i(h)$ so that for all $t, r' \geq 0$ there is a $C > 0$ so that

$$\sup_{h \in U(\epsilon), a_1 \in A_+^P(b), \nu \in \mathfrak{a}^*_{F_0}} \Xi_P^{-1}(a_1)(1 + \sigma(a_1))^r d_Q^{-1}(a_1)$$
$$\cdot \| f_{Q,s_i s}(F : \epsilon : h : \nu ; D_1 : \beta_1 ; a_1 ; \beta_2) \|$$
$$\leq C \sum_{D_2 \in S} T^0_{D_2, r', t}(f_{Q,s_i s}(F : \epsilon)) \sup_{h \in U(\epsilon), a_1 \in A_+^P(b)} (1 + d_j(h)^{-1})^{-t}$$
$$\cdot e^{-\epsilon_i(h) T} d_Q(a_1)^{-1} \Xi_P^{-1}(a_1) \Xi_Q(a_1)(1 + \sigma(a_1))^{r + r' + r_1}.$$

But as before, for all $h \in U(\epsilon), a_1 \in A_+^P(b)$,

$$d_Q(a_1)^{-1} \Xi_P^{-1}(a_1) \Xi_Q(a_1) e^{-\epsilon_i(h) T} \leq D \Xi_P(a_1)^{b \epsilon_i(h)} (1 + \sigma(a_1))^q$$

for some constants $D, q \geq 0$. But as in [**HW5**, 8.7], since $\epsilon_i(h) \geq 0$ for $h \in U(\epsilon)$ and $\epsilon_i(h_0) = 0$ for $h_0 \in U(\epsilon)$ only when $h_0 \in \mathcal{H}_j$ so that $d_j(h_0) = 0$ also, there exist $C, t \geq 0$ so that $\sup_{h \in U(\epsilon), a_1 \in A_+^P(b)} (1 + \sigma(a_1))^{r + r' + r_1 + q} \Xi_P(a_1)^{b \epsilon_i(h)} \leq C(1 + d_j(h)^{-1})^t$. Now use (6.21) to obtain an estimate involving terms of the form $T^0_{D', r'}(f_{Q,s_i s}(F : \epsilon))$. Finally, as in (6.18), we use (3.19) to obtain estimates with terms of the form $T^0_{D'', r'}(f(F : \epsilon))$. $\square$

7. Calculus lemmas.

If f is a locally integrable function on $\mathbf{R}$, define

$$\text{P.V.} \int_{\mathbf{R}} f(x)\, dx = \lim_{M \to +\infty} \int_{-M}^{M} f(x)\, dx$$

whenever the limit exists. Let $S^{\pm} = \{(h, t) \in \mathbf{R}^2 : \pm ht > 0\}$.

Lemma 7.1. *Suppose* $(h, t) \in S^-$. *Then*

$$\text{P.V.} \int_{\mathbf{R}} \frac{e^{ixt}}{x + ih} dx = 2\pi i (sgn\ t) e^{ht}.$$

If $(h,t) \in S^+$ *then for all integers* r, k *with* $0 \le r \le k$,

$$\text{P.V.} \int_{\mathbf{R}} \frac{e^{ixt}x^r}{(x+ih)^{k+1}} dx = 0.$$

Proof. We will prove the formula when $t > 0$. The result for $t < 0$ follows by making the change of variables $x \mapsto -x$.

Fix $t > 0, h \neq 0, 0 \le r \le k$, and define $f(z) = e^{izt}z^r$. For $M > 2|h|$, let C_M be the contour in the complex plane which is the union of the four line segments $C_i, 1 \le i \le 4$, where C_1 runs from $-M$ to M, C_2 runs from M to $M + iM$, C_3 runs from $M + iM$ to $-M + iM$, and C_4 runs from $-M + iM$ to $-M$. Then it is easy to calculate that for $2 \le i \le 4$,

$$\lim_{M \to +\infty} \int_{C_i} \frac{f(z)}{(z+ih)^{k+1}} dz = 0.$$

Now if $h > 0$, then $-ih$ is outside the contour C_M so that

$$\text{P.V.} \int_{\mathbf{R}} \frac{e^{ixt}x^r}{(x+ih)^{k+1}} dx = 0.$$

If $h < 0$, then $-ih$ is inside the contour C_M so that

$$\text{P.V.} \int_{\mathbf{R}} \frac{e^{ixt}}{x+ih} dx = 2\pi i e^{ht}.$$

□

Lemma 7.2.

$$\sup_{(h,t)\in S^-} \left| \int_{-1}^{1} \frac{e^{ixt}}{x+ih} dx \right| < \infty.$$

For all $0 \le r \le k$,

$$\sup_{(h,t)\in S^+} \left| \int_{-1}^{1} \frac{e^{ixt}x^r}{(x+ih)^{k+1}} dx \right| < \infty.$$

Proof. Use the contour C_M with $M = 1$ to estimate the integrals as in (7.1). □

Lemma 7.3. *For every integer* $k \ge 0$,

$$\sup_{(h,t)\in S^-} \left| \text{P.V.} \int_{|x|\ge 1} \frac{e^{ixt}}{(x+ih)^{k+1}} dx \right| < \infty.$$

Proof. Suppose $k \geq 1$. Then for all $(h,t) \in S^-$,

$$\left|\text{P.V.} \int_{|x|\geq 1} \frac{e^{ixt}}{(x+ih)^{k+1}} dx\right| \leq \int_{|x|\geq 1} |x|^{-k-1} dx < \infty.$$

Now if $k=0$, using (7.1), (7.2),

$$\begin{aligned}\sup_{(h,t)\in S^-} \left|\text{P.V.} \int_{|x|\geq 1} \frac{e^{ixt}}{(x+ih)} dx\right| &\leq \sup_{(h,t)\in S^-} \left|\text{P.V.} \int_{\mathbf{R}} \frac{e^{ixt}}{(x+ih)} dx\right| \\ &+ \sup_{(h,t)\in S^-} \left|\int_{-1}^{1} \frac{e^{ixt}}{x+ih} dx\right| < \infty.\end{aligned}$$

□

Let $a > 0$ and write $I^+(0,a] = (0,a], I^-(0,a] = [-a,0), I^\pm[0,a] = I^\pm(0,a] \cup \{0\}$. Define $\mathbf{R}^\pm = \{x \in \mathbf{R} : \pm x > 0\}$. For any Banach space W define $\mathcal{C}(I^\pm[0,a] \times \mathbf{R} : W) =$

$$\{g \in C^\infty(I^\pm[0,a] \times \mathbf{R} : W) : \|g\|_{D,r} < \infty \quad \text{for all } D \in D(\mathbf{R}^2), r \geq 0\}$$

where

$$\|g\|_{D,r} = \sup_{(h,x)\in I^\pm[0,a]\times\mathbf{R}} (1+|x|)^r |Dg(x:h)|.$$

Lemma 7.4. *Suppose $f \in \mathcal{C}(I^\pm[0,a] \times \mathbf{R} : W), k \geq 0$. Then there are $C > 0$ and a finite subset F of $D(\mathbf{R}^2)$ so that*

$$\sup_{(h,t)\in I^\pm(0,a]\times\mathbf{R}^\pm} \left|\int_{\mathbf{R}} \frac{f(h:x)e^{ixt}}{(x+ih)^{k+1}} dx\right| \leq C \sum_{D\in F} \|f\|_{D,2}.$$

Proof. First, for all (h,t),

$$\begin{aligned}&\left|\int_{\mathbf{R}} \frac{f(h:x)e^{ixt}}{(x+ih)^{k+1}} dx\right| \\ &\leq \left|\int_{-1}^{1} \frac{f(h:x)e^{ixt}}{(x+ih)^{k+1}} dx\right| + \left|\int_{|x|\geq 1} \frac{f(h:x)e^{ixt}}{(x+ih)^{k+1}} dx\right|.\end{aligned}$$

Now

$$\left|\int_{|x|\geq 1} \frac{f(h:x)e^{ixt}}{(x+ih)^{k+1}} dx\right| \leq \int_{|x|\geq 1} |f(h:x)| dx \leq C\|f\|_{1,2}.$$

Fix h and expand f in its Taylor series at $x = 0$ as

$$f(h:x) = \sum_{r=0}^{k} \frac{(\partial/\partial x)^r f(h:0) x^r}{r!} + R(h:x)$$

where the remainder term $R(h:x)$ satisfies

$$\sup_{h,x} |R(h:x)x^{-k-1}| \leq \sup_{h,x} \frac{|(\partial/\partial x)^{k+1} f(h:x)|}{(k+1)!}.$$

Now

$$\left| \int_{-1}^{1} \frac{f(h:x)e^{ixt}}{(x+ih)^{k+1}} dx \right| \leq \sum_{r=0}^{k} \left| \frac{(\partial/\partial x)^r f(h:0)}{r!} \right| \left| \int_{-1}^{1} \frac{e^{ixt} x^r}{(x+ih)^{k+1}} dx \right| + \left| \int_{-1}^{1} \frac{e^{ixt} R(h:x)}{(x+ih)^{k+1}} dx \right|.$$

Using (7.2), for each $0 \leq r \leq k$ there are $C_r > 0$ and $D_r = (\partial/\partial x)^r$ so that for every $(h,t) \in I^{\pm}(0,a] \times \mathbf{R}^{\pm} \subset S^+$,

$$\left| \frac{(\partial/\partial x)^r f(h:0)}{r!} \right| \left| \int_{-1}^{1} \frac{e^{ixt} x^r}{(x+ih)^{k+1}} dx \right| \leq C_r \|f\|_{D_r,0}.$$

Finally,

$$\left| \int_{-1}^{1} \frac{e^{ixt} R(h:x)}{(x+ih)^{k+1}} dx \right| \leq 2 \sup_{h,x} \frac{|(\partial/\partial x)^{k+1} f(h:x)|}{(k+1)!} = C_{k+1} \|f\|_{D_{k+1},0}.$$

□

Lemma 7.5. *Suppose $f \in \mathcal{C}(I^{\pm}[0,a] \times \mathbf{R} : W)$. Then given $s, k \geq 0$ there are $C > 0$ and a finite subset F of $D(\mathbf{R}^2)$ so that*

$$\sup_{(h,t) \in I^{\pm}(0,a] \times \mathbf{R}^{\pm}} (1+|t|)^s \left| (\partial/\partial h)^k \int_{\mathbf{R}} \frac{f(h:x)e^{ixt}}{(x+ih)} dx \right| \leq C \sum_{D \in F} \|f\|_{D,2}.$$

Proof. Since $h \neq 0$, we can differentiate under the integral so that

$$(\partial/\partial h)^k \int_{\mathbf{R}} \frac{f(h:x)e^{ixt}}{(x+ih)} dx = \int_{\mathbf{R}} e^{ixt} (\partial/\partial h)^k (f(h:x)(x+ih)^{-1}) dx.$$

But there are $D_r \in D(\mathbf{R}), 0 \leq r \leq k$, so that

$$(\partial/\partial h)^k (f(h:x)(x+ih)^{-1}) = \sum_{r=0}^{k} \frac{f(h;D_r:x)}{(x+ih)^{r+1}}.$$

Further, for each r and each integer $s \geq 0$ we can integrate by parts to write

$$\int_{\mathbf{R}} \frac{f(h; D_r : x)e^{ixt}}{(x+ih)^{r+1}} dx = (it)^{-s} \int_{\mathbf{R}} e^{ixt} (\partial/\partial x)^s (f(h; D_r : x)(x+ih)^{-r-1}) dx.$$

But again, there are $D'_j \in D(\mathbf{R}), 0 \leq j \leq s$, so that

$$(\partial/\partial x)^s (f(h; D_r : x)(x+ih)^{-r-1}) = \sum_{j=0}^{s} f(h; D_r : x; D'_j)(x+ih)^{-r-j-1}.$$

Now the result follows from (7.4). □

Lemma 7.6. *Suppose that* $f \in \mathcal{C}(I^{\pm}[0,a] \times \mathbf{R} : W)$ *and* $g \in C^{\infty}(I^{\pm}[0,a] : W)$ *such that*

$$(h,x) \mapsto \phi(h:x) = \frac{f(h:x) + g(h)}{x+ih}$$

is jointly smooth for $(h,x) \in I^{\pm}[0,a] \times \mathbf{R}$. *Then given* $s, k \geq 0$ *there are* $t \geq 0$, *and a finite subset* F *of* $D(\mathbf{R}^2)$ *so that*

$$\sup_{(h,t) \in I^{\pm}(0,a] \times \mathbf{R}^{\mp}} (1+|t|)^s \left| (\partial/\partial h)^k \, \mathrm{P.V.} \int_{\mathbf{R}} e^{ixt} \phi(h:x) dx \right| \leq \sum_{D \in F} \left(\|f\|_{D,t} + \sup_{h \in I^{\pm}[0,a]} |Dg(h)| \right).$$

Proof. As in (7.5) it is enough to estimate terms of the form

$$\left| \mathrm{P.V.} \int_{\mathbf{R}} e^{ixt} \phi(h:x;D) dx \right| \quad \text{where } D \in D(\mathbf{R}^2).$$

Now

$$\sup_{(h,t)} \left| \int_{|x| \leq 1} e^{ixt} \phi(h:x;D) dx \right| \leq 2 \sup_{h \in I^{\pm}[0,a], |x| \leq 1} |\phi(h:x;D)|.$$

Now, for each $D \in D(\mathbf{R}^2)$ it follows from [**H2**, 6.7] that there is a finite subset F of $D(\mathbf{R}^2)$ so that

$$\sup_{h \in I^{\pm}[0,a], |x| \leq 1} |\phi(h:x;D)| \leq \sum_{D' \in F} \sup_{h \in I^{\pm}[0,a], |x| \leq 1} |f(h:x;D') + g(h;D')| \leq \sum_{D' \in F} \left(\|f\|_{D',0} + \sup_{h \in I^{\pm}[0,a]} |D'g(h)| \right).$$

Now there are finitely many $k \geq 0, D_k \in D(\mathbf{R}^2)$, so that

$$\phi(h : x; D) = \sum_k \frac{f(h : x; D_k) + g(h; D_k)}{(x + ih)^{k+1}}.$$

For each such $k \geq 0$,

$$\left| \text{P.V.} \int_{|x| \geq 1} \frac{e^{ixt}(f(h : x; D_k) + g(h; D_k))}{(x + ih)^{k+1}} dx \right|$$
$$\leq \int_{|x| \geq 1} |f(h : x; D_k)| dx + |g(h; D_k)| \left| \text{P.V.} \int_{|x| \geq 1} \frac{e^{ixt}}{(x + ih)^{k+1}} dx \right|.$$

Thus using (7.3) there are $C_k \geq 0$ so that

$$\sup_{(h,t)} \left| \text{P.V.} \int_{|x| \geq 1} e^{ixt} \phi(h : x; D) dx \right| \leq \sum_k C_k \left(\|f\|_{D_k, 2} + \sup_{h \in I^{\pm}[0,a]} |D_k g(h)| \right).$$

□

We now turn to the notation of §6, in particular (6.7) and (6.10). For any $F_0'(s) \subseteq F' \subseteq F_0'$, define $\mathcal{C}(U(\epsilon) \times \mathfrak{a}_{F'}^*) =$

$$\{g \in C^\infty(U(\epsilon) \times \mathfrak{a}_{F'}^*) : \|g\|_{D,r} < \infty \quad \text{for all } D \in D(i\mathfrak{v}^* \times \mathfrak{a}_{F'}^*), r \geq 0\}$$

where

$$\|g\|_{D,r} = \sup_{(h,\nu) \in U(\epsilon) \times \mathfrak{a}_{F'}^*} (1 + |\nu|)^r \|g(h : \nu; D)\|.$$

Given $g \in \mathcal{C}(U(\epsilon) \times \mathfrak{a}_{F'}^*)$, extend g to $U(\epsilon) \times \mathfrak{a}_{F_0'}^*$ by $g(h : \nu_{F_0'}) = g(h : \nu_{F'})$ where for $\nu_{F_0'} \in \mathfrak{a}_{F_0'}^*, \nu_{F'}$ is the restriction of $\nu_{F_0'}$ to $\mathfrak{a}_{F'}$.

Theorem 7.7. *Suppose for $F_0'(s) \subseteq F' \subseteq F_0'$ there are $g(F') \in \mathcal{C}(U(\epsilon) \times \mathfrak{a}_{F'}^*)$ such that*

$$(h, \nu) \mapsto p_{F_0' \setminus F_0'(s)}(h : \nu)^{-1} \sum_{F_0'(s) \subseteq F' \subseteq F_0'} g(F' : h : \nu)$$

is jointly smooth on $U(\epsilon) \times \mathfrak{a}_{F_0'}^*$. Then given $D \in D(i\mathfrak{v}^*), r \geq 0$, there are $t \geq 0$ and a finite subset E of $D(i\mathfrak{v}^* \times \mathfrak{a}_{F_0'}^*)$ so that

$$\sup_{h \in U^0(\epsilon), a \in A_P^+} (1 + \sigma(a))^r \left\| D \ \text{P.V.} \int_{\mathfrak{a}_{F_0'}^*} \frac{e^{i\nu}(s^{-1}a) \sum_{F_0'(s) \subseteq F' \subseteq F_0'} g(F' : h : \nu)}{p_{F_0'}(h : \nu)} d\nu \right\|$$
$$\leq \sum_{D' \in E} \sum_{F_0'(s) \subseteq F' \subseteq F_0'} \|g(F')\|_{D',t}.$$

Proof. As in (6.5) we write $s^{-1}a = a_\emptyset \exp\left(\sum_{\alpha\in F_0'} t_\alpha H^*_{c_{F_0}\alpha}\right)$. Now there is $C_1 > 0$ so that

$$(1+\sigma(a))^r \le C_1(1+\sigma(a_\emptyset))^r \prod_{\alpha\in F_0'} (1+|t_\alpha|)^r.$$

Write

$$\begin{aligned} &\mathrm{P.V.}\int_{\mathfrak{a}^*_{F_0'}} \frac{e^{i\nu}(s^{-1}a)\sum_{F'} g(F':h:\nu)}{p_{F_0'}(h:\nu)}d\nu \\ &= \int_{\mathfrak{a}^*_\emptyset} d\nu_\emptyset e^{i\nu_\emptyset}(a_\emptyset) \prod_{\alpha\in F_0'} \mathrm{P.V.}\int_{\mathbf{R}} d\nu_\alpha \frac{e^{i\nu_\alpha t_\alpha}}{(\nu_\alpha + ih_\alpha)} \sum_{F'} g(F':h:\nu). \end{aligned}$$

Now since each $g(F':h:\nu)$ is Schwartz as a function of $\nu_\emptyset$, we have $t_1 \ge 0$ and a finite subset E_1 of $D(i\mathfrak{b}^* \times \mathfrak{a}^*_\emptyset)$ so that

$$\begin{aligned} &\sup_{(h,a_\emptyset)} (1+\sigma(a_\emptyset))^r \\ &\cdot \left\| D\int_{\mathfrak{a}^*_\emptyset} d\nu_\emptyset e^{i\nu_\emptyset}(a_\emptyset) \prod_{\alpha\in F_0'} \mathrm{P.V.}\int_{\mathbf{R}} d\nu_\alpha \frac{e^{i\nu_\alpha t_\alpha}}{(\nu_\alpha + ih_\alpha)} \sum_{F'} g(F':h:\nu)\right\| \\ &\le \sum_{D_1\in E_1} \sup_{(h,\nu_\emptyset)} (1+|\nu_\emptyset|)^{t_1} \left\| D_1 \prod_{\alpha\in F_0'} \mathrm{P.V.}\int_{\mathbf{R}} d\nu_\alpha \frac{e^{i\nu_\alpha t_\alpha}}{(\nu_\alpha + ih_\alpha)} \sum_{F'} g(F':h:\nu)\right\|. \end{aligned}$$

For each $\alpha \in F_0'(s)$ we have $t_\alpha h_\alpha > 0$ for all $a \in A_P^+, h \in U^0(\epsilon)$. Further, each $g(F':h:\nu)$ is Schwartz as a function of ν_α. Thus applying (7.5), there are $t_2 \ge 0$ and a finite subset E_2 of $D(i\mathfrak{b}^* \times \mathfrak{a}_{F_0'(s)})$ so that

$$\begin{aligned} &\sum_{D_1\in E_1} \sup_{(h,\nu_\emptyset)} (1+|\nu_\emptyset|)^{t_1} \left\| D_1 \prod_{\alpha\in F_0'} \mathrm{P.V.}\int_{\mathbf{R}} d\nu_\alpha \frac{e^{i\nu_\alpha t_\alpha}}{(\nu_\alpha + ih_\alpha)} \sum_{F'} g(F':h:\nu)\right\| \\ &\le \sum_{D_2\in E_2} \sup_{h,\nu_{F_0'(s)}} (1+|\nu_{F_0'(s)}|)^{t_2} \\ &\cdot \left\| D_2 \prod_{\alpha\in F_0'\setminus F_0'(s)} \mathrm{P.V.}\int_{\mathbf{R}} d\nu_\alpha \frac{e^{i\nu_\alpha t_\alpha}}{(\nu_\alpha + ih_\alpha)} \sum_{F'} g(F':h:\nu)\right\|. \end{aligned}$$

Finally, for each $\alpha \in F_0'\setminus F_0'(s)$ we have $t_\alpha h_\alpha < 0$ for all $a \in A_P^+, h \in U^0(\epsilon)$, and $(h,\nu) \mapsto (\nu_\alpha + ih_\alpha)^{-1}\sum_{F'} g(F':h:\nu)$ is jointly smooth on $U(\epsilon) \times \mathfrak{a}^*_{F_0'}$ by hypothesis. Let S_α be the set of all F' such that $F_0'(s) \subseteq F' \subseteq F_0'$ and $\alpha \in F'$ and let S^α be the set of all F' such that $F_0'(s) \subseteq F' \subseteq F_0'$ and $\alpha \notin F'$. Then we can write

$$\sum_{F_0'(s)\subseteq F'\subseteq F_0'} g(F':h:\nu) = \sum_{F'\in S_\alpha} g(F':h:\nu) + \sum_{F'\in S^\alpha} g(F':h:\nu)$$

where $\sum_{F'\in S_\alpha} g(F' : h : \nu)$ is Schwartz as a function of ν_α and $\sum_{F'\in S^\alpha}$ $g(F' : h : \nu)$ is independent of ν_α. Thus applying (7.6) to each $\alpha \in F_0'\backslash F_0'(s)$ we have $t \geq 0$ and a finite subset E of $D(i\mathfrak{v}^* \times \mathfrak{a}^*_{F_0'})$ so that

$$\sum_{D_2\in E_2} \sup_{h,\nu_{F_0'(s)}} (1 + |\nu_{F_0'(s)}|)^{t_2}$$
$$\cdot \left\| D_2 \prod_{\alpha\in F_0'\backslash F_0'(s)} \text{P.V.} \int_{\mathbf{R}} d\nu_\alpha \frac{e^{i\nu_\alpha t_\alpha}}{(\nu_\alpha + ih_\alpha)} \sum_{F'} g(F' : h : \nu) \right\|$$
$$\leq \sum_{D'\in E} \sum_{F_0'(s)\subseteq F'\subseteq F_0'} \sup_{(h,\nu_{F'})\in U(\epsilon)\times\mathfrak{a}^*_{F'}} (1 + |\nu_{F'}|)^t \|g(F' : h : \nu_{F'}; D')\|.$$

□

References

[CM] W. Casselman and D. Miličić, *Asymptotic behavior of matrix coefficients of admissible representations*, Duke Math. J., **49** (1982), 869-930.

[HC1] Harish-Chandra, *Harmonic analysis on real reductive groups I*, J. Funct. Anal., **19** (1975), 104 -204.

[HC2] ———, *Harmonic analysis on real reductive groups II*, Inv. Math., **36** (1976), 1-55.

[HC3] ———, *Harmonic analysis on real reductive groups III*, Annals of Math., **104** (1976), 117-201.

[H1] R. Herb, *The Schwartz space of a general semisimple Lie group II, Wave packets associated to Schwartz functions*, Trans. AMS., **327** (1991), 1-69.

[H2] ———, *The Schwartz space of a general semisimple Lie group III, c-functions*, Advances in Math., **99** (1993), 1-25.

[H3] ———, *The Schwartz space of a general semisimple Lie group IV, Elementary mixed wave packets*, Compositio Math., **84** (1992), 115-209.

[HW1] R. Herb and J. Wolf, *The Plancherel theorem for general semisimple groups*, Compositio Math., **57** (1986), 271-355.

[HW2] ———, *Rapidly decreasing functions on general semisimple groups*, Compositio Math., **58** (1986), 73-110.

[HW3] ———, *Wave packets for the relative discrete series I: The holomorphic case*, J. Funct. Anal., **73** (1987), 1-37.

[HW4] ———, *Wave packets for the relative discrete series II: The non-holomorphic case*, J. Funct. Anal., **73** (1987), 38-106.

[HW5] ———, *The Schwartz space of a general semisimple group I, Wave packets of Eisenstein integrals*, Advances in Math., **80** (1990), 164-224.

[KM] H. Kraljević and D. Miličić, *The C^*-algebra of the universal covering group of $SL(2,\mathbf{R})$*, Glasnik Mat. Ser. III, **7(27)** (1972), 35-48.

[S] W. Schmid, *Two character identities for semisimple Lie groups*, (Proc. Marseille Conf., 1976) Lecture Notes in Math., Vol. 587, Springer-Verlag, Berlin and New York, 1977.

[W] J. Wolf, *Unitary representations on partially holomorphic cohomology spaces*, Memoirs A.M.S., **138** (1974).

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